Entanglement entropy in (simple) 2D critical wave functions

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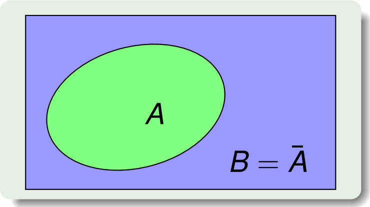
Les Houches 2010 - Physics in the plane: From condensed matter to string theory

Outline

1. Entanglement entropy
   - Definition and basic properties
   - Why do we study it?
   - Scaling properties : classic results
2. 2D Critical Rokhsar-Kivelson wave functions
   - Choice of geometry and wave function
   - Summary of the mappings
   - Scaling behaviour
3. General results
   - c=1 critical case
   - c=1/2 critical case
   - Replica trick
4. Summary

What is entanglement entropy?



Entanglement for a pure state $|\psi\rangle$

- Reduced density matrix :

$$\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$$

- Entanglement entropy :

$$S_A \overset{def}{=} -\mathrm{Tr}\, \rho_A \log \rho_A$$

Basic properties

- $S_A = S_B$
- $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$: $S_A = 0$
- $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$: $S_A = \log 2$

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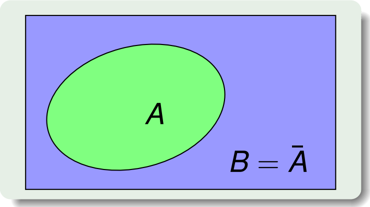

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Why do we study it?

It is interesting

- Quantifying entanglement in complex quantum many-body wave-functions.
- How to store efficiently quantum states in a computer?
- It is useful to distinguich between subtly different phases of matter.
- Scaling properties.

But

- Not clear how to measure experimentally.
- Difficult to compute in general :

$$\hat{\mathcal{H}} \longrightarrow |\psi\rangle \longrightarrow \rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi| \longrightarrow S_A = -\mathrm{Tr}\, \rho_A \log \rho_A$$

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Scaling properties : classic results

- Boundary law : $S_A = \; L^{d-1} + \text{lower order terms}$.
- $1D$ quantum critical spin chains : long segment of length L $S_A \simeq \frac{c}{3} \log L$ (see Wilczek & al, Calabrese & Cardy).
- Topological order in gapped systems : $S_A = \alpha L + \gamma_{\text{topo}}$ (see Levin & Wen, Kitaev & Preskill).
- Fractionnal quantum Hall effect (Haque, Zozulya & Shoutens)

- Entanglement entropy in 2D : challenging problem.

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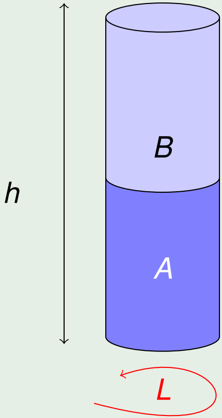
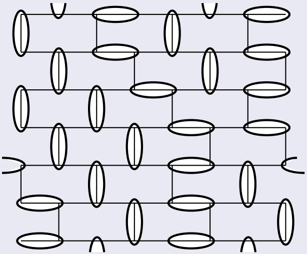
Choice of geometry and Wave function

Geometry : long cylinder $h \to \infty$



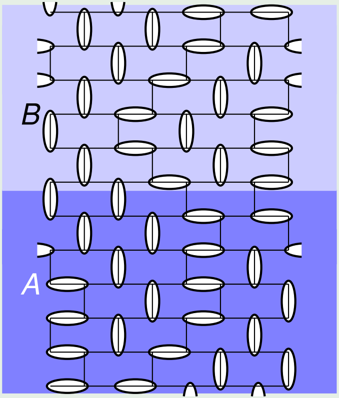
Quantum dimers on the honeycomb

- RK wave function : $|\psi\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_c |c\rangle$
- Configurations : fully-packed dimers on the honeycomb lattice.



Entanglement entropy as a Shannon entropy

Geometry : long cylinder



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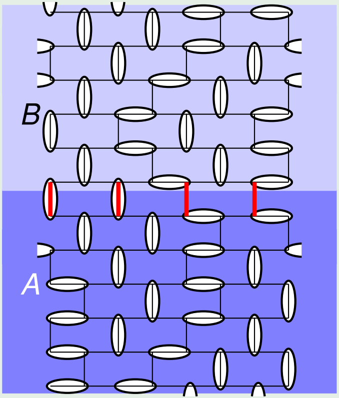
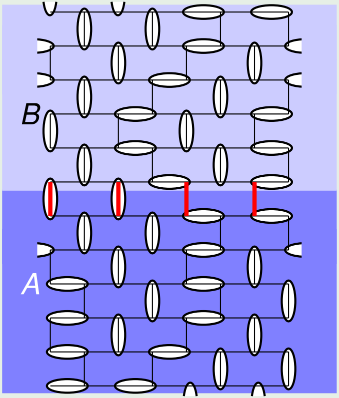


- $|i\rangle$: configuration of the boundary between A and B
- $|i\rangle \in A$ but B "knows" it too, because of the hardcore constraint.

Entanglement entropy as a Shannon entropy

Geometry : long cylinder



Schmidt decomposition

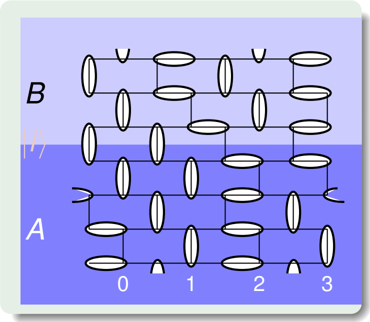
$$|\psi\rangle = \sum_i \sum_{c_i^A} |c_i^A\rangle \sum_{c_i^B | c_i^B \cup c_i^A = c_i} |c_i^B\rangle$$

$$= \sum_i \sqrt{p_i} |\psi_i^A\rangle |\psi_i^B\rangle$$

- $p_i = \frac{\mathcal{Z}_i^A \mathcal{Z}_i^B}{\mathcal{Z}} \longrightarrow$ Combinatorial problem
- $S_A = -\sum_i p_i \log p_i$

=> Shannon/configuration entropy

Entanglement entropy as a Shannon entropy



$$|i\rangle = c_2^\dagger c_3^\dagger |0\rangle$$

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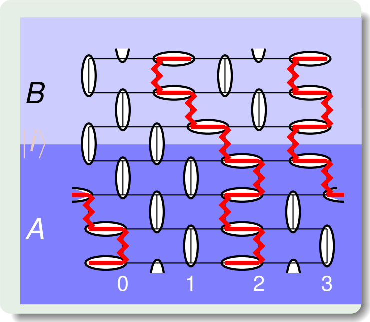

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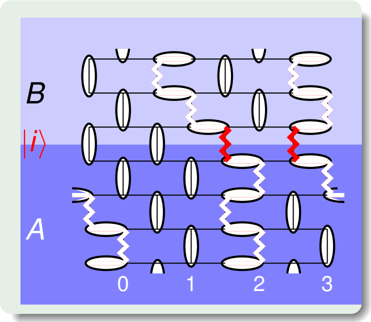

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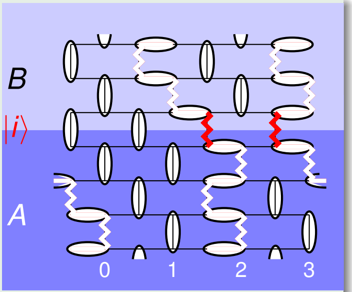

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Transfer matrix as free fermions

- Fermionic representation $|i\rangle = c_{x_1}^\dagger \dots c_{x_n}^\dagger |0\rangle$
- Each fermion can go to the left or to the right with equal amplitude.
- Conserved number of fermions.
- Long cylinder : only the dominant eigenvector $|g\rangle$ matters.
- Exact formula for $p(i) = |\langle i|g\rangle|^2$:

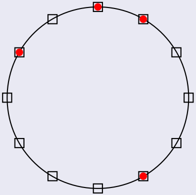
$$p(i) = \frac{1}{L^n} \prod_{j<\ell} \left| e^{2i\pi x_j/L} - e^{2i\pi x_\ell/L} \right|^2$$

Summary of the mappings

2D Entanglement entropy $S_{VN} = -\mathrm{Tr}\,\rho \log \rho$

- $|\psi\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_c |c\rangle$
- L : circumference of the cylinder
- n : number of fermions (fixed by the boundary)

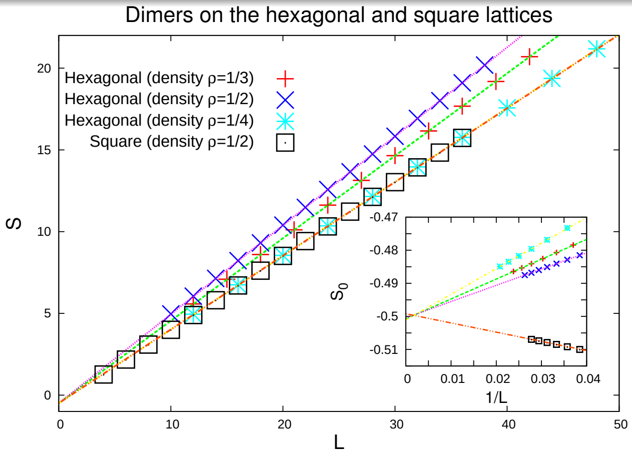
1D configuration entropy : Discretized Dyson gas



$n = 4$, $L = 12$

- $E(i) = -\sum_{j<\ell} \log \left| e^{2i\pi x_j/L} - e^{2i\pi x_\ell/L} \right|$
- $Z(\beta) = \sum_i e^{-\beta E(i)}$
- $S(\beta) = (1 - \beta\partial_\beta) \log Z(\beta)$
- $S_A = S(\beta = 2)$

Scaling behaviour for the entropy



- Very good agreement with $S_A = \mu L - 1/2$

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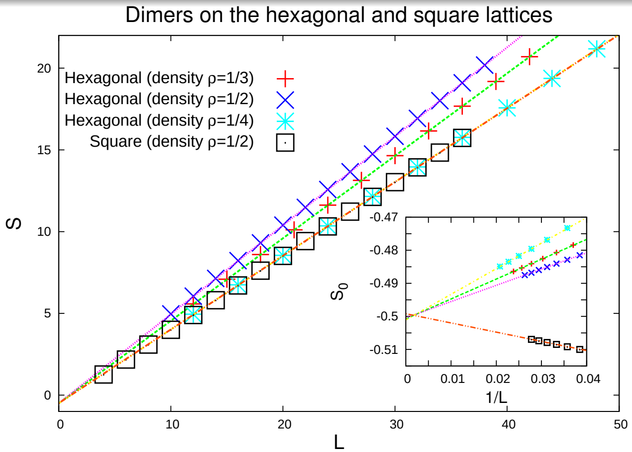

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$c = 1$: Quantum Dimers, Quantum six-vertex,...

Quantum six-vertex model

- $p(i) = |\langle i|g\rangle|^2$, $|g\rangle$ ground-state of the XXZ chain

$$\mathcal{H} = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) - h \sum_j \sigma_j^z$$

- Compactification radius $R = f(\Delta, h)$
- $S_0 = \log R - 1/2$

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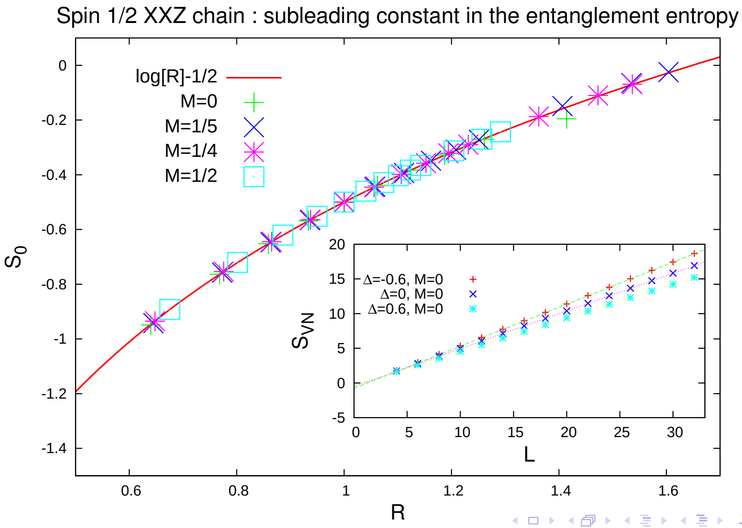

$c = 1$: Quantum Dimers, Quantum six-vertex,...

Discretized Calogero-Sutherland ground-state wave function

- Renyi entropies for dimers
  $(1-\alpha)^{-1} \log \mathrm{Tr}\, \rho^{\alpha} = (1-\alpha)^{-1} \log \sum_i (p_i)^{\alpha}$
  =>Dyson-Gaudin gas at inverse temperature $\beta = 2\alpha$.
- $R = \sqrt{\alpha}$:
  The Renyi parameters allows to change the compactification radius.
- $S_0 = \log R - 1/2$

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massive states

- $S_0 = \log d$
  d: deganeracy of the 1D classical model.

2D Ising model

2D Ising model

Dominant eigenvector $|g\rangle$ of the transfer matrix for :

- 2D Ising model on the square/triangular lattice
- Ising chain in transverse field at the critical point :

$$\mathcal{H} = -\underbrace{\mu}_{=1} \sum_{j=0}^{L-1} \sigma_j^x \sigma_{j+1}^x - \sum_{j=0}^{L-1} \sigma_j^z$$

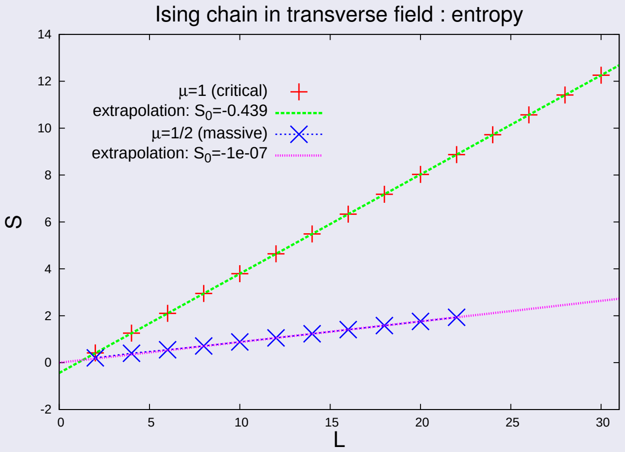
$$|i\rangle = |\sigma_0^z, \ldots, \sigma_{L-1}^z\rangle \qquad \sigma_j^z = \pm 1, j = 0 \ldots L-1$$

$$\boxed{p_i = |\langle i|g\rangle|^2 = \frac{1}{2^L} \left| \det_{j,\ell} \left(\delta_{j\ell} + \frac{(-1)^{j-\ell} \sigma_j^z}{L \sin\left[(j-\ell+1/2)\pi/L\right]} \right) \right|}$$

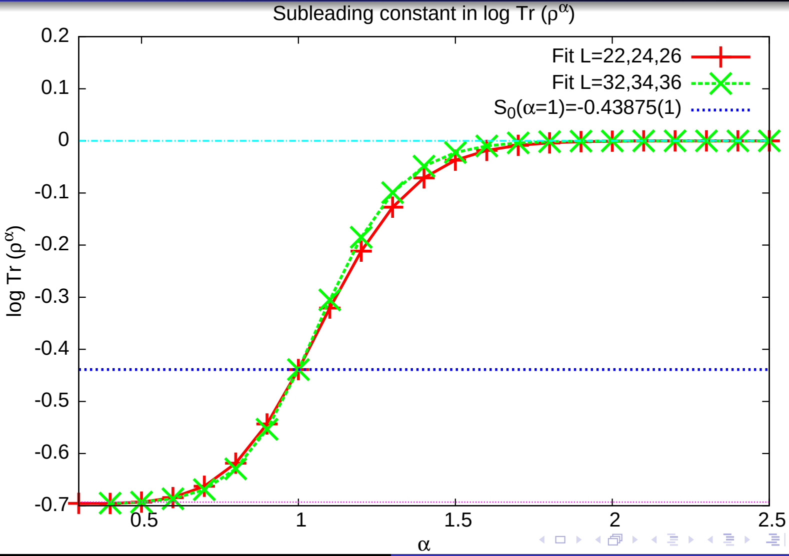
- $S_0 = -0.43875(1)$

2D Ising model

Ising chain in transverse field : $\mathcal{H} = -\mu \sum_j \sigma_j^x \sigma_{j+1}^x - \sum_j \sigma_j^z$



Replica trick in the $c = 1/2$ case : $(1-\alpha)^{-1} \log \mathrm{Tr}\, \rho^{\alpha}$



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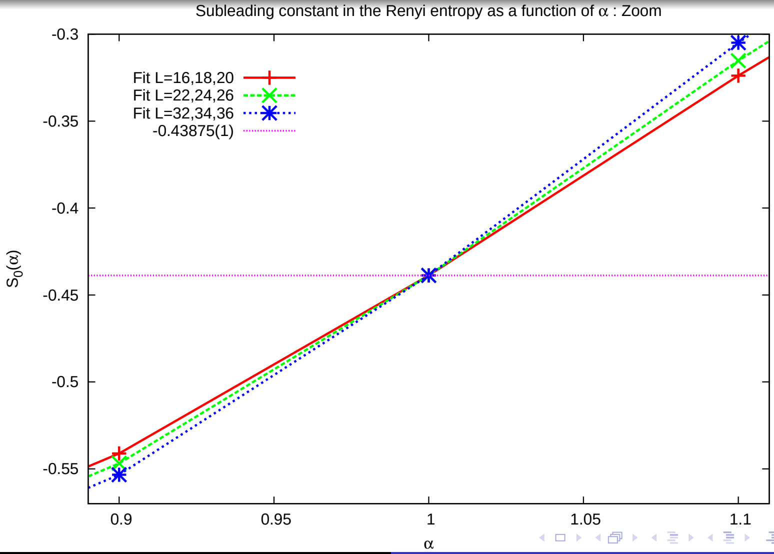

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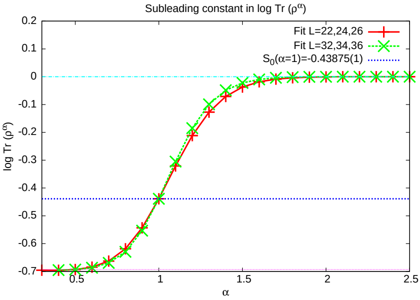

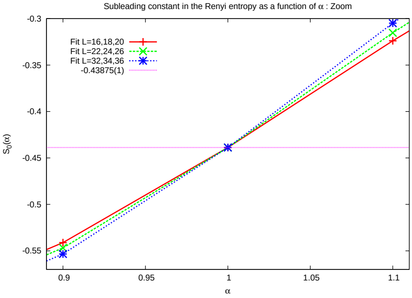

- $$S_0(\alpha) = \begin{cases} -\log 2 & , \quad \alpha < 1 \\ -0.43875(1) & , \quad \alpha = 1 \\ 0 & , \quad \alpha > 1 \end{cases}$$
- Non analytic behaviour : Replica trick should fail.

Results

- Entanglement entropy of 2D critical wave-functions.
- Cylinder geometry + RK wave functions :
  Exact numerical results up to $L \simeq 40$.
- Free bosonic case $S_0 = \log R - 1/2$.
- New tool for determinig R, detecting phase transitions, ...
- See more on Phys. Rev. B 80, 184421 (2009)

What next?

- $S_0 = -0.43875(1)$ in CFT?
- Other geometries : torus, band, ...
- Validity of the replica trick $S_A \stackrel{?}{=} \lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} \mathrm{Tr} \rho^\alpha$ in 2D?

Thank you for your attention.

Simple free field derivation of S_0

In the continuum limit (ρ is the density of particles) :

$$E[\rho] = -\frac{1}{2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \rho(\theta)\rho(\theta') \log|e^{i\theta} - e^{i\theta'}|$$

$$\phi(\theta) = 2\pi \int_0^{\theta} \rho(\sigma) d\sigma = 2\pi \sum_{m\geq 1} x_m e^{im\theta} + \bar{x}_m e^{-im\theta}$$

$$E = \frac{(2\pi)^2}{2} \sum_{m\geq 1} m|x_m|^2$$

$$Z[\beta] = \int \mathcal{D}\rho e^{-\beta E[\rho]} = \prod_{m\geq 1} \frac{1}{\pi\beta m} \overbrace{=}^{\zeta \text{ reg.}} \sqrt{\beta/2}$$

$$S(\beta) = (1 - \beta\partial_\beta) \log Z(\beta) = \log\sqrt{\beta/2} - 1/2 = \log R - 12$$