Boundary conditions of the $O(n)$ model on a dynamical lattice

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Outline

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- We consider a lattice Γ, to each point $r \in \Gamma$ we associate an $O(n)$ spin $S_a(r)$ with $a = 1 \cdots n$ and normalized such that $\mathrm{tr}\ S_a(r)S_b(r') = \delta_{ab}\delta_{rr'}$.
- The 'geometric' partition function reads

$$Z_\Gamma(T) = \mathrm{tr} \prod_{<rr'>} \left(1 + \frac{1}{T}\sum_{a=1}^{n} S_a(r)S_a(r')\right).$$

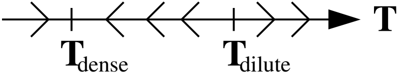
Introduction: the $O(n)$ lattice model

Reformulation as a loop gas model

- The $O(n)$ model can be reformulated as a sum over configurations of self-avoiding, mutually avoiding loops of weight n,

$$Z_\Gamma(T) = \sum_{\text{loops}} T^{-\text{length}} n^{\#\text{ loops}}.$$

- This formulation makes sense for arbitrary n. It exhibits a critical behavior when $\mid n \mid \leq 2$.
- The phase diagram has two critical points:



Introduction: the $O(n)$ lattice model

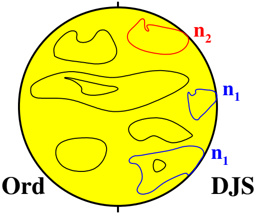
Boundary conditions

Ordinary boundary conditions:

Loops avoid to touch the boundary.

JS boundary conditions:

Loops with weight k on the boundary.

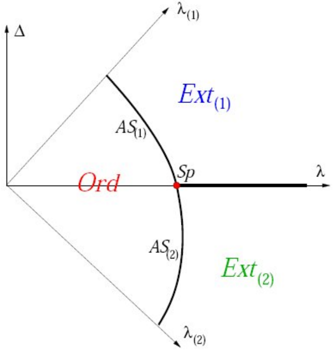


Dilute JS boundary conditions:

Split the spins component in two orthogonal sets $\overrightarrow{S} = \overrightarrow{S}_1 + \overrightarrow{S}_2$.
Leads to two kinds of loops, with weight n_1 and n_2 ($n = n_1 + n_2$), and coupling constants λ_1 and λ_2.

Flat lattice results from DJS

DJS BC in the dilute phase



- **Ord**: Loops avoids the boundary.
- **AS**$_{(1)}$: Loops of weight $n^{(1)}$ criticaly enhanced.
- **AS**$_{(2)}$: Loops of weight $n^{(2)}$ criticaly enhanced.
- **Sp**: Both loops touch the boundary.

Perturbation : Boundary thermal operator $\Phi_{1,3}$

Boundary anisotropic operator $\Phi_{3,3}$

Study of the DJS BC using the matrix model

In collaboration with K. Hosomichi and I. Kostov

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   - Conformal weight of Ord/JS operators.
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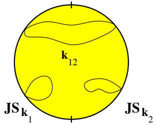

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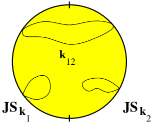


3. Ord/DJS bcc op. in the dilute phase arXiv:0910.1581
   - Phase diagram of DJS boundary conditions (Δ, λ).
   - Conformal weight of Ord/Sp and Ord/AS bcc operators.
   - Conformal weight of operators generating the flows.
   - Bulk thermal flow of conformal b.c. $(r, s) \to (s-1, r)$.

Outline

The $O(n)$ model on a dynamical lattice

Introduction of the matrix model I

- The partition function on the dynamical lattice is obtained as a sum over random lattices

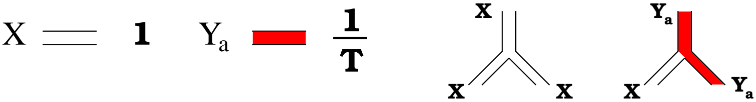
$$Z_{\text{dyn}}(\kappa, T) = \sum_{\Gamma} \kappa^{-A(\Gamma)} Z_{\Gamma}(T)$$

- It can be generated as an expansion of the $O(n)$ matrix model

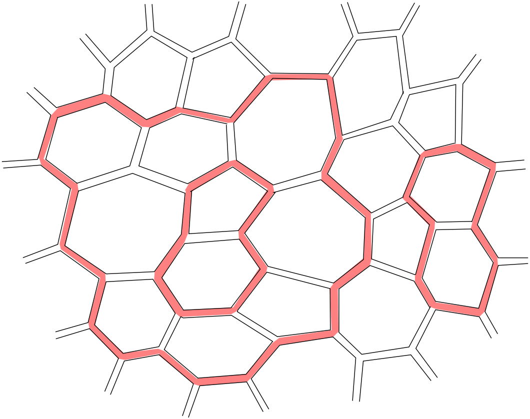
$$Z = \int dX \prod_{a=1}^{n} dY_a e^{\beta \mathrm{tr} \left(-\frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{T}{2}\sum_{a=1}^{n} Y_a^2 + \sum_{a=1}^{n} XY_a^2 \right)}.$$

where X and Y_a are $N \times N$ hermitian matrices.

- Propagators and vertices:



- A loop configuration:



- Planar limit (disc corr): $(\beta, N) \to \infty$, $\beta/N = \kappa^2$.

The $O(n)$ model on a dynamical lattice

The disc partition function

- The partition function on the disc is

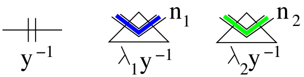
$$Z_{\text{dyn}}(\kappa, x, T) = \sum_{\Gamma:\ \text{disc}} \frac{1}{L(\Gamma)} \kappa^{-A(\Gamma)} x^{-L(\Gamma)} Z_\Gamma(T),$$

- It is generated by correlators of the matrix model

$$\text{Ord}: \quad W(x) = -\frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{x - X} \right\rangle$$

$$\text{JS}: \quad \tilde{W}(y) = -\frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{y - Y_k^2} \right\rangle, \qquad Y_k^2 = \sum_{a=1}^{k} Y_a^2$$

$$\text{DJS}: \quad H(y|\lambda_1, \lambda_2) = -\frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{y - X - \lambda_1 Y_{n_1}^2 - \lambda_2 Y_{n_2}^2} \right\rangle$$



The $O(n)$ model on a dynamical lattice

The disc partition function with two boundaries

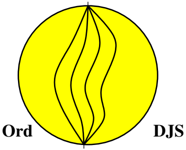
We consider a disc with mixed Ord-DJS boundary conditions,

$$D_L^{(i)}(x,y) = \frac{1}{\beta}\left\langle \mathrm{tr}\,\left(\frac{1}{x-X}\mathbb{S}_L^{(i)}\frac{1}{y-X-\lambda_1 Y_{n_1}^2-\lambda_2 Y_{n_2}^2}\mathbb{S}_L^{(i)\dagger}\right)\right\rangle$$

between both boundaries L open lines are inserted by the operators

$$\mathbb{S}_L^{(1)} = \sum_{\{a_1,\cdots,a_L\}\subset\{1,\cdots,n_1\}} Y_{a_1}\cdots Y_{a_L}$$

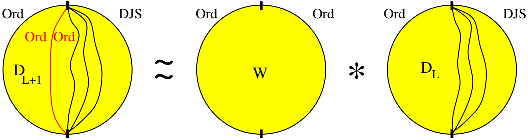
$$\mathbb{S}_L^{(2)} = \sum_{\{a_1,\cdots,a_L\}\subset\{n_1+1,\cdots,n\}} Y_{a_1}\cdots Y_{a_L}$$



Loop equations

Getting rid of loops

- The loop equations are obtained using the invariance of the matrix measure.
- They describe the removing of loops.
- Correlators satisfy a recursion relation $D_{L+1} = W * D_L$



- All the physics is contained in the 0th order equation which couples D_0 and $D_1^{(i)}$.
- This equation will be studied in the continuum limit.

- Taking the continuum limit leads to:

Statistical model at the critical point on a flat lattice.	$\longrightarrow$	CFT, operators $\Phi_{r,s}$, conformal dimension $\delta_{r,s}$
Statistical model at the critical point on a dynamical lattice.	$\longrightarrow$	CFT$\otimes$Liouville$\otimes$ghost, dressed operators $e^{2\alpha_{r,s}\phi}\Phi_{r,s}$, gravitational dimension $\Delta_{r,s}$

- The KPZ formulas relate the central charges and the dimensions $\delta_{r,s}$ and $\Delta_{r,s}$.
- The Liouville action has a boundary term (FZZT brane).

- We adjust the parameter to their critical value where the mean length and area of loops diverge.

$$\epsilon^2 \mu = \kappa - \kappa^*, \quad \epsilon^{1/g} \xi = x - x^*, \quad \epsilon^{\theta/g} t_B = \lambda - \lambda_c, \quad \cdots$$

- Critical correlators correspond to boundary 2pts functions of

$$\begin{aligned} d_L^{(i)}(\xi,\zeta) \quad &\to (Ord|S_L^{(i)}|AS_{(1)}), \qquad \Delta > 0 \\ d_L^{(i)}(\xi,\zeta) \quad &\to (Ord|S_L^{(i)}|AS_{(2)}), \qquad \Delta < 0 \\ d_L^{(i)}(\xi,\zeta) \quad &\to (Ord|S_L^{(i)}|Sp), \qquad \Delta = 0, \ \lambda = \lambda^* \end{aligned}$$

- Loop equations are shift equations on boundary parameters $\xi(\tau)$, $\mu_B(\sigma)$, e.g. on the $\mathrm{AS}_{(1)}$ branch (for any ξ, μ_B, t, t_B):

$$d_0^{(2)}(\tau,\sigma) d_1^{(1)}(\tau \pm i\pi, \sigma) + w(\tau) + n_1 w(\tau \pm i\pi) = \mu_B - t_B \xi$$

The continuum limit

Brief summary of the method

- How to find the scaling dimension of boundary operators ?
  1. Construct the matrix model correlators.
  2. Derive the loop equations from the invariance of the matrix measure.
  3. Take the continuum limit.
  4. Read the gravitationnal dimensions from the critical loop equation.
  5. Recover the scaling dimension of bare operators using the KPZ formula.

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- And moreover :
  1. The phase diagram is be obtained from criticality conditions.
  2. The dimension of the perturbations give the operators that generate the flows.
  3. The evolution of b.c. under thermal flows can be tracked down.

Outline

Results

Results in the Liouville context

- Dense phase, Ord/JS bcc operators:
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- Dense phase, JS/JS bcc operators :
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- Dilute phase, Ord/DJS bcc operators :
  1. Loop equation for QFT coupled to 2D gravity (t, t_B).
  2. Solution on the critical curves AS (Liouville boundary 2pts functions).
  3. Solution at $\mu = \mu_B = 0$, perturbed Liouville gravity

$$\delta\mathcal{S} = t\int_{\text{bulk}} \mathcal{O}_{1,3} + \xi\int_{\text{Ord.}} \mathcal{O}^B_{1,1} + t_B\int_{\text{DJS}} \mathcal{O}^B_{1,3}$$

Perspectives and open problems

Work in progress...

1. Open problems:
   - Calculation of the DJS disc partition function $H(y)$.
   - Dimension of the DJS boundary.
   - Explicit expression for the AS curve.
2. Perspectives:
   - Bulk anisotropy

$$\mathcal{S}[X, Y_a] = \mathrm{tr}\left(-\frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{T_1}{2}\sum_{a=1}^{n^{(1)}} Y_a^2 - \frac{T_2}{2}\sum_{a=n^{(1)}+1}^{n} Y_a^2 + \sum_{a=1}^{n} XY_a^2\right)$$

   - ADE models with boundaries.
   - SLE, Liouville gravity and Matrix Models...