The integrable $\mathbb{Z}_2$ staggered model

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Les Houches

This talk is based on:

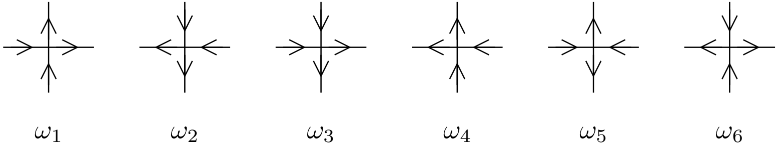
- YI, J.L Jacobsen, H. Saleur
  A Temperley-Lieb quantum Hamiltonian with two and three-site interactions
  J. Phys. **A 42**, 292002 (2009)

- YI, J.L Jacobsen, H. Saleur, *The* $\mathbb{Z}_2$ *staggered model and its applications*
  `arxiv:0911.3003`

Outline

I. The six-vertex model and related physical models

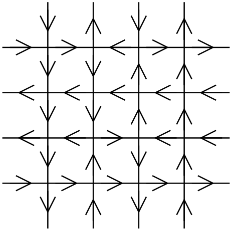
1. The six-vertex model



$$\omega_1, \ldots, \omega_6 = \sin(\gamma - u),\ \sin(\gamma - u),\ \sin u,\ \sin u,\ e^{iu}\sin\gamma,\ e^{-iu}\sin\gamma$$

$$\Delta = -\cos\gamma$$

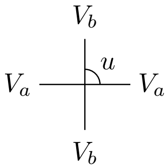
$$Z_{6\mathrm{V}} = \sum_{\text{config.}} \omega_1^{n_1}\omega_2^{n_2}\ldots\omega_6^{n_6}$$



2. Building a solvable, inhomogeneous six-vertex model

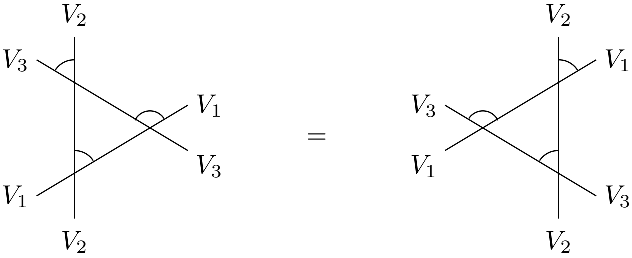
- R-matrix acting on two edges:

$$V_a = \mathrm{span}(\uparrow, \downarrow)\,, \qquad R_{ab}(u) : V_a \otimes V_b \to V_a \otimes V_b$$



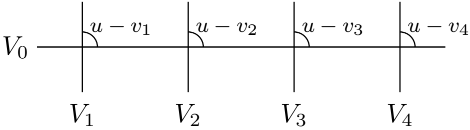
- R satisfies the Yang-Baxter Equations (YBE):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$



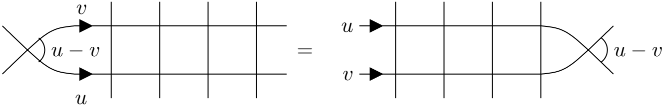
- Inhomogeneous transfer matrix (with periodic BC):

$$t(u) := \mathrm{Tr}_0 \left[R_{01}(u - v_1) R_{02}(u - v_2) \dots R_{0,2L}(u - v_{2L}) \right]$$



- Commutation of transfer matrices:

$$\forall u, v \qquad [t(u), t(v)] = 0$$



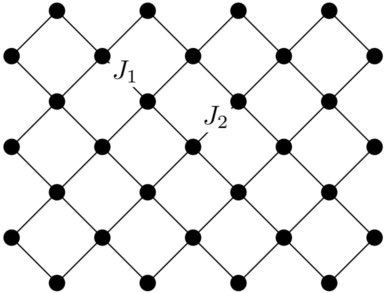
- Inhomogeneous partition function:

$$Z_{2L,2M} = \mathrm{Tr}_{1,2,\dots 2L} \left[t(u_1) t(u_2) \dots t(u_{2M}) \right]$$

3. Relation to the Potts model

- Partition function

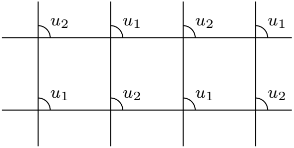
$$Z_{\rm Potts}(Q) = \sum_{\{S_j=1,\ldots,Q\}} \prod_{i \diagdown j} \exp\left(J_1 \delta_{S_i,S_j}\right) \prod_{i \diagup j} \exp\left(J_2 \delta_{S_i,S_j}\right)$$



- Potts/6V equivalence ($\sqrt{Q} = 2\cos\gamma$)
  - Coupling constants:

$$x_1 = \frac{e^{J_1} - 1}{\sqrt{Q}} = \frac{\sin u_1}{\sin(\gamma - u_1)} \qquad x_2 = \frac{e^{J_2} - 1}{\sqrt{Q}} = \frac{\sin(\gamma - u_2)}{\sin u_2}$$

  - Transfer matrix:



- Integrable cases
  - $u_1 = u_2 \quad \Leftrightarrow \quad x_1 x_2 = 1$
    - $x_1, x_2 > 0$: ferromagnetic critical transition
    - $x_1, x_2 < 0$: ‘non-physical’ self-dual line
  - $u_1 = u_2 + \frac{\pi}{2} \quad \Leftrightarrow \quad (2 + x_1\sqrt{Q})(2 + x_2\sqrt{Q}) = 4 - Q$
    - $x_1, x_2 < 0$: antiferromagnetic critical transition
    - $(x_1 x_2) < 0$: 'totally anisotropic' regime

4. Relation to quantum spin chains

The 'standard' case: homogeneous 6V transfer matrix

- Very anisotropic limit

$$t(0) \propto \exp(-iP)$$

- Hamiltonian

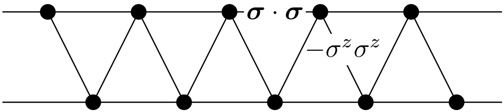
$$\begin{aligned} H_{\mathrm{XXZ}} &:= -\left.\frac{\partial \log t(u)}{\partial u}\right|_{u=0} \\ &= \sum_{j=1}^{2L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos\gamma \ \sigma_j^z \sigma_{j+1}^z \right) \end{aligned}$$

4. Relation to quantum spin chains *(cont.)*

Staggered transfer matrix

- Spin-chain Hamiltonian:

$$
\begin{aligned}
H \quad &:= \quad - \left. \frac{\partial \log t(u) t(u + \frac{\pi}{2})}{\partial u} \right|_{u=0} \\
&= \quad \sum_{j=1}^{2L} \left[\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} - \mu^2 \ \sigma_j^z \sigma_{j+1}^z \right. \\
&\qquad \left. + i\mu \ (\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \right] , \quad \mu = \sin\gamma
\end{aligned}
$$



- Expression in terms of Temperley-Lieb generators

$$
H = -\sum_{j=1}^{2L} e_j + K \sum_{j=1}^{2L} (e_j e_{j+1} + e_{j+1} e_j) \, , \quad K = \frac{1}{2\cos\gamma}
$$

Bethe Ansatz solution

Symmetries of the Hamiltonian

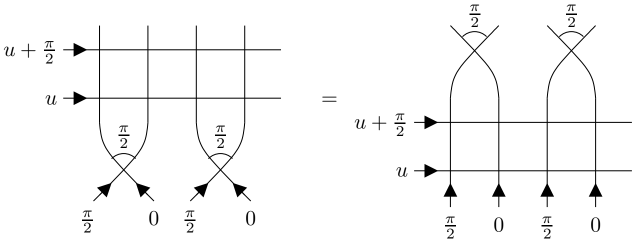
- Total magnetisation

$$S^z := \frac{1}{2}\sum_{j=1}^{2L}\sigma_j^z\,, \quad [H, S^z] = 0$$

- $\mathbb{Z}_2$ charge

$$C = \frac{R_{12}(\pi/2)}{\cos\gamma}\frac{R_{34}(\pi/2)}{\cos\gamma}\dots\frac{R_{2L-1,2L}(\pi/2)}{\cos\gamma}\,, \quad C^2 = 1$$

$$[H, C] = 0 \qquad \text{(Yang-Baxter)}$$



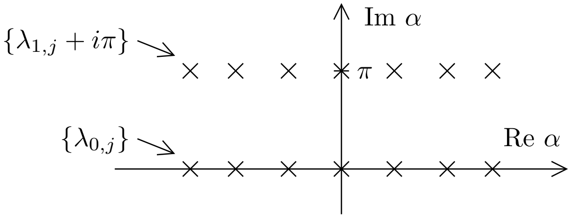
Bethe Ansatz Equations

- BAE and energy (periodic BC):

$$\left[\frac{\sinh(\alpha_j - i\gamma)}{\sinh(\alpha_j + i\gamma)}\right]^L = -\prod_{\ell=1}^{r} \frac{\sinh\frac{1}{2}(\alpha_j - \alpha_\ell - 2i\gamma)}{\sinh\frac{1}{2}(\alpha_j - \alpha_\ell + 2i\gamma)}$$

$$E = -\sum_{j=1}^{r} \frac{2\sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}$$

- Position of roots (ground state)



Bethe integers, excitations

- Logarithmic form of the BAE

$$
\begin{aligned}
Lk(\lambda_{0j}) &= 2\pi I_{0j} - \sum_{\ell=1}^{r_0} \Theta_0(\lambda_{0j} - \lambda_{0\ell}) - \sum_{\ell=1}^{r_1} \Theta_{-1}(\lambda_{0j} - \lambda_{1\ell}) \\
Lk(\lambda_{1j}) &= 2\pi I_{1j} - \sum_{\ell=1}^{r_0} \Theta_1(\lambda_{1j} - \lambda_{0\ell}) - \sum_{\ell=1}^{r_1} \Theta_0(\lambda_{1j} - \lambda_{1\ell})
\end{aligned}
$$

- Magnetic excitations: $\begin{cases} r_0 = r_0^{\mathrm{gs}} - m_0 \\ r_1 = r_1^{\mathrm{gs}} - m_1 \end{cases}$
- Electric excitations: $\begin{cases} I_{0j} = I_{0j}^{\mathrm{gs}} + e_0 \\ I_{1j} = I_{1j}^{\mathrm{gs}} + e_1 \end{cases}$
- Other: holes, strings.

Realisation of the symmetries

- Total magnetisation sectors:

$$S^z = L - r_0 - r_1$$

- Action of $\mathbb{Z}_2$ charge on Bethe eigenstates (from YBE):

$$C|\psi(\alpha_1, \dots \alpha_r)\rangle = |\psi(\alpha_1 + i\pi, \dots \alpha_r + i\pi)\rangle$$

$$C : \{\lambda_{0j}\} \leftrightarrow \{\lambda_{1j}\}$$

Conformal Field Theory in the continuum limit

Electromagnetic spectrum

- Conformal dimensions from finite-size eigenvalues [Cardy 86]:

$$
\begin{aligned}
E_{h,\bar{h}} &\simeq E^{\mathrm{gs}} + \frac{2\pi v_F}{L}(h+\bar{h}) \\
P_{h,\bar{h}} &= \frac{2\pi}{L}(h-\bar{h}) \mod 2\pi
\end{aligned}
$$

- Wiener-Hopf technique $\rightarrow$ finite-size effects in Bethe Ansatz:

$$
\begin{aligned}
h &= \frac{1}{8}\left(\frac{e}{\sqrt{2g}} + m\sqrt{2g}\right)^2 + \frac{1}{8}(\widetilde{e}+\widetilde{m})^2 \\
\bar{h} &= \frac{1}{8}\left(\frac{e}{\sqrt{2g}} - m\sqrt{2g}\right)^2 + \frac{1}{8}(\widetilde{e}-\widetilde{m})^2
\end{aligned}
$$

$$
g = \frac{\pi - 2\gamma}{2\pi}\,, \quad \left\{ \begin{array}{ll} e = e_0 + e_1 & \widetilde{e} = e_0 - e_1 \\ m = m_0 + m_1 & \widetilde{m} = m_0 - m_1 \end{array} \right.
$$

Two-boson theory: Bethe Ansatz *vs* 'naive bosonisation'

- From Bethe-Ansatz results, continuum action is:

$$\mathcal{A} = \frac{g(\gamma)}{4\pi} \int \mathrm{d}^2 x \, |\nabla \varphi|^2 \; + \; \frac{(1/2)}{4\pi} \int \mathrm{d}^2 x \, |\nabla \widetilde{\varphi}|^2$$

with $\varphi \equiv \varphi + 2\pi, \; \widetilde{\varphi} \equiv \widetilde{\varphi} + 2\pi$.

- Bosonisation of the spin-chain Hamiltonian:

$$H = \sum_{j=1}^{2L} \left[\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} - \mu^2 \; \sigma_j^z \sigma_{j+1}^z + i\mu \; \underbrace{(\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)}_{\text{RG irrelevant!}} \right]$$

$$H_b = v_F \int \mathrm{d}x \left[(\partial_x \phi_0)^2 + (\partial_x \phi_1)^2 - \mu^2 \mathcal{J}_0 \mathcal{J}_1 \right] , \qquad \mathcal{J}_a := \partial_x \phi_a$$

Eigenmodes: $\begin{cases} \phi = \phi_0 + \phi_1 \;\leftrightarrow\; g(\mu) \\ \widetilde{\phi} = \phi_0 - \phi_1 \;\leftrightarrow\; \widetilde{g}(\mu) \end{cases} \Rightarrow g, \widetilde{g}$ depend on γ

- Bethe Ansatz results show that irrelevant terms play a role!

Operator content

a. Reminder: bosonic and fermionic partition sums

- Toroidal partition sums for the free boson

$$Z_{m,m'}(g) := \int_{\substack{\delta\varphi=2\pi m \\ \delta'\varphi=2\pi m'}} D\varphi \ \exp\left(-\frac{g}{4\pi}\int \mathrm{d}^2x|\nabla\varphi|^2\right)$$

$$Z_b(g) := \sum_{m,m'} Z_{m,m'}(g)$$

- Toroidal partition sums for the Ising model
  $\mathcal{Z}_{r,r'}$:= partition sum with boundary conditions on spins:

$$\sigma \to (-1)^r\sigma \qquad \text{(horizontal BC)}$$
$$\sigma \to (-1)^{r'}\sigma \qquad \text{(vertical BC)}$$

Operator content

b. Partition sum of the $\mathbb{Z}_2$ model

- Sum contributions from $h, \bar{h}$:

$$Z(g) = \sum_{\substack{m+\widetilde{m}\equiv 0\ [2] \\ m'+\widetilde{m}'\equiv 0\ [2]}} Z_{m,m'}(g)\ Z_{\widetilde{m},\widetilde{m}'}(1/2)$$

- Rewrite as:

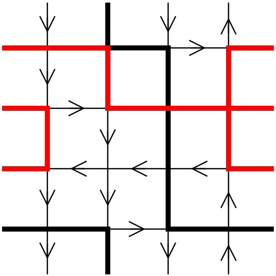
$$Z(g) = \frac{1}{2} \sum_{\substack{m+r_1+r_2\equiv 0\ [2] \\ m'+r_1'+r_2'\equiv 0\ [2]}} (-1)^{r_1 r_2' + r_1' r_2}\ Z_{m,m'}(g)\ \mathcal{Z}_{r_1,r_1'}\ \mathcal{Z}_{r_2,r_2'}$$

Lattice interpretation

- States on a double-edge:

$$\begin{aligned} \uparrow\uparrow &\rightarrow \uparrow \\ \downarrow\downarrow &\rightarrow \downarrow \\ \{\uparrow\downarrow,\ \downarrow\uparrow\} &\rightarrow \{|,\ {\color{red}|}\} \end{aligned}$$

- Configuration on the lattice:



Conclusion

- Summary
  - Integrable, inhomogeneous 6V model with $\mathbb{Z}_2$ symmetry
  - Related to Potts model and quantum spin chain
  - Bethe Ansatz solution $\rightarrow$ low-energy spectrum
  - CFT = '1 boson + 2 Majorana fermions'
  - Coupling of excitation sectors: parity of defects

- Perspectives
  - Construct higher-order models $\mathbb{Z}_N$
  - Interpret conformal spectrum in terms of extended CFTs?
  - Pseudo-Hermitian Hamiltonian [Bender], [Korff,Weston], [Castro-Alvaredo,Fring]

Thank you for your attention!