"La Physique dans le Plan "

Les Houches, 5 Mars , 2010

Y-system for the spectrum of AdS/CFT

Vladimir Kazakov

(ENS, Paris)

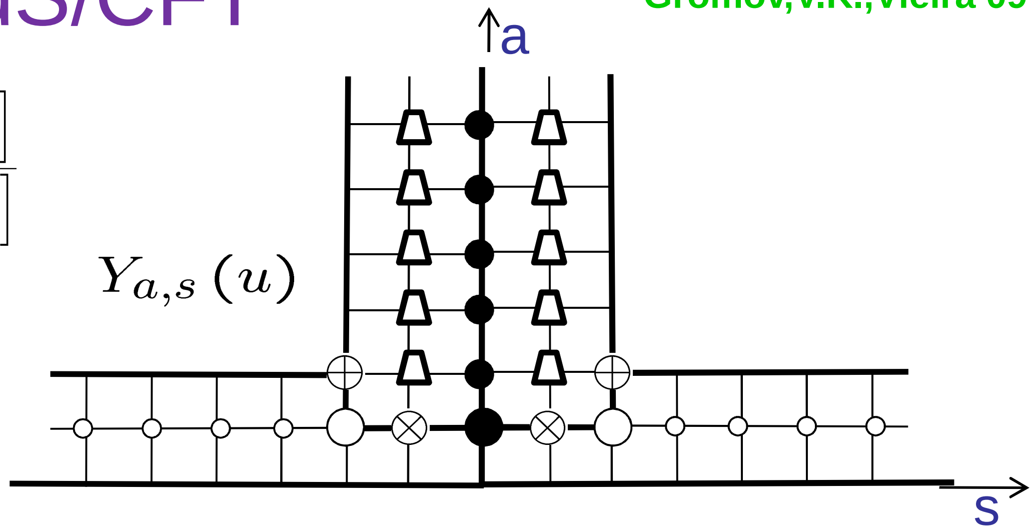
with N.Gromov and P.Vieira,	arXiv:0812.5091
	arXiv:0901.3753
	arXiv:0906.4240
with N.Gromov, A.Kozak and P.Vieira	arXiv:0902.4458
with N.Gromov and Z.Tsuboi	arXiv:1002.3981
with S.Leurent	to appear

Outline

- Integrability for QFT's in dimensions D>(1+1)?
- Possible for some CFT's at D=3,4 (and may be more...)
- Based on AdS/CFT duality: (1+1)D superstring sigma models on AdS-type backgrounds
- String σ-models are 2D reparametrization invariant (more then conformal): β-function = 0, c=26
- In light-cone gauge it looks like a "massive" non-relativistic σ-model
- All popular 2D integrability tools applicable: S-matrix, asymptotic Bethe ansatz (ABA), TBA for finite size spectrum, ...
- .... Y-system for AdS/CFT

Gromov,V.K.,Vieira'09

$$\frac{Y^+_{a,s}Y^-_{a,s}}{Y_{a+1,s}Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\left[1+Y_{a+1,s}\right]}$$



Calculates anomalous dimensions of all local operators in planar N=4 SYM theory at any coupling

Integrability on SYM side

- Origins of YM integrability: Lipatov's reggeon Hamiltonian

Lipatov
Faddeev,Korchemsky

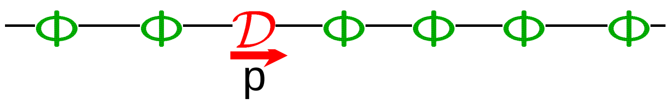
- 1,2,3,4,..∞ –loop integrability in N=4 SYM

Minahan,Zarembo,

Beisert,Kristijanssen, Staudacher

- Exact spectrum of impurities on the SYM spin chain

Fiamberti,Zieg, Zanon, Santambogio
Beisert,Dippel,Staudacher



$$\epsilon(p) = \sqrt{1 + \lambda \sin^2 \frac{p}{2}}$$

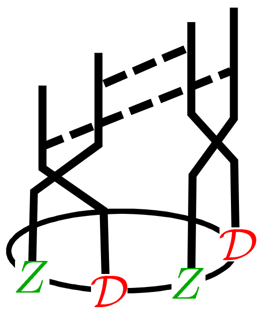
- All loop asymptotic Bethe ansatz for long operators. PSU(2,2|4) S-matrix, Cusp dimention at all loops.

Staudacher, Beisert
Beisert,Eden,Staudacher,
Janik, Hernandez,Lopez

- Direct 4-loop calculations and wrapping for short operators

Bern,Kosover,Smirnov,Dixon
Santambrogio, Zanon,
Velizhanin

$$\Delta_{Konishi} = 2 + 12\lambda - 48\lambda^2 + 336\lambda^3 - [(2820 + 288\zeta(3)) + (324 + 864\zeta(3) - 1440\zeta(5))]\,\lambda^4$$
$$+ \left[(26508 + 4320\zeta(3) + 2880\zeta(5)) + \left(-11340 + 2592\zeta(3) - 5184\zeta(3)^2 - 11520\zeta(5) + 30240\zeta(7)\right)\right]\lambda^5 + O\left(\lambda^6\right)$$



+ ....

$$\mathcal{O}_{Konishi} = \mathrm{Tr}\ [\mathcal{D}, Z]^2$$

… reproduced from Y-system

Gromov,V.K.,Vieira
Arutyunov, Frolov
Balog, Hegedus

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Integrability on string side

- Classical integrability of Green-Schwarz-Metsaev-Tseytlin string σ-model on $AdS_5 \times S^5$ and Lax pair — Berna,Roiban,Polchinski
- Finite gap solution of string σ-model and quasilcassics — V.K.,Marshakov, Minahan, Zarembo, Beisert,V.K.,Sakai,Zarembo Gromov,Vieira
- Early attempts of non-perturbative quantization — Arutyunov,Frolov,Staudacher Beisert,Staudacher
- Dressing factor — Janik, Hernandez,Lopez Beisert,Eden,Staudacher,
- Konishi dimension 4- and 5-loop calculations from string (agrees with direct 4-loop SYM perturbation theory) — Bajnok,Janik,Lukowski
- Y-system for AdS/CFT: the spectral equation for exact dimensions of planar N=4 SYM at any YM coupling. — Gromov , V.K, Vieira
- TBA for AdS/CFT, confirming and clarifying the Y-system — Bombardelli,Fioravanti, Tateo Gromov , V.K, Vieira Arutyunov, Frolov
- New integrable AdS/CFT's: AdS4/CFT3, AdS3/CFT2,..... — Minahan,Zarembo, Zarembo

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N=4 SYM and a string on $AdS_5 \times S^5$

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

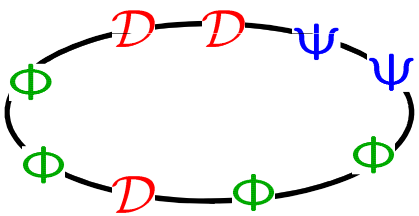
$$\mathcal{S}_{sigma} = \sqrt{\lambda}\int d\tau \int_0^L d\sigma \left[\left(\partial\vec{X}\right)^2 + \Lambda(\vec{X}^2 - I) + \text{fermions}\right]$$

Metsaev-Tseytlin sigma model of GS superstring

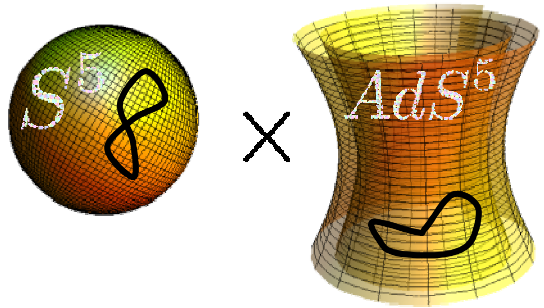
Maldacena'97
Gubser,Polyakov,Klebanov'98
Witten'98

CFT/AdS duality

$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots](x) \quad \longleftrightarrow$$



$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$



$$\mathcal{O}_A(\Lambda x) \rightarrow \Lambda^{\Delta_A(\lambda)}\mathcal{O}_A(x)$$

$$\hat{D}\,\mathcal{O} = \Delta\,\mathcal{O}$$

Dilatation operator
as a spin chain hamiltonian

Dimension of a local operator = Energy of a string state

Global superconformal symmetry → psu(2,2|4) ← isometry of background

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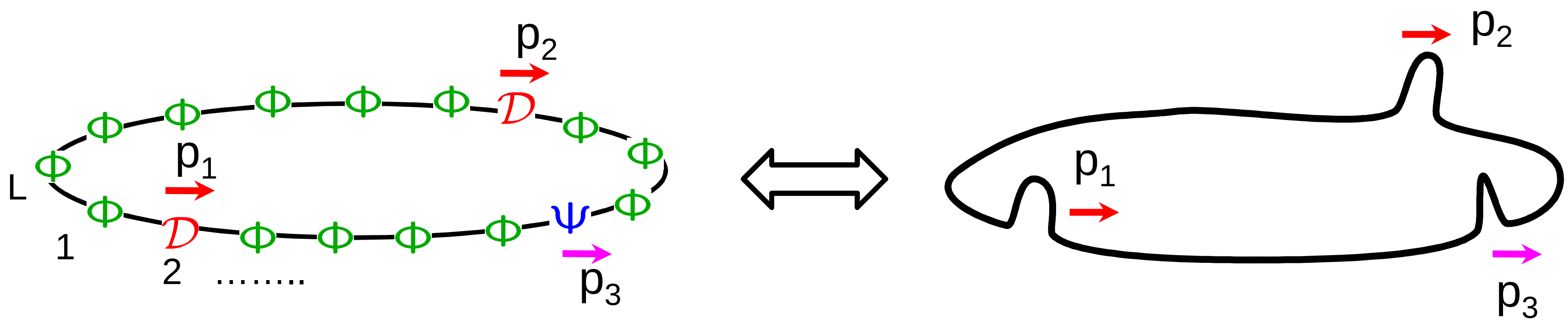
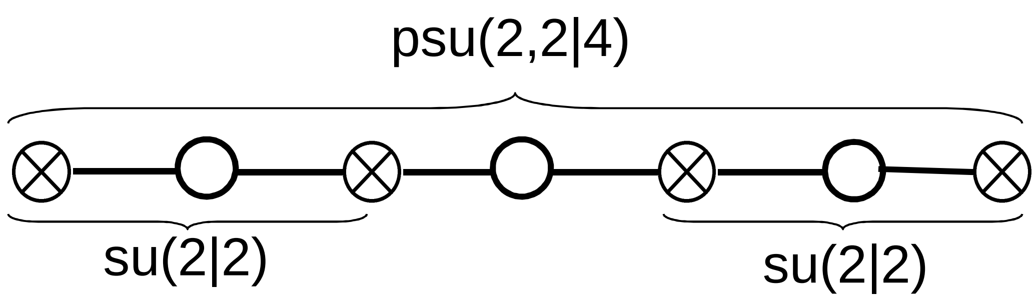
Integrability and S-matrix



- Light cone gauge breaks the global and world-sheet Lorentz symmetries :

$$psu(2,2|4) \Longrightarrow su(2|2) \times su(2|2)$$

Beisert,
Staudacher
Janik
Eden,Beisert,Staudacher...



- S-matrix (Hubbard R!) of AdS/CFT via bootstrap of A.&Al.Zamolodchikov:

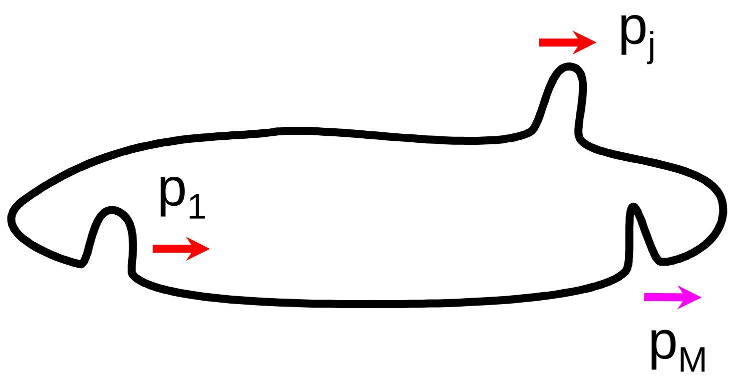
$$\hat{S}_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2)$$

- Asymptotic integrability : factorized scattering, asymptotique Bethe ansatz…

Z *

Asymptotic Bethe Ansatz (ABA)

$$0 = \left(e^{iLp_j} \prod_{k \neq j}^{M} \hat{S}(p_j, p_k) - 1 \right) \Big| \quad \Big\rangle$$

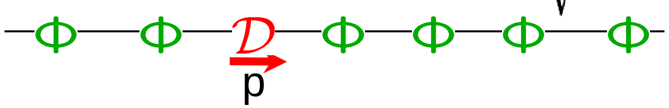


- This periodicity condition is diagonalized by nested Bethe ansatz
- Asymptotic dispersion relation for dimensions of one particle states

$$E = \Delta - L = \sum_{j=1}^{M} \epsilon(p_j) \quad + \quad O\left(e^{-cL}\right)$$

$$\epsilon(p) = \sqrt{1 + \lambda \sin^2 \frac{p}{2}}$$

Santambrogio,Zanon'02
Beisert,Dippel,Staudacher'04



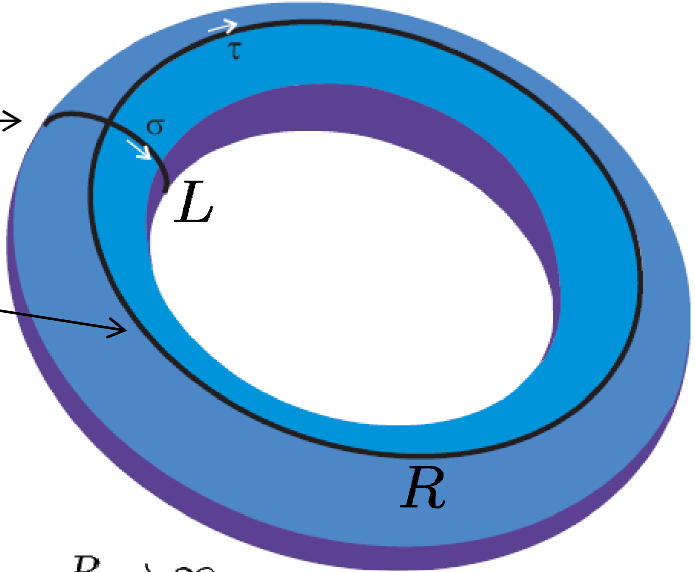
finite size corrections
Important for short operators!

Zamolodchikov TBA for short operators

original model
on L circle

”mirror” model
on R circle

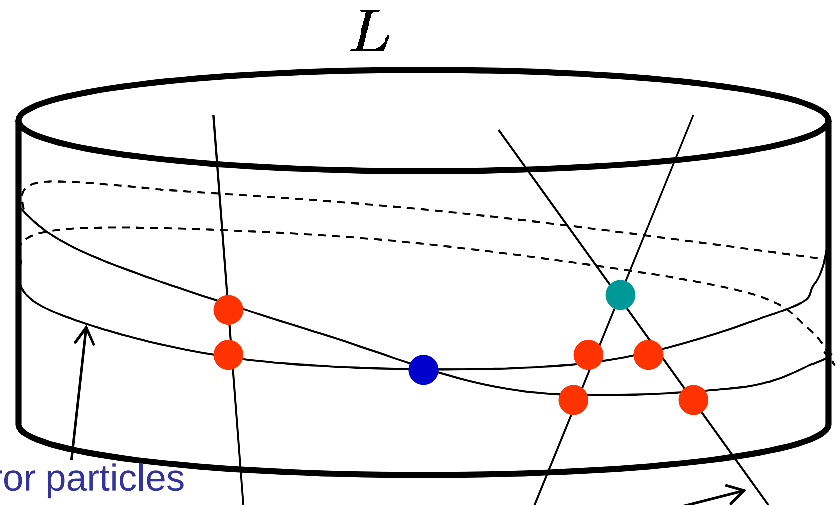
Ambjorn,Janik,Chrisijansen’07
Arutyunov,Frolov’08



$$\mathcal{Z}=\sum_n e^{-E_n^{mir}(R)L}=\sum_n e^{-E_n^{ph}(L)R} \quad\rightarrow\quad e^{-E_0^{ph}(L)R}, \qquad R\rightarrow\infty$$

- 2d vacuum consists of a dense gas of virtual (mirror) “particles” and a few physical particles.
- Scattering: mir-mir, mir-phys, phys-phys
- Fusion is simple on a u-plane (rapidity)
- Large R : mirrror rapidities localize on poles of S-matrix $\rightarrow$ bound states

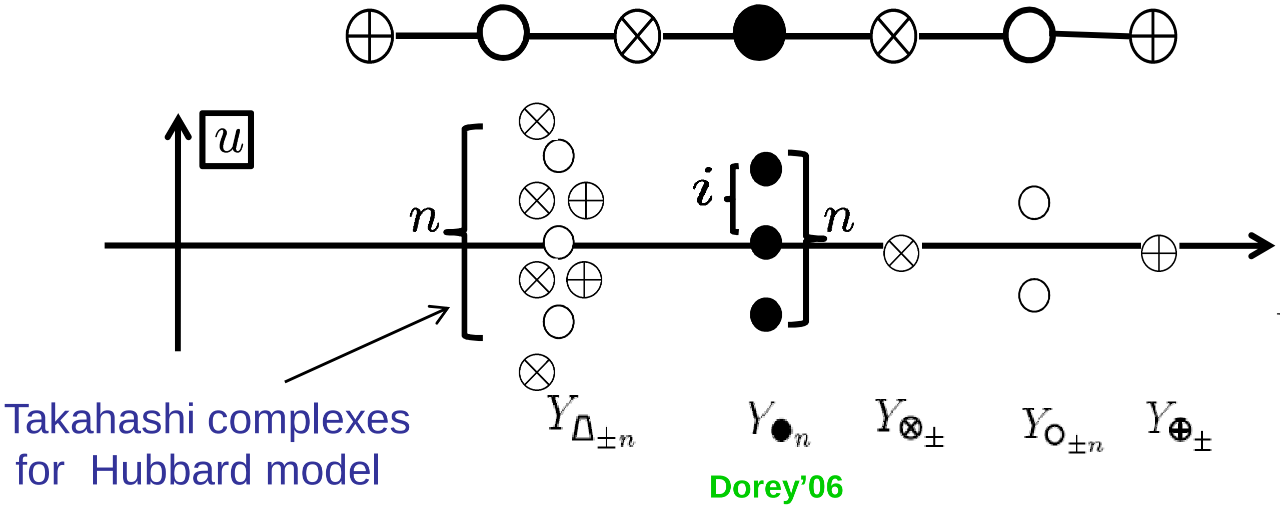
$$S_{ab}^{cd}(p(u),p(v))\sim\frac{1}{u-v+i}$$



reminds the spin chain transfer matrix

Bound states and TBA in AdS/CFT

Gromov,V.K.,Vieira'09
Bombardelli,Fioravanti,Tateo'09
Gromov,V.K.,Kozak,Vieira'09
Arutyunov,Frolov'09



$$Y(u) = \frac{\bar\rho(u)}{\rho(u)}$$

- Dispersion in Zhukovski parametrization:

$$\epsilon(u) = \sqrt{\lambda}\left(\frac{i}{x^-(u)} - \frac{i}{x^+(u)}\right), \qquad p(u) = i\log\left(\frac{x^-(u)}{x^+(u)}\right)$$

- Mirror (Wick) transformation :

$$\epsilon \to i\tilde p \equiv \epsilon^*, \qquad p \to i\tilde\epsilon \equiv p^*$$

$$\epsilon = \sqrt{1 + 8\lambda \sin^2\frac{p}{2}} \quad \to \quad \sqrt{8\lambda}\sinh\frac{\tilde\epsilon}{2} = \sqrt{1+\tilde p^2}$$

$$x^+(u) \to x^+(u), \qquad x^-(u) \to x^-(u^*) \sim \frac{1}{x^-(u)}$$

Zhukovski map:

$$x + \frac{1}{x} = \frac{u}{\sqrt{\lambda}}$$

$$x^\pm + \frac{1}{x^\pm} = \frac{u \pm i/2}{\sqrt{\lambda}}$$

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Y-system for the spectrum of AdS/CFT

• TBA: Minimizing free energy at finite temperature T=1/L : TBA eqs. and Y-system

$$f = \sum_{(a,s)\in \mathrm{T-hook}} \int \left[\delta_{s,0} L\, \rho_{a,0}\, \tilde{p}_a - \rho_{a,s} \log\left(1 + \frac{\bar{\rho}_{a,s}}{\rho_{a,s}}\right) - \bar{\rho}_{a,s} \log\left(1 + \frac{\rho_{a,s}}{\bar{\rho}_{a,s}}\right)\right]$$

where $\rho_{a,s} + \bar{\rho}_{a,s} = \delta_{s,0}\tilde{\epsilon}'_a + K_{a',s';a,s} * \rho_{a',s'}$

• Extremum condition

$$\log Y_{a,s} = \delta_{s,0} L \tilde{p}'_a + K_{a,s;a',s'} * \log\left(1 + Y_{a',s'}(u)\right)$$

only ratios enter!

$$Y_{a,s}(u) = \frac{\bar{\rho}_{a,s}(u)}{\rho_{a,s}(u)}$$

• Act by discrete Laplace operator on both parts of this eq.

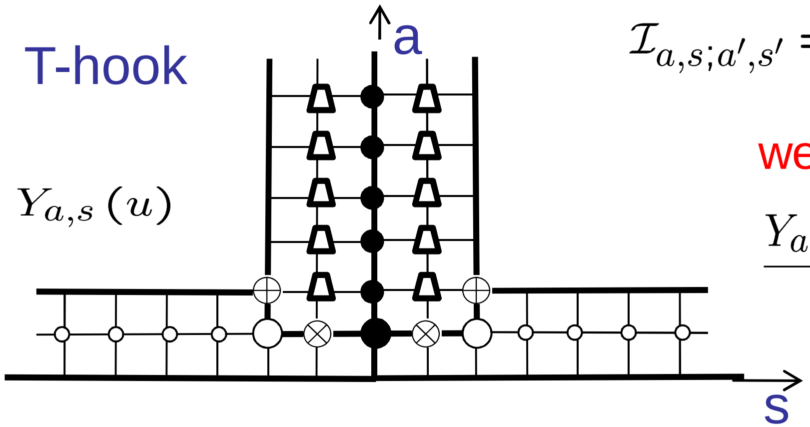
$$\Delta f_a(u) \equiv f_a(u + i/2) + f_a(u - i/2) - f_{a+1}(u) - f_{a-1}(u)$$

• Using $\Delta K(u,v) = \delta(u - v)\mathcal{I}$ where $\mathcal{I}$ is the incidence matrix of T-hook

$$\mathcal{I}_{a,s;a',s'} = \delta_{a,a'}\left(\delta_{s',s+1} + \delta_{s',s-1}\right) - \delta_{s,s'}\left(\delta_{a',a+1} + \delta_{a',a-1}\right)$$

T-hook

$Y_{a,s}(u)$



we get the universal Y-system!

$$\frac{Y_{a,s}(u+i/2)\, Y_{a,s}(u-i/2)}{Y_{a+1,s}\, Y_{a-1,s}} = \frac{\left[1 + Y_{a,s+1}\right]\ \left[1 + Y_{a,s-1}\right]}{\left[1 + Y_{a+1,s}\right]\ \left[1 + Y_{a+1,s}\right]}$$

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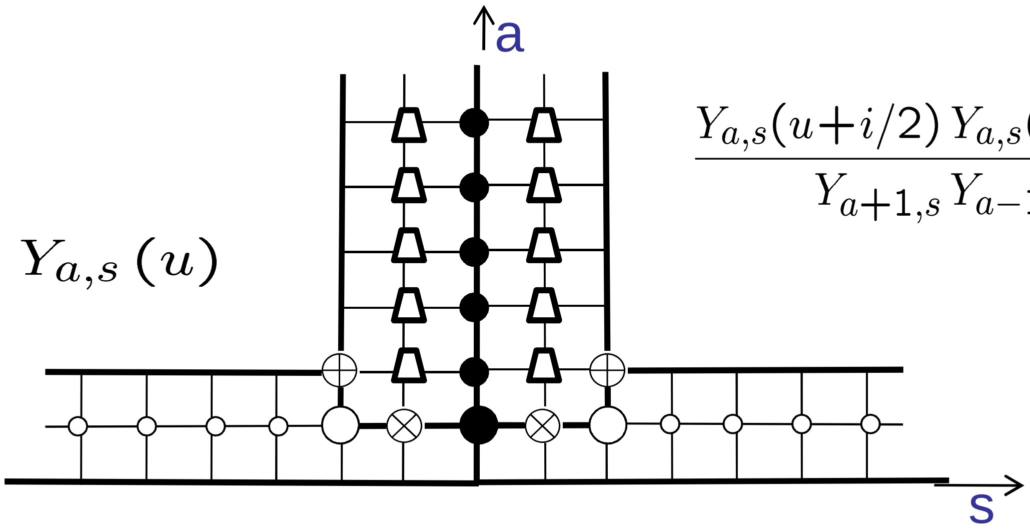
Y-system for excited states of AdS/CFT



$$\frac{Y_{a,s}(u+i/2)\,Y_{a,s}(u-i/2)}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

- Anomalous dimensions: $\Delta = f_{min} = \int \frac{du}{2\pi i}\,\partial_u \epsilon_a(u) \log\left(1+Y_{a,0}(u)\right)$

- u_j obey the exact Bethe eqs: $Y_{1,0}(u_j)+1=0, \qquad j=1,2,\ldots,M$

Bazhanov, Lukyanov, Zamolodchikov
Dorey,Tateo
Fioravanti,Quattrini,Ravanini

For AdS/CFT **Gromov,V.K.,Kozak,Vieira**

Y-system and Hirota eq.: discrete integrable dynamics

- Relation of Y-system to T-system (Hirota equation)
  . (the Master Equation of Integrability!)

$$Y_{a,s} = \frac{T_{a,s+1}\, T_{a,s-1}}{T_{a+1,s}\, T_{a-1,s}}$$

$$T_{a,s}(u+i/2)T_{a,s}(u-i/2) = T_{a,s-1}(u)\, T_{a,s+1}(u) + T_{a+1,s}(u)\, T_{a-1,s}(u)$$

- Discrete classical integrable dynamics!

- General solution of Hirota equations with Dirichlet b.c. on a line:

$$T_{a,s}(u) = \mathrm{Det}_{\,1\le k,j\le a}\, T_{1,s+k-j}(u+k+j)$$

- General solution in a strip of width N:

$$T_{a,s}(u) = \begin{vmatrix} \bar q_1(u-s-1) & \bar q_2(u-s-1) & \cdots & \bar q_N(u-s-1) \\ \cdots & \cdots & \cdots & \cdots \\ \bar q_1(u-s-a) & \bar q_2(u-s-a) & \cdots & \bar q_N(u-s-a) \\ q_1(u+s-a-1) & q_2(u+s-a-1) & \cdots & q_N(u+s-a-1) \\ \cdots & \cdots & \cdots & \cdots \\ q_1(u+s-N) & q_2(u+s-N) & \cdots & q_N(u+s-N) \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} a \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} N-a \end{matrix}$$

Krichever,Lipan, Zabrodin,Wiegmann

- Gauge transformations of T...

$$Ta,s(u) \to G_1\left(u+i\frac{a+s}{2}\right) G_2\left(u+i\frac{a-s}{2}\right) G_3\left(u-i\frac{a+s}{2}\right) G_4\left(u-i\frac{a-s}{2}\right) T_{a,s}(u)$$

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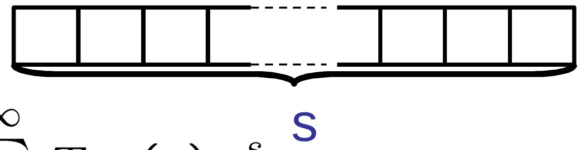
Hirota equation and characters of supergroups

- Jacobi-Trudi formula for SL(K|M) characters general irrep $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_a\}$

$$T_{\{\lambda\}}[g] = \det_{1\le i,j\le a} T_{1,\lambda_i-i+j}[g]\,, \qquad g = \text{diag}\{x_1,\cdots,x_K|y_1,\cdots,y_M\} \in SL(K|M).$$

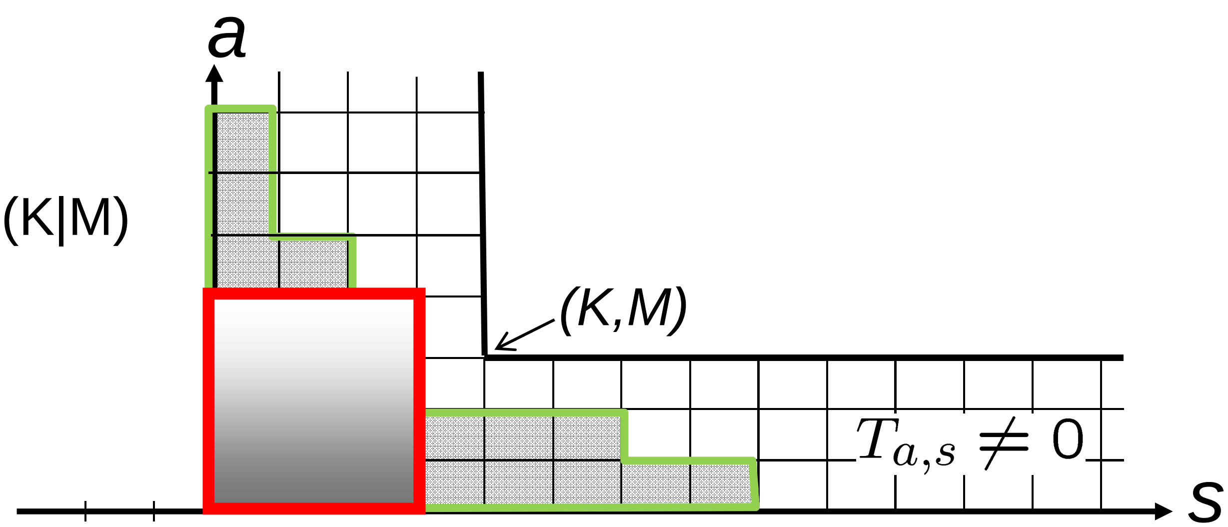
$T_{1,s}[g]$ are super-characters in symmetric irreps generated by

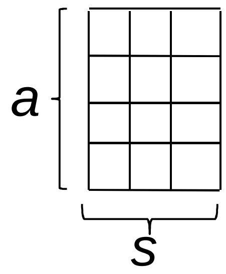


$$w(z) \equiv \text{sdet}\,(1-zg)^{-1} = \sum_{s=1}^{\infty} T_{1,s}(g)\,z^s$$

- For rectangular Young tableaux (a,s) Hirota eq. with fat hook b.c.:

$$T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$





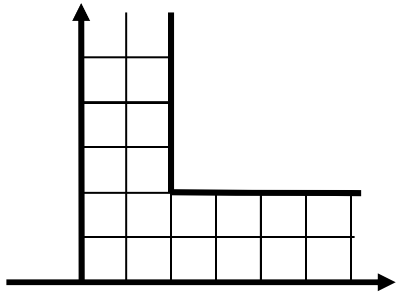
Generalization to super-spin chains:

V.K.,Zabrodin,Sorin'07
Tsuboi'97

Supergroups: Fat Hook of SU(4|4) and T-hook of PSU(2,2|4)

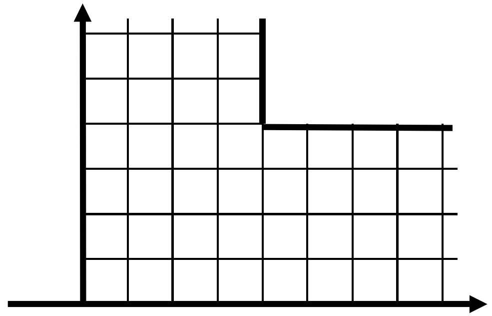
SU(2|2)

$$w_{SU(2|2)}(z;g) = \frac{(1-zy_1)(1-zy_2)}{(1-zx_1)(1-zx_2)} = \sum_{j=0}^{\infty} T_{1,s} z^s$$



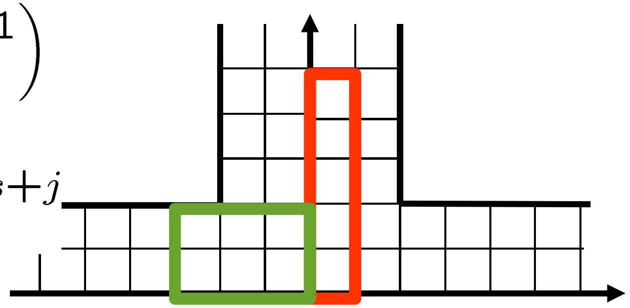
SU(4|4)

$$\begin{aligned} w_{SU(4|4)}(z; g \otimes g') &= w_{SU(2|2)}(z;g) \times w_{SU(2|2)}\left(z;g'\right) \\ &= \sum_{s=0}^{\infty} z^s \sum_{j=0}^{s} T_{1,j} T'_{1,s-j} \end{aligned}$$



SU(2,2|4)

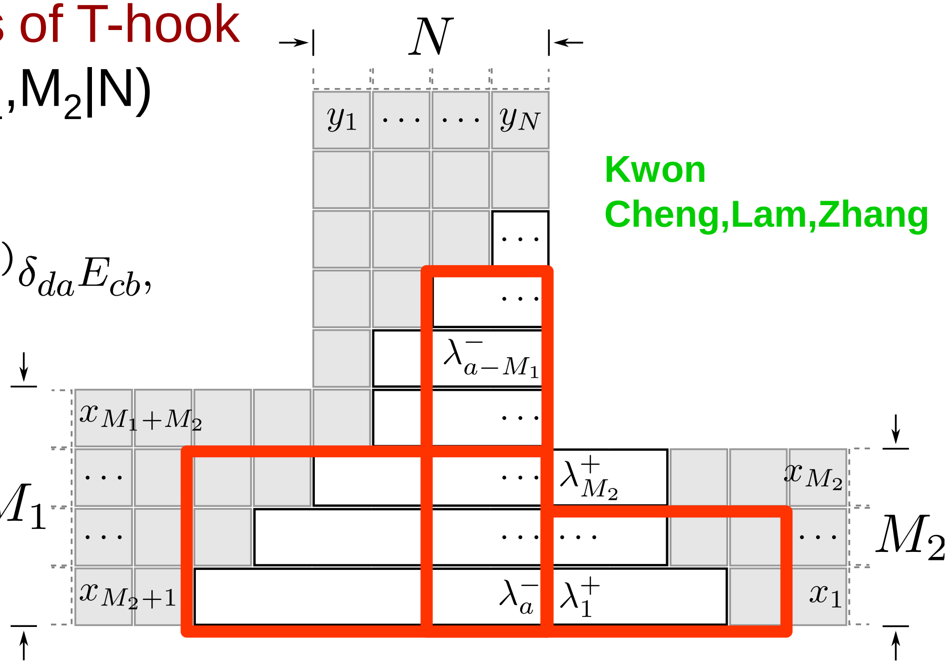
$$\begin{aligned} w_{SU(2,2|4)}(z; g \otimes g') &= w_{SU(2|2)}(z;g) \times w_{SU(2|2)}\left(\frac{1}{z}; g'^{-1}\right) \\ &= \frac{y_3 y_4}{x_3 x_4} \sum_{s=-\infty}^{\infty} z^s \sum_{j=\mathrm{Min}(-s,0)}^{\infty} \bar{T}'_{1,j} T_{1,s+j} \end{aligned}$$



- Using these characters one solves AdS/CFT quasiclassically
- Generalization to full quantum case (u-dependence) possible!

Gromov
V.K.
Tsuboi

Unitary highest weight representations of T-hook of non-compact superalgebra u(M$_1$,M$_2$|N)



Kwon
Cheng,Lam,Zhang

Generators:

$$[E_{ab}, E_{cd}\} = \delta_{bc}E_{ad} - (-1)^{(p_a+p_b)(p_c+p_d)}\delta_{da}E_{cb},$$

Dual basis : $\varepsilon_i(E_{jj}) = \delta_{ij}$

$$(\varepsilon_i|\varepsilon_j) = p_i\delta_{ij}, \quad p_i = \left(+1_{M_1}, -1_N, +1_{M_2}\right)$$

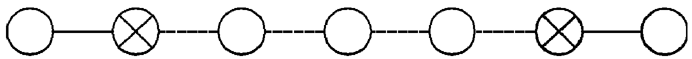
Simple roots:

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \ldots, N+M-1$

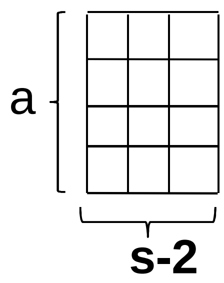
Cartan matrix :

$$C_{ij} = (\alpha_i, \alpha_j)$$

- Highest weight Λ for u(2,|4|,2)



$$\Lambda = \begin{cases} a(-\varepsilon_1-\varepsilon_2) + (s+2)\sum_{i=7}^{a+6}\varepsilon_i , & s<-2 , \quad 0\le a\le 2 \\ a(-\varepsilon_1-\varepsilon_2+\sum_{i=3}^{s+4}\varepsilon_i) , & -2\le s\le 2 , \quad 0\le a \\ a(-\varepsilon_1-\varepsilon_2+\sum_{i=3}^{6}\varepsilon_i) + (s-2)\sum_{i=7}^{a+6}\varepsilon_i , & 2<s , \quad 0\le a\le 2 \end{cases}$$



- Unitarity : $\eta E^\dagger_{ab}\eta = E_{ba}$, $\quad \eta = \mathrm{diag}(-1_{M_1}, +1_N, +1_{M_2})$

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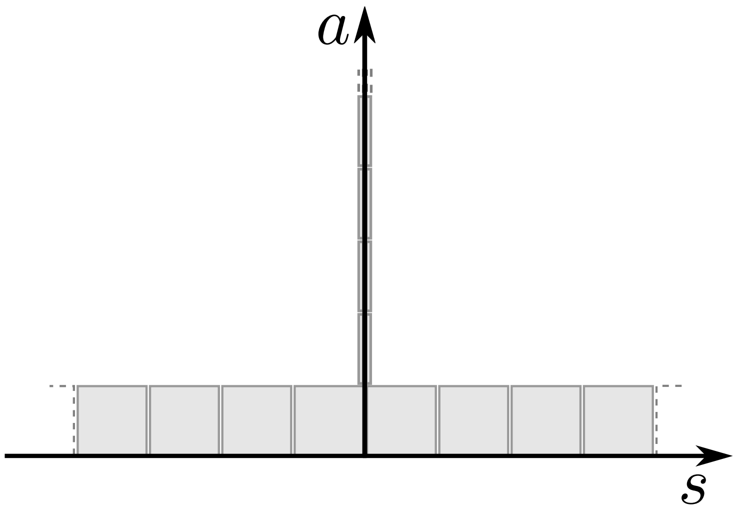
Unitary non-compact representations of T-hook of u(1,1)

$$[h_3, h_\pm] = \pm 2h_\pm \ , \ [h_-, h_+] = h_3$$



highest weight

$$\Lambda = \begin{cases} -\varepsilon_1 + |s|\varepsilon_2 & , \ s > 0 \ , \ a = 1 \\ -\varepsilon_1 - |s|\varepsilon_1 & , \ s < 0 \ , \ a = 1 \\ -a\varepsilon_1 & , \ s = 0 \ , \ a > 0 \end{cases}$$

$$h_+ = \begin{pmatrix} 0 & a_0 & 0 & 0 & \cdots \\ 0 & 0 & a_1 & 0 & \cdots \\ 0 & 0 & 0 & a_2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad h_- = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -a_0 & 0 & 0 & 0 & \cdots \\ 0 & -a_1 & 0 & 0 & \cdots \\ 0 & 0 & -a_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad h_3 = \begin{pmatrix} -|s|-1 & 0 & 0 & \cdots \\ 0 & -|s|-3 & 0 & \cdots \\ 0 & 0 & -|s|-5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$a_j = i\sqrt{(j+1)(j+|s|+1)}$$

- Solution of Hirota for U(1,1) characters

$$T_{1,s} \equiv T_s^{(1+1)} = \frac{1}{x_2} \sum_{j=\max(0,-s)}^{\infty} T_{s+j}^{(1)}(x_1) T_j^{(1)}(1/x_2) = \sum_{j=\max(0,-s)}^{\infty} x_1^{s+j} x_2^{-j-1}$$

$$T_{1,s} \equiv T_s^{(1+1)}(x_1, x_2) = \begin{cases} \frac{x_1^s}{x_2 - x_1} & , \ s > 0 \\ \frac{x_2^s}{x_2 - x_1} & , \ s \le 0 \end{cases} \qquad T_{a,0} = \frac{x_2^{1-a}}{x_2 - x_1}$$

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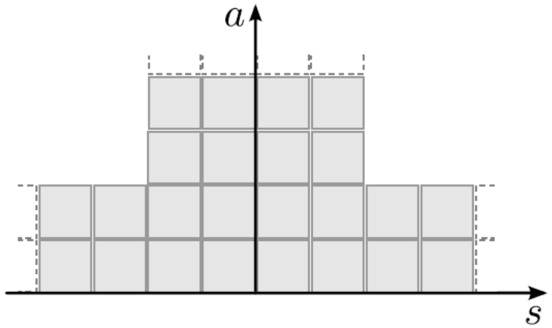
Character solution of T-hook for u(2,2|4)



- Solution in finite determinants
  (analogue of the 1-st Weyl formula)

$$T_{a,s} = \begin{cases} (-1)^{(a+1)s} \left(\dfrac{x_3 x_4}{y_1 y_2 y_3 y_4}\right)^{s-a} \dfrac{\det\left(S_i^{\theta_{j,s+2}} y_i^{j-4-(a+2)\theta_{j,s+2}}\right)_{1\le i,j\le 4}}{\det\left(S_i^{\theta_{j,0+2}} y_i^{j-4-(0+2)\theta_{j,0+2}}\right)_{1\le i,j\le 4}} , & a \ge |s| \\ \dfrac{\det\left(Z_i^{(1-\theta_{j,a})} x_i^{2-j+(s-2)(1-\theta_{j,a})}\right)_{1\le i,j\le 2}}{\det\left(Z_i^{(1-\theta_{j,0})} x_i^{2-j+(0-2)(1-\theta_{j,0})}\right)_{1\le i,j\le 2}} , & s \ge +a \end{cases}$$

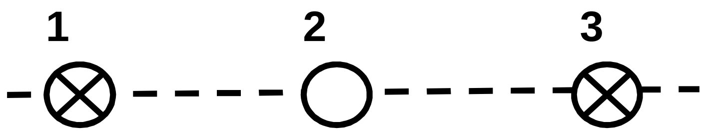
$$\theta_{j,s} = \begin{cases} 1 , & j > s \\ 0 , & j \le s \end{cases}$$

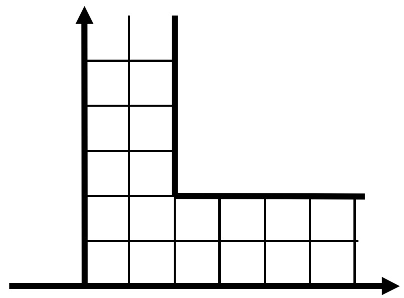
$$S_i = \frac{(y_i - x_3)(y_i - x_4)}{(y_i - x_1)(y_i - x_2)}$$

$$Z_i = \frac{(x_i - y_1)(x_i - y_2)(x_i - y_3)(x_i - y_4)}{(x_i - x_3)(x_i - x_4)} ,$$

Solving full quantum Hirota: SU(2|2) example

- Generating functional of SU(2|2) irreps:





$$\mathcal{W} = \left[1 - \mathbf{T}^1 D\right] \cdot \left[1 - \mathbf{T}^2 D\right]^{-1} \cdot \left[1 - \mathbf{T}^3 D\right]^{-1} \cdot \left[1 - \mathbf{T}^4 D\right] , \quad (1) \qquad D = e^{-i\partial_u}$$

$$\mathcal{W}_{\mathsf{R}} = \sum_{s=0}^{\infty} \mathbf{T}^{\mathsf{R}}_{1,s}\left(u + i\frac{1-s}{2}\right) D^s , \qquad \mathcal{W}_{\mathsf{R}}^{-1} = \sum_{a=0}^{\infty} (-1)^a \mathbf{T}^{\mathsf{R}}_{a,1}\left(u + i\frac{1-a}{2}\right) D^a .$$

- T's are parametrized through 4(+1) functions

$$\begin{aligned}
\mathbf{T}^1(u) &= h\frac{Q_1^-}{Q_1^+}\frac{B_4^+}{B_4^-}, \\
\mathbf{T}^2(u) &= h\frac{Q_1^- Q_2^{++}}{Q_1^+ Q_2}, \\
\mathbf{T}^3(u) &= h\frac{Q_2^{--} Q_3^+}{Q_2 Q_3^-} \\
\mathbf{T}^4(u) &= h\frac{Q_3^+}{Q_3^-}\frac{R_4^-}{R_4^+}
\end{aligned}$$

Back to asymptotic Bethe eqs.and S-matrix: L→∞

Gromov,V.K.,Vieira'09

- From TBA:

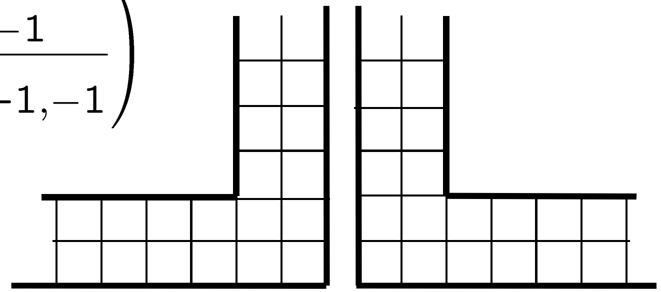
$$Y_{a\geq 1,0} \sim \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \to 0$$

$$1 + Y_{a,s} = \frac{T^+_{a,s} T^-_{a,s}}{T_{a+1,s} T_{a-1,s}}$$

- It is a spin chain limit:

$$\frac{Y^+_{a,0} Y^-_{a,0}}{Y_{a-1,0} Y_{a+1,0}} \simeq \left(\frac{T^+_{a,1} T^-_{a,1}}{T_{a-1,1} T_{a+1,1}}\right)\left(\frac{T^+_{a,-1} T^-_{a,-1}}{T_{a-1,-1} T_{a+1,-1}}\right)$$

- T-system splits into two SU(2|2)$_{L,R}$ wings:



$T_{a,s<0}$, $T_{a,s>0}$

- Solving this discrete Laplace eq. in (a,u)-variables we get

$$Y_{a,0}(u) \simeq \underbrace{\left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \frac{\phi^{[-a]}}{\phi^{[+a]}}} T^L_{a,-1} T^R_{a,1}$$

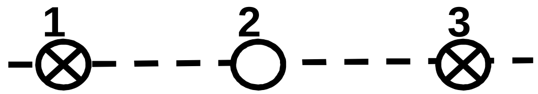
zero mode (dressing factor…)

transfer matrices of SU(2|2) (fixed from analyticity)

Asymptotic Bethe Ansatz $L \to \infty$: concise form

- Fundamental transfer matrix for $SU(2|2)_{L,R}$

$$\mathbf{T}_{L,R}(u) = -\mathbf{T}^1_{L,R}(u) + \mathbf{T}^2_{L,R}(u) + \mathbf{T}^3_{L,R}(u) - \mathbf{T}^4_{L,R}(u)$$



$$\begin{aligned}
\mathbf{T}^1(u) &= h\frac{Q_1^-}{Q_1^+}\prod_{j=1}^{K_4}\frac{\frac{1}{x^+}-x^-_{4,j}}{\frac{1}{x^+}-x^+_{4,j}},\\
\mathbf{T}^2(u) &= h\frac{Q_1^- Q_2^{++}}{Q_1^+ Q_2},\\
\mathbf{T}^3(u) &= h\frac{Q_2^{--} Q_3^+}{Q_2 Q_3^-}\\
\mathbf{T}^4(u) &= h\frac{Q_3^+}{Q_3^-}\prod_{j=1}^{K_4}\frac{x^- - x^+_{4,j}}{x^- - x^-_{4,j}}
\end{aligned}$$

- Scalar dressing factor and T-functions:
1. Analyticity (each T has ≤2 cuts)
2. Y's are real functions

$$\frac{\phi^-}{\phi^+} = \frac{Q_4^{--}}{Q_4^{++}} \cdot \prod_{j=1}^{K_4}\left[\frac{\frac{1}{x^+}-x^-_{4,j}}{\frac{1}{x^-}-x^+_{4,j}}\right] \cdot \mathcal{S}_0^2$$

$$\mathcal{S}_0(u) = \prod_{j=1}^{K_4}\sigma(x(u), x_{4,j})$$

Defs:

$$\begin{aligned}
Q_l(u) &= \prod_{j=1}^{K_l}(u - u_{l,j})\\
l &= 1,2,3(5,6,7)\\
h &= \prod_{j=1}^{K_4}\frac{x \;\; - x^-_{4,j}}{x^- - x^+_{4,j}}\prod_{j=1}^{K_1}\frac{\frac{1}{x^+}-x_{1,j}}{\frac{1}{x^-}-x_{1,j}}\prod_{j=1}^{K_3}\frac{\frac{1}{x^-}-x_{3,j}}{\frac{1}{x^+}-x_{3,j}}
\end{aligned}$$

- Crossing equation (from reality of Y's)

$$\sigma(x,y)\cdot\sigma(x^\star,y) = \frac{y^-}{y^+}\frac{x^- - y^-}{x^+ - y^-}\cdot\frac{1/x^- - y^+}{1/x^+ - y^+}$$

Janik
Beisert, Eden, Staudacher,
Volin

- Beisert-Staudacher ABA – from $Y_{1,0}(u_j) + 1 = 0$
- Auxiliary Bethe eqs. from regularity of $\mathbf{T}_{L,R}(u)$

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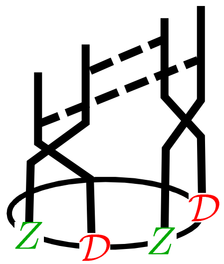
Konishi at 4 loops (one wrapping) from **Y-system**

- We have to calculate $Y_{a,0}$ in mirror for L=2
- Dressing factor does not contribute at this order
- Two physical roots defined from central ABA eq.

$$\mathcal{O}_{Konishi} = \mathrm{Tr}\ [\mathcal{D}, Z]^2$$



+ ....

$$\left(\frac{x^-(u_\pm)}{x^+(u_\pm)}\right)^2 \frac{u_\pm - i/2}{u_\pm + i/2} = 1 \Rightarrow u_\pm = \pm\frac{1}{2\sqrt{3}}$$

- Mirror dispersion:

$$\epsilon^{mir}(u) = -\log\left(x^{[-a]}x^{[+a]}\right) = \log\frac{g^4}{(u^2+a^2/4)^2} + \mathcal{O}(g^5)$$

$$Y_{a,0}^{mir}(u) = g^8\left(3\cdot 2^7\frac{3a^2+12au^2-4a}{(a^2+4u^2)^2}\right)^2\frac{1}{y_a(u)y_{-a}(u)}$$

$$y_a(u) = 9a^2 - 36a^3 + 72u^2a^2 + 60a^2 - 144u^2a - 48a + 144u^4 + 48u + 16$$

$$(\Delta_K - 2)_{wrap} \simeq \sum_{a=1}^{\infty}\int_{-\infty}^{\infty} du\ \epsilon_a^{mir}(u)\cdot Y_{a,0}^{mir}(u) =$$

$$g^8(324 + 864\zeta(3) - 1440\zeta_5) + \mathcal{O}(g^{10})$$

~131.000 4D Feynman graphs!

- Now 5 loops available:

$$\begin{aligned}\Delta_{Konishi} &= 2 + 12\lambda - 48\lambda^2 + 336\lambda^3 - [(2820 + 288\zeta(3)) + \underline{(324 + 864\zeta(3) - 1440\zeta(5))}]\,\lambda^4 \\ &+ \left[(26508 + 4320\zeta(3) + 2880\zeta(5)) + \underline{\left(-11340 + 2592\zeta(3) - 5184\zeta(3)^2 - 11520\zeta(5) + 30240\zeta(7)\right)}\right]\lambda^5 + O\left(\lambda^6\right)\end{aligned}$$

Minahan, Zarembo,

Beisert, Kristijanssen, Staudacher

Beisert, Dippel, Staudacher
Beisert, Eden, Staudacher,

Bajnok, Janik, Lukowski

Bern, Dixon, Kosover, Smirnov
Fiamberti, Zieg, Zanon, Santambogio
Velizhanin

Gromov, V.K., Vieira
Arutyunov, Frolov
Balog, Hegedus

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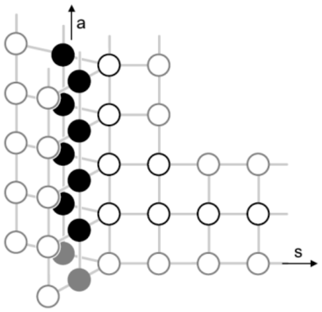
Conclusions and problems

Done:

- Y-système and TBA for exact spectrum of AdS/CFT
- Konishi dimension at “all” couplings (numerics)
- Tested at weak (4 loops) and strong (one loop) coupling
- Two loops at strong coupling predicted
- 5-loops done

$$\Delta_{Konishi} = 2\lambda^{1/4} + 2\lambda^{-1/4} + O\left(\lambda^{-3/4}\right)$$

- Y-system for CP3xAdS4 available and checked



Gromov,V.K.,Vieira
Minahan,OhlssonSax,Sieg
Bajnok, Janik
Bombardelli,Fioravanti,Tateo
Gromov,Levkovich-Maslyuk

To do:

- Derivation of Y-system from SYM?

  It would prove both integrability and AdS/CFT correspondance
- Equation of Destri-DeVega-type, using Hirota discrete integrable dynamics
- Regular weak and strong coupling expansions for Konishi and other operators
- BFKL via Y-system?
- Lessons for QCD?
- Classification and solution of «all» integrable sigma models using Hirota dynamics
- New AdS/CFT’s **Zarembo**

End