Logarithmic behavior on a lattice and in the continuum: the quantum group approach

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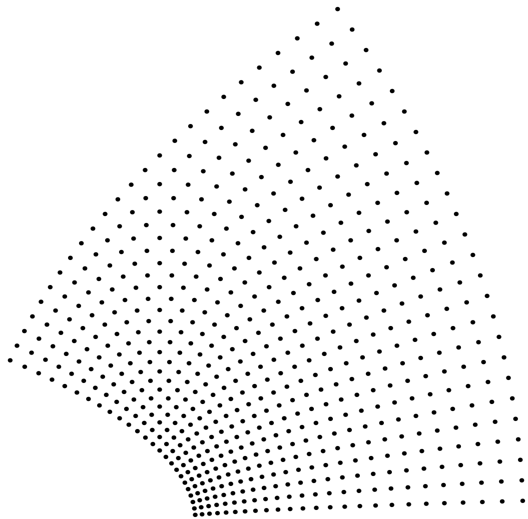
Why conformal?



To describe universality classes of 2D statistaical systems

at a **critical point** $(\xi \to \infty)$

Rational models of CFT

Minimal models $\mathsf{M}(p, p')$ (exactly solvable):

- finite number of physical observables (primary fields)
- with semisimple fusion algebra
- and Casimir effect

$$c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}, \qquad \text{where} \qquad \frac{p}{p'} \in \mathbb{Q}.$$

describe critical points of well-known 2D statistical systems:

- $\mathsf{M}(3, 4)$ describes the critical point of the **Ising** model;
- $\mathsf{M}(5, 6)$ – critical point of the **Potts** model with $\mathbb{Z}_3$-symmetry.

On which questions **rational** models give an answer?

- to describe correlations of local objects (e.g., spins $s(\mathbf{x})$, heights and so on):

Rational CFT:

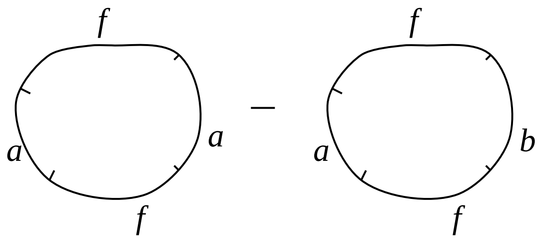
$$\text{scaling fields} \quad + \quad \text{conformal Ward identities} \quad \longrightarrow$$

$$\longrightarrow \qquad \langle s(\mathbf{x}) s(0) \rangle, \quad \langle s(\mathbf{x}) s(\mathbf{y}) s(0) \rangle, \quad \ldots$$

Why we need logarithmic models of CFT?

2D percolation

Cardy, 1992



Crossing probability $P = Z_{aa} - Z_{ab}$ of percolation cluster formation between two boundaries (horizontal crossing)

$$Z_{aa} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle$$

$$Z_{ab} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|b}(z_3)\phi_{b|f}(z_4)\rangle$$

2D percolation

Crossing probability $P = Z_{aa} - Z_{ab}$ of percolation

$$Z_{aa} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle$$

$$Z_{ab} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|b}(z_3)\phi_{b|f}(z_4)\rangle$$

Conformal dimension of the boundary field $\phi_{f|a}(z)$ should be zero:

$$L_0\,\phi_{f|a}(z) = 0 \quad \& \quad c = 0 \qquad \longrightarrow \qquad \phi_{f|a}(z) = \phi_{1,2}(z)$$

$$(L_{-2} - \tfrac{3}{2}L_{-1}^2)\phi_{1,2}(z) = 0 \qquad \longrightarrow \qquad \text{Cardy formula}$$

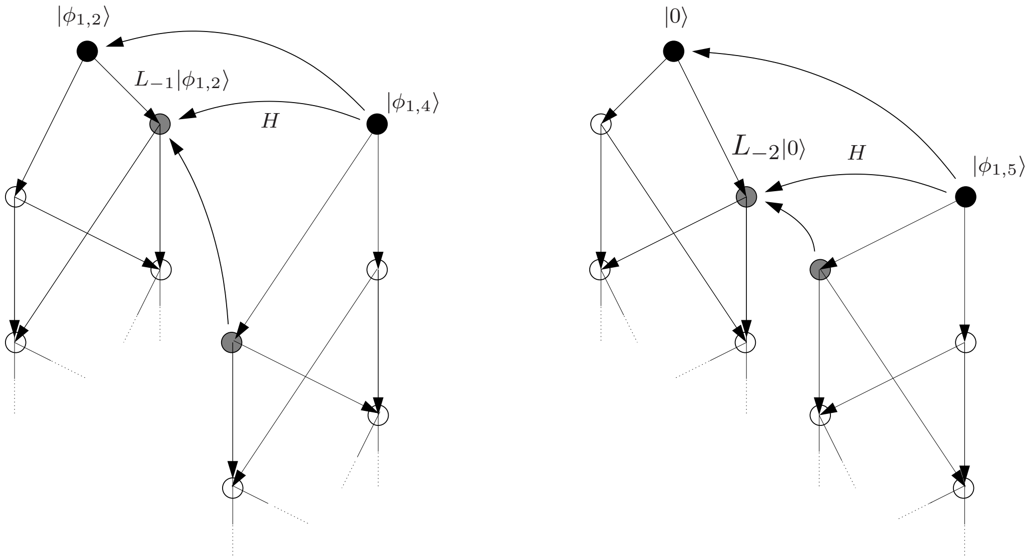
2D percolation

Mathieu, Ridout, 2007

Boundary fields $\phi_{1,2}(z)$ + operator algebra to be closed

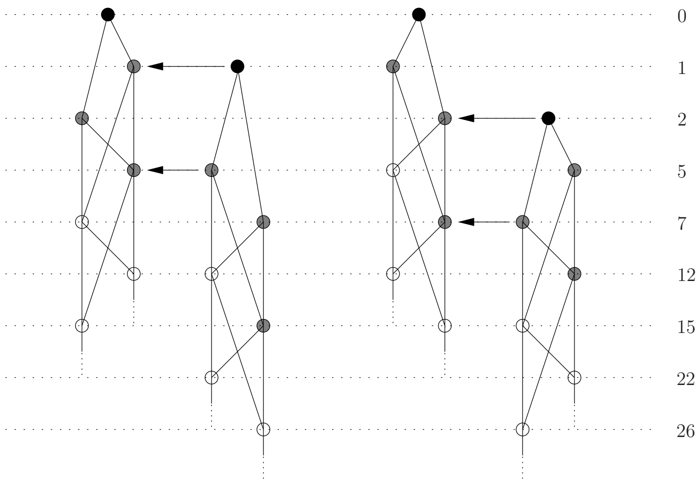
$\Longrightarrow$ appearance of nontrivial Jordan cells

in spectrum of hamiltonian L_0:

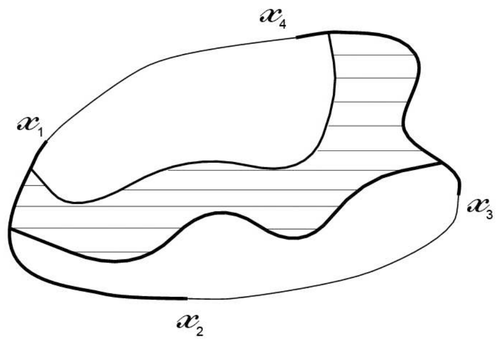


2D percolation

Watts probability requires introducing more primary fields with vanishing conformal dimension and singular vectors up to **5th level** — pictures are even “worse” than in Cardy-case.



2D percolation



Scaling fields $s(x)$ of a rational CFT

+ new scaling fields $\phi(x)$ giving

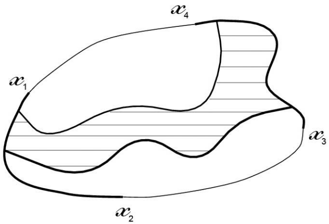
$$P_{\text{percolation}} \sim \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle$$

+ condition on operator algebra to be closed $\longrightarrow$

$\longrightarrow$ **logarithmic** singularities in OPEs

+ **indecomposable** reps with **non-diag** L_0.

On which questions **logarithmic** models give an answer?



Studying of non-locality properties

— presence of a number of clusters —

requires to extend **rational** family of models

to more general, **logarithmic** models.

Logarithmic models
of 2D
conformal field theory

- singularities $\log(z-w)$ in OPEs;
- *"nonunitary"* evolution — appearance of Jordan cells for L_0;
- *infinitely* many primary fields (usually) —

— in contrast with

Rational conformal field theories characterized

- by *diagonalizable* evolution operator
- space of states $=$ finite set of *irreducible* representations

Example. Log models with extended (W-)symmetry

Extension of the field content

of the minimal models $\mathsf{M}(p, p')$

to logarithmic theories $\mathsf{WM}(p, p')$

by introducing <u>additional</u> (to conformal symmetry) generators —

— primary boson fields $W^+(z)$ and $W^-(z)$:

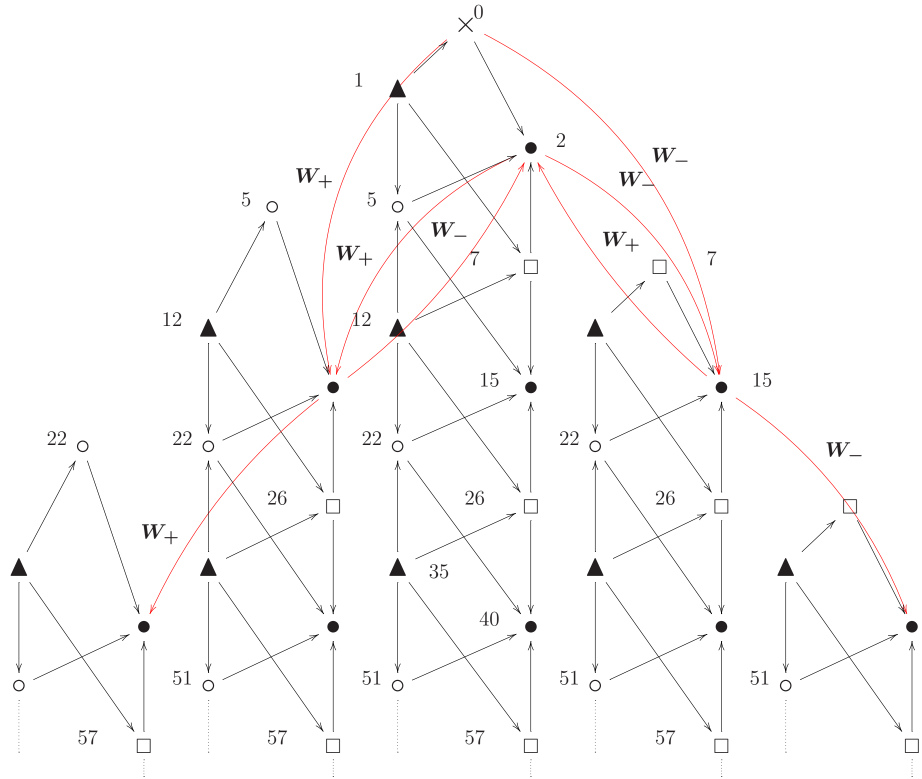
$$W^+(z) = \mathcal{P}^+_{3pp'-3p-p'+1}(\partial\varphi(z))\, :e^{p\alpha\varphi(z)}:\,,$$

$$W^-(z) = \mathcal{P}^-_{3pp'-p-3p'+1}(\partial\varphi(z))\, :e^{p'\alpha'\varphi(z)}:\,,$$

with OPE

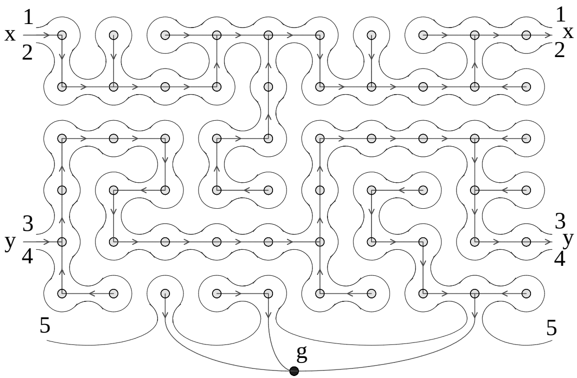
$$W^+(z)W^-(w) = \frac{T(w)}{(z-w)^{7pp'-3p-3p'+1}} + \text{ less singular terms.}$$

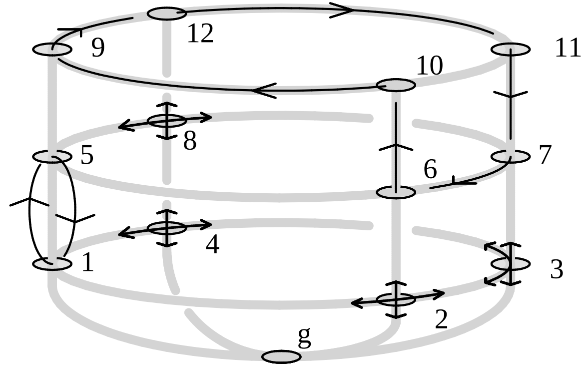
Example. Log models with extended (W-)symmetry — WM$(2,3)$ with $c=0$



Priezzhev et al, 2008

Spanning webs on the cylinder with open (closed) b.c.





Spanning webs model in the scaling limit is described by the Log model WM$(1,2)$

In a log CFT model, we encounter a number of **difficulties**:

(1) The space of states is a direct sum of **projectives** but their structure is not known apriori – a problem in constructing the full space of states (non-chiral theory).

(2) The space of characters is **not closed** under modular group transformations.

(3) **non**semisimple fusion rules – a problem in construction **Verlinde** formula which is important for boundary theory.

It might be better to begin studying logarithmic behaviour on a lattice and getting thus some intuition for the continuum.

Pasquier, Saleur, 1990

Anisotropic Heisinberg model (XXZ-model, open case) with **non-hermitian** b.t.:

$$H_{\mathsf{XXZ}} = \sum_{j=1}^{N-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2} \sigma_j^z \sigma_{j+1}^z \right) + \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{2} (\sigma_1^z - \sigma_N^z)$$

- deformation of isotropic XXX-model ($\mathfrak{q} \to 1$) with $SU(2)$-symmetry:

$$\left[S_N, SU(2) \right] = 0 \quad \Rightarrow \quad \left[H_{\mathsf{XXX}}, SU(2) \right] = 0$$

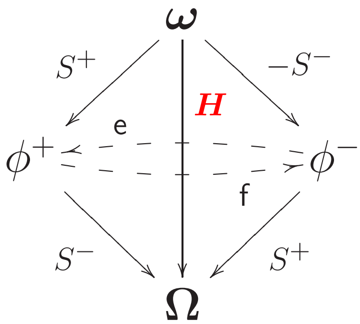
- "gener. spectrum" algebra S_N and symmetry of H_{XXZ} are deformed

$H_{\mathsf{XXX}} \mapsto H_{\mathsf{XXZ}}$: $\qquad S_N \approx TL_N(1) \mapsto TL_N(\mathfrak{q})$ and $SU(2) \mapsto SU_{\mathfrak{q}}(2)$.

$$\longrightarrow \quad \left[TL_N(\mathfrak{q}), SU_{\mathfrak{q}}(2) \right] = 0 \quad \Rightarrow \quad \left[H_{\mathsf{XXZ}}, SU_{\mathfrak{q}}(2) \right] = 0$$

Logarithmic behaviour of the hamiltonian H_{XXZ}

at a **root of unity** ($\mathfrak{q} = e^{i\pi/p}$, $p \geqslant 2$)



$$\mathsf{e} = \frac{(S^+)^2}{[2]!}, \qquad \mathsf{f} = \frac{(S^-)^2}{[2]!}$$

the vacuum $\boldsymbol{\Omega}$ and the state $\boldsymbol{\omega}$ form a 2-dim Jordan cell of the lowest eigenvalue for $\boldsymbol{H}_{\mathsf{XX}}$

The full space of states is a **bimodule** over $TL_N(\mathfrak{q}) \otimes SU_\mathfrak{q}(2)$

- the states are organized into indecomposables for $SU_\mathfrak{q}(2)$ (or $TL_N(\mathfrak{q})$)
- with nodiag. action of the Casimir operator (or the hamiltonian H_{XXZ})

What is going on with such a structure when the thermodynamic limit is taken?

Thermodynamic limit $N \to \infty$

- ground state $\leftrightarrow$ $|0\rangle$;
- low-lying excitations $\leftrightarrow$ many-particles states in Fock spaces;
- correspondence between generating spectrum algebras:

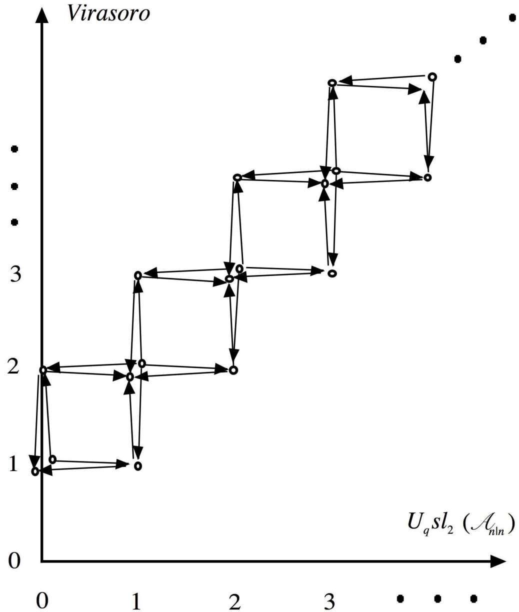
(periodic)Temperley–Lieb $\leftrightarrow$ Virasoro $(+\ \overline{\text{Virasoro}})$

Fourier modes (1) $\sum_k \mathrm{e}^{ik\mathbf{n}\frac{\pi}{N}} \boldsymbol{e_k} \xrightarrow[\text{asymptotic}]{\text{leading}}$ Virasoro $\frac{\pi}{N}\left(L_{\mathbf{n}} + \bar{L}_{-\mathbf{n}}\right)$

(2) $\sum_k \mathrm{e}^{ik\mathbf{n}\frac{\pi}{N}} [\boldsymbol{e_k}, \boldsymbol{e_{k+1}}] \longrightarrow \frac{\pi}{N}\left(L_{\mathbf{n}} - \bar{L}_{-\mathbf{n}}\right)$

Read, Saleur, 2007

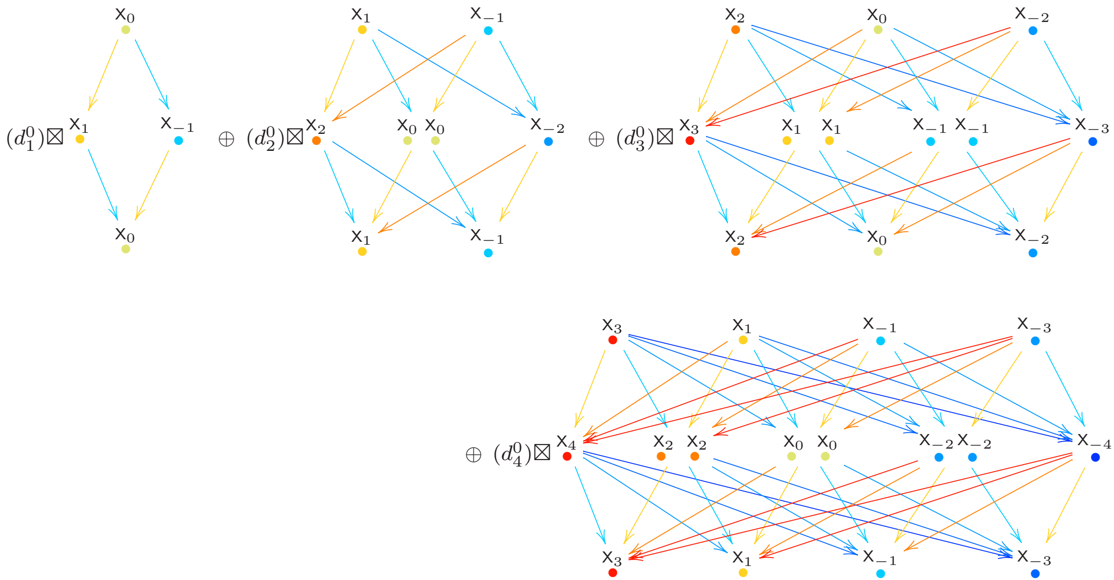
Open case for the XX model



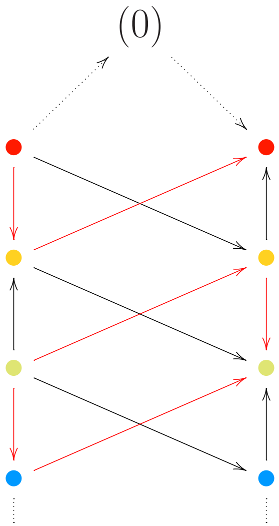
More formally, the structure is equivalent to the **regular bimodule** for $U_{\mathfrak{q}}sl(2)$

QG-symmetry for **periodic** TL on closed XX spin chain

Decomposition w.r.t. the odd quantum group $U_{\mathfrak{q}}^{\mathsf{odd}}(s\ell(2))$



periodic Temperley–Lieb $\widehat{TL}_N(\mathfrak{q})$ on closed XX spin chain



Each sector with fixed number of fermions is a **Feigin–Fuchs** type module over $\widehat{TL}_N(\mathfrak{q})$ (without the "top")

Quantum group

- is “**insensitive**” to lattice-size increasing,
- multiplicities for the QG-reps are only changed

  but the symmetry and QG-multiplets are **same** for any number N of sites.

$\longrightarrow$ possible **survival** of the quantum-group symmetry in the limit $N \to \infty$.

We hope to find **quantum-group** symmetries in conformal models realized as the thermodynamic limit of 2D integrable systems at a critical point.

Example. Quantum-group symmetry for $\mathrm{WM}(p,1)$

Algebra **commuting** with the W-symmetry

- is realized by the restricted quantum group $\overline{SU}_{\mathfrak{q}}(2)$ relations:

$$(1)\quad [S^z, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = [2S^z]_{\mathfrak{q}}, \quad \text{where} \quad [x]_{\mathfrak{q}} = \frac{\mathfrak{q}^x - \mathfrak{q}^{-x}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

$$(2)\quad (S^{\pm})^p = 0, \qquad \mathfrak{q}^{2pS^z} = 1$$

— deformation of usual relations

for generators of $SU(2)$ ($\mathfrak{q} \to 1$)

with additional constraints (2).

- Restricted quantum group $\overline{SU}_{\mathfrak{q}}(2)$ **commutes**
  - with conformal symmetry $T(z)$,
  - and generators $W^{+}(z)$ and $W^{-}(z)$.

- in conformal-fileds terms, is constructed using the screenings

$$S^{-}(\cdot) = \oint dz \,\mathbf{:} e^{-\sqrt{\frac{2}{p}}\varphi(z)} \cdot \mathbf{:} \qquad \longrightarrow$$

$\xrightarrow[\text{double}]{\text{quantum}}$ conjugate operator S^{+}

Quantum group $\overline{SU}_{\mathfrak{q}}(2)$ — additional symmetry of the log model WM$(p,1)$ $\longrightarrow$

$\longrightarrow$ structure of log reps, fusion rules, modular action, braiding, ...

- Described lattice approach suggest us to consider **LCFTs** (their chiral algebras) as centralizers of suitable **quantum groups** in an appropriate space of fields.
- primary fields $\leftrightarrow$ QG's highest-weights;

  logarithmic partners $\leftrightarrow$ cyclic vectors in QG-reps.
- Such a correspondence between chiral-algebra data and QG-stuff let us study representation theory of chiral algebras (Virasoro, W-algebras, etc.) $\longrightarrow$

  $\longrightarrow$ structure of the space of states in LCFT.
- But even more — allow us to calculate **fusion** rules
- and for LCFTs with rational properties — **Verlinde** formula

  and even **boundary theory**.

Logarithmic Verlinde formula from quantum groups

The existence of a **nondegenerate quantum Fourier** transformation on the quantum group **center** has led to a generalization of the Verlinde formula for the $(p, 1)$ logarithmic theories.

Logarithmic Verlinde formula from quantum groups

The structure constants in the QG center $\mathfrak{Z}$

- with respect to the Drinfeld basis are integer numbers $=$ LCFT fusion rules
- while in the Radford basis the multiplication is block-diagonal, and the structure constants are expressed in terms of the $\mathcal{S}$ matrix vacuum row.

We then use (i) this fact and

(ii) the equivalence of $SL(2,\mathbb{Z})$-actions on $\mathfrak{Z}$ and $\mathfrak{Z}_{\mathsf{cft}}$

to obtain a log version of the Verlinde-formula for the WM$(1,p)$ models.

Logarithmic Verlinde formula from quantum groups

The fusion structure constants are reproduced from the $\mathcal{S}$-matrix action and we thus get a generalized Verlinde-formula for the $\mathsf{WM}(1,p)$ models.

- The structure constants in $\mathfrak{Z}_{\mathsf{cft}}$ with respect to the basis of the characters and pseudocharacters are given by

$$N_{[r;\alpha][s;\beta]}^{[k;\gamma]} = \sum_{l=1}^{p+1}\sum_{\lambda=1}^{n_l} \frac{S_{[l;1]}^{\mathsf{vac}} S_{[l;1]}^{[r;\alpha]} S_{[l;\lambda]}^{[s;\beta]} + S_{[l;1]}^{\mathsf{vac}} S_{[l;\lambda]}^{[r;\alpha]} S_{[l;1]}^{[s;\beta]} - S_{[l;\lambda]}^{\mathsf{vac}} S_{[l;1]}^{[r;\alpha]} S_{[l;1]}^{[s;\beta]}}{\left(S_{[l;1]}^{\mathsf{vac}}\right)^2} S_{[k;\gamma]}^{[l;\lambda]},$$

where $S_{[r;\alpha]}^{\mathsf{vac}}$ are the "vacuum" row elements.

Boundary theory.

Many algebraic aspects of the problem of boundary conditions in LCFTs reduces to a study of the center of the corresponding quantum group.

- for the Log models $\mathrm{WM}(p, 1)$, analysis of the center has yielded **all amplitudes** between possible boundary states on a cylinder.
- a natural generalization of the **Cardy formula** for boundary states localized in a coordinate space was proposed for the $(p, 1)$ logarithmic models, where $(3p - 1)$ Cardy states were identified.

Good cooorespondence with the **sand-pile model** (or spanning-webs) calculations!

Conclusion.

In bridging the gap between lattice and continuum logarithmic conformal field theories, the guiding principle may be sought in quantum group symmetries. Being finite-dimensional objects, factorizable ribbon quantum groups at roots of unity are capable of capturing a number of very essential features normally extracted from vertex-operator algebras, such as the representation category, modular representations, fusion, braiding and monodromy properties.