Cohomological Subsectors in Sigma Models on Superspaces

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Outline

1. Motivation
   - Observations
   - Questions
2. Cohomological reduction of
   - Superalgebras
   - Modules
   - Spin chains
   - Sigma models
3. Conclusions and Outlook

Observations in σ-models

"Subsectors"

Calculation of some correlators in the σ-models on the l.h.s where mapped onto the correlators of the free theories on the r.h.s.:

σ-model	some kind of subsector
$S^{3\|2} = \frac{\mathrm{OSp}(4\|2)}{\mathrm{OSp}(3\|2)}$	S^1 or free compact boson [CC, Saleur, 08; Mitev, Quela, Schomerus 08]
$\mathbb{C}P^{1\|2} = \frac{\mathrm{U}(2\|2)}{\mathrm{U}(1) \times \mathrm{U}(1\|2)}$	$\mathbb{C}P^{0\|1}$ or free symplectic fermions [CC, Read, Jacobsen, Saleur 09; CC, Mitev, Quella, Saleur, Schomerus 09]

Observations in spin chains
Inspiring technicalities

$\mathrm{OSp}(2N+2|2N)$ chain $V_{2N+2|2N}^{\otimes L}$

$$H_N^{\mathrm{OSp}} = \mathrm{rep}_N\left(H_{\mathrm{Brauer}}\right)$$
$$H_{\mathrm{Brauer}} = \sum E_{i,i+1} + wP_{i,i+1}$$

$\mathrm{GL}(N|N)$ chain $(V_{N|N} \otimes V^*{}_{N|N})^{\otimes L}$

$$H_N^{\mathrm{GL}} = \mathrm{rep}_N\left(H_{\mathrm{Brauer}}^{\mathrm{walled}}\right)$$
$$H_{\mathrm{Brauer}}^{\mathrm{walled}} = \sum E_{i,i+1} + wP_{i,i+2}$$

have been extensively studied as discretizations of boundary σ-models

$$S^{2N+1|2N} = \frac{\mathrm{OSp}(2N+2|2N)}{\mathrm{OSp}(2N+1|2N)}$$

[CC, Saleur 08]

$$\mathbb{C}P^{N-1|N} = \frac{\mathrm{U}(N|N)}{\mathrm{U}(1)\times \mathrm{U}(N-1|N)}$$

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$$\mathrm{spec}\, H_0 \subset \mathrm{spec}\, H_1 \subset \cdots \subset \mathrm{spec}\, H_{\mathrm{Brauer}}^{(\mathrm{walled})}$$

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again “subsectors”

Motivation

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“Subsectors” appear generic in physical models with supergroup symmetry.

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Questions

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2. How much of the structure (conformal invariance, integrability) of the subsector theory lifts to the full theory?

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3. Concerning σ-models on superspaces
   - How to characterize the set of fields in the $S^{3|2}$ and $\mathbb{C}P^{1|2}$ σ-models whose correlators can be computed within the simpler theories?
   - Is the existence of simplified subsectors a general feature of σ-models?
   - Are the simplified subsectors equivalent again to σ-models?

Mathematical definitions and constructions

Lie superalgebras

Cohomological reduction of a Lie superalgebra $\mathfrak{g}$

with respect to an odd element Q, $[Q,Q] = 2Q^2 = 0$, is the Lie superalgebra defined as

$$\mathsf{H}_Q(\mathfrak{g}) := \frac{\mathsf{Ker}[Q,\cdot]}{\mathsf{Im}[Q,\cdot]} = \frac{\mathsf{Ker}_Q\,\mathfrak{g}}{\mathsf{Im}_Q\,\mathfrak{g}}$$

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Classification — $r_Q = \mathrm{rank}(Q)$

For any Q, such that $Q^2 = 0$, one has natural isomorphisms

$$\mathsf{H}_Q\left(\mathrm{gl}(M|N)\right) \simeq \mathrm{gl}(M - r_Q|N - r_Q)$$
$$\mathsf{H}_Q\left(\mathrm{sl}(M|N)\right) \simeq \mathrm{sl}(M - r_Q|N - r_Q)$$
$$\mathsf{H}_Q\left(\mathrm{osp}(M|2N)\right) \simeq \mathrm{osp}(M - 2r_Q|2N - 2r_Q)$$

Mathematical definitions and constructions
Modules

Cohomological reduction of a $\mathfrak{g}$-module V is the $\mathsf{H}_Q(\mathfrak{g})$-module defined as

$$\mathsf{H}_Q(V) = \frac{\mathsf{Ker}\, Q : V \mapsto V}{\mathsf{Im}\, Q : V \mapsto V} = \frac{\mathsf{Ker}_Q\, V}{\mathsf{Im}_Q\, V}$$

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Properties

- if $V \simeq V^*$ then

  $$V|_{\mathsf{H}_Q(\mathfrak{g})} \simeq W \oplus E \oplus F$$

  $$W \simeq \mathsf{H}_Q(V),\ E = \mathsf{Im}_Q\, V$$

- $\mathsf{H}_Q(U \oplus V) \simeq \mathsf{H}_Q(U) \oplus \mathsf{H}_Q(V)$
- $\mathsf{H}_Q(U \otimes V) \simeq \mathsf{H}_Q(U) \otimes \mathsf{H}_Q(V)$
- $\mathrm{sdim}\, \mathsf{H}_Q(V) = \mathrm{sdim}\, V$

Spin chains
Applications

Fundamental representation $n = N - r_Q$

$V_{2N+2|2N}$ of $\mathrm{OSp}(2N+2|2N)$ $V_{N|N}$ of $\mathrm{GL}(N|N)$

$$\mathsf{H}_Q(V_{2N+2|2N}) \simeq V_{2n+2|2n} \qquad \mathsf{H}_Q(V_{N|N}) \simeq V_{n|n}$$

Spin chains $n = N - r_Q$

$$\mathsf{H}_Q\left((V_{2N+2|2N})^{\otimes L}\right) \simeq (V_{2n+2|2n})^{\otimes L} \qquad \mathsf{H}_Q\left((V_{N|N})^{\otimes L}\right) \simeq (V_{n|n})^{\otimes L}$$

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“Subsectors” have cohomological nature!

Cohomological reduction of σ-models on G/G'

Set-up

1. Pick target space supersymmetry Q, $Q^2 = 0$. Correlation functions of Q-invariant local fields depend only on their Q-cohomology.
2. Compute the Q-cohomology of the space of local fields. Interpret the result as the space of local fields of a reduced field theory.
3. Map the correlators of Q-invariant local fields to correlators in the reduced theory (localization).

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Cohomological reduction as a geometrical problem

geometrical object		field theory object
• $T(G/G')^{\otimes n} \otimes L_2(G/G')$	$\Rightarrow$	• n-worldsheet derivative fields
• G-invariant symmetric, antisymmetric form of rank 2		• kinetic term in the action, B-field or θ-terms

Notations

Cohomological reduction of σ-models on G/G'

Define the superalgebras

$Q \in \mathfrak{g}' \subset \mathfrak{g}$

$$
\begin{array}{lcl}
\mathfrak{g}' \simeq \mathfrak{h}' \oplus \mathfrak{e}' \oplus \mathfrak{f}' & \subset & \mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{e} \oplus \mathfrak{f} \\
\mathfrak{h}' \simeq \mathsf{H}_Q(\mathfrak{g}') & \subset & \mathfrak{h} \simeq \mathsf{H}_Q(\mathfrak{g}) \\
\mathfrak{e}' = \mathsf{Im}_Q\, \mathfrak{g}' & \subset & \mathfrak{e} = \mathsf{Im}_Q\, \mathfrak{g}
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G' & \subset & G \\
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Then one has

$$H/H' \subset G/G' \,.$$

GEOMETRICAL RESULTS
Cohomological reduction of σ-models on G/G'

Cohomology evaluation

$$\mathsf{H}_Q\left(T^{\otimes n}(G/G')\otimes L_2(G/G')\right) \simeq T^{\otimes n}(H/H')\otimes L_2(H/H')$$

$$\omega \stackrel{\rho}{\mapsto} \rho(\omega)$$

Q-invariant tensor form ω of rank n on G/G'

restriction $\rho(\omega)$ of ω to

1. submanifold $H/H' \subset G/G'$
2. $T^{\otimes n}(H/H') \subset T^{\otimes n}(G/G')|_{H/H'}$

Geometrical results

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1. submanifold $H/H' \subset G/G'$
2. $T^{\otimes n}(H/H') \subset T^{\otimes n}(G/G')|_{H/H'}$

Localization formula

$$\int_{G/G'} \omega = \int_{H/H'} \rho(\omega)$$

Field theory results

Cohomological reduction of σ-models

Space of local fields

Q-cohomology of the space of local fields in the σ-model on G/G' identified with the space of local fields in the σ-model on H/H'.

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Action

Restriction of a G-invariant metric/2-form on G/G' to

- the points of $H/H' \subset G/G'$
- the tensor space $T^{\otimes 2}(H/H') \subset T^{\otimes 2}(G/G')|_{H/H'}$

obviously gives an H-invariant metric/2-form on H/H'

$$S_{H/H'} = \rho(S_{G/G'})$$

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Q-cohomology of the space of local fields in the σ-model on G/G' identified with the space of local fields in the σ-model on H/H'.

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$$\boxed{S_{H/H'} = \rho(S_{G/G'})}$$

Correlation functions

$$\left\langle \prod_i O_i(x_i) \right\rangle_{G/G'} = \left\langle \prod_i \rho(O_i)(x_i) \right\rangle_{H/H'}$$

CFT RESULTS

Applications of cohomological reduction of σ-models

reduced model H/H' conformal invariant

1. G/G' admits a single radius only
2. $c_{H/H'} \neq 0$

$\Rightarrow$ σ-model G/G' is conformal invariant

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Classification of CFT σ-models on G/G' superspaces with one radius only

σ-model	maximal reduction
$\frac{\mathrm{OSp}(2M+2N+2\|2M+2N)}{\mathrm{OSp}(2M+1\|2M)\times\mathrm{OSp}(2N+1\|2N)}$	free
$\mathrm{OSp}(2N+2\|2N)$	compact
$D(2,1;\alpha)$	boson
$\frac{\mathrm{GL}(M+N+1\|M+N+1)}{\mathrm{GL}(M+1\|N)\times\mathrm{GL}(M\|N+1)}$	free
$\frac{\mathrm{PSL}(2N\|2N)}{\mathrm{OSp}(2N\|2N)}$	symplectic
$\mathrm{PSL}(N\|N)$	fermions

Extension

1. WZW σ-models can be reduced with the same tools: restriction of a G-invariant 3-form on G is again an H-invariant 3-form on H.
2. Models like

OSp$(M|2N)$ Landau-Ginsburg	OSp$(M|2N)$ Gross-Neveu
$\mathcal{L} = (\partial_\mu \Phi, \partial_\mu \Phi) + g(\Phi, \Phi)^2$	$\mathcal{L} = (\Psi, \bar{\partial}\Psi) + (\bar{\Psi}, \partial\bar{\Psi}) + g(\Psi, \bar{\Psi})^2$
Φ an even field in the fundamental representation	$\Psi, \bar{\Psi}$ odd fields in the fundamental representation

   can be reduced to corresponding OSp$(M - 2r_Q|N - 2r_Q)$ models.
3. Spin chains.

Conclusions

Results

- Mapping of Q-invariant correlation functions of local fields in the G/G' σ-model to correlation functions in a *simpler* H/H' σ-model.
- Classification of CFT σ-models with *one radius*.
- Extension of cohomological reduction to WZW, Landau-Ginsburg, Gross-Neveu models and spin chains.

Outlook

1. Define a suitable complex to characterize exactly how much more complicated is the full theory w.r.t. the subsector theory.
2. Reduction with respect to $Q \notin \mathfrak{g}'$
3. Proof of conformality of string theory in the pure spinor formalism.
4. How does the integrability of the subsector theory constrain the integrability of the full theory?