Les Houches 4th Mar 2010

Workshop

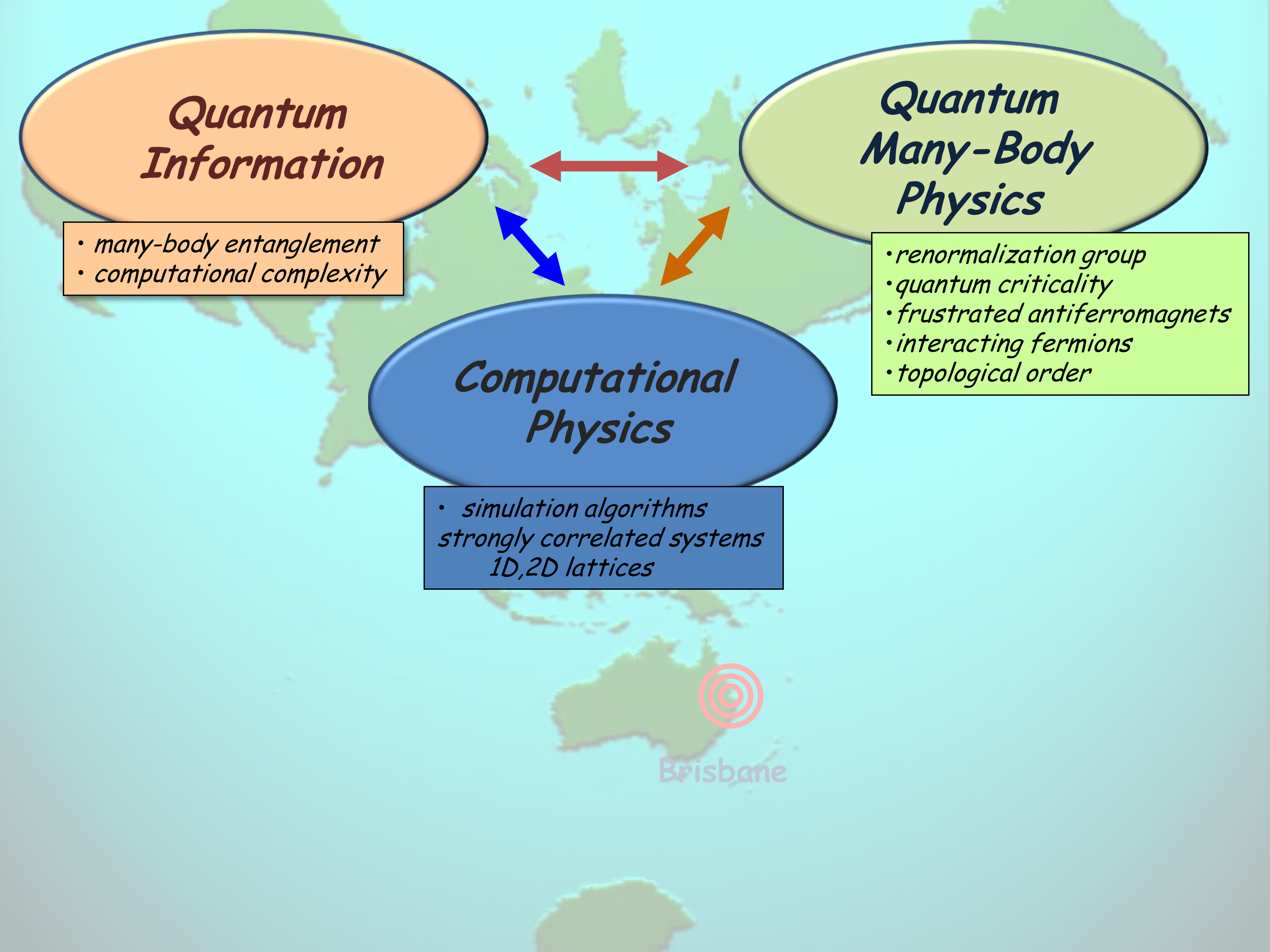
Physics in the plane

Entanglement Renormalization and Conformal Field Theory

Guifre Vidal

The University of Queensland
Australia


Brisbane


Quantum Information
• many-body entanglement
• computational complexity
Quantum Many-Body Physics
• renormalization group
• quantum criticality
• frustrated antiferromagnets
• interacting fermions
• topological order
Computational Physics
• simulation algorithms
strongly correlated systems
1D,2D lattices
Brisbane

Research group at the University of Queensland

faculty	postdocs	students
Guifre Vidal	Philippe Corboz	Andrew Birrell
Ian McCulloch	Karen Dancer	Glen Evenbly
	Andy Ferris	Xiaoli Huang
	Roman Orus	Jacob Jordan
	Luca Tagliacozzo	Shahla Nikbakht
		Robert Pfeifer
		Sukhi Singh

announcement: postdoctoral position available

Outline

MERA = **M**ulti-scale **E**ntanglement **R**enormalization **A**nsatz

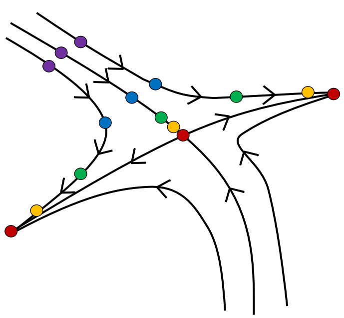
Entanglement Renormalization/MERA

- Renormalization Group (RG) transformation
- Quantum Computation

Scale invariant MERA ⟺ RG fixed point

- critical fixed point ⟺ continuous quantum phase transition ⟺ CFT
  - bulk (local/non-local)
  - boundary
  - defect

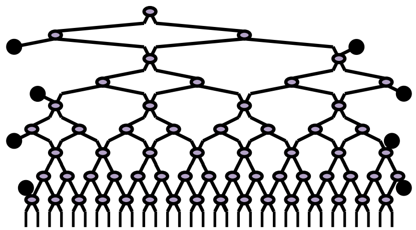
Scale invariant MERA ⟺ Holographic principle



collaboration with

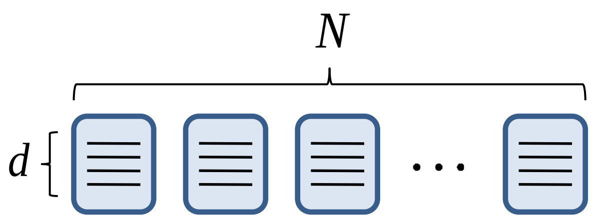
Glen Evenbly,
R. Pfeifer, P. Corboz,
L. Tagliacozzo, I.P. McCulloch

V. Pico (U. Barcelona)
S. Iblisdir (U. Barcelona)



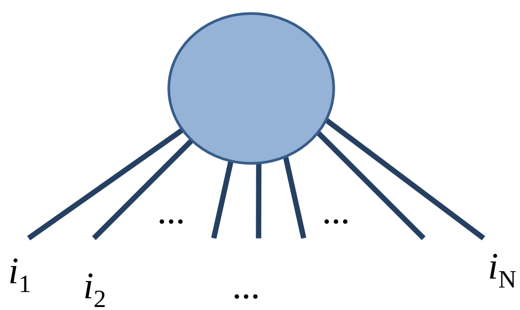
tensor network ansatz/state

- Lattice with N sites

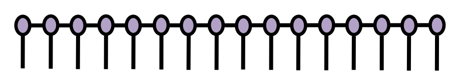


$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

$$\left|\Psi_{GS}\right\rangle = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_N=1}^{d} c_{i_1 i_2 \ldots i_N} \left|i_1\right\rangle \left|i_2\right\rangle \ldots \left|i_N\right\rangle$$

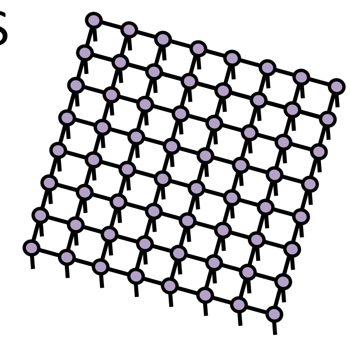


MPS (1D)



$O(N)$ coefficients

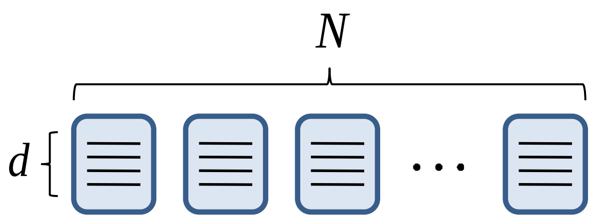
PEPS (2D)



MERA (multi-scale entanglement renormalization ansatz)

Vidal, Phys. Rev. Lett. 99, 220405 (2007); ibid 101, 110501 (2008)

- Lattice with N sites



$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$$

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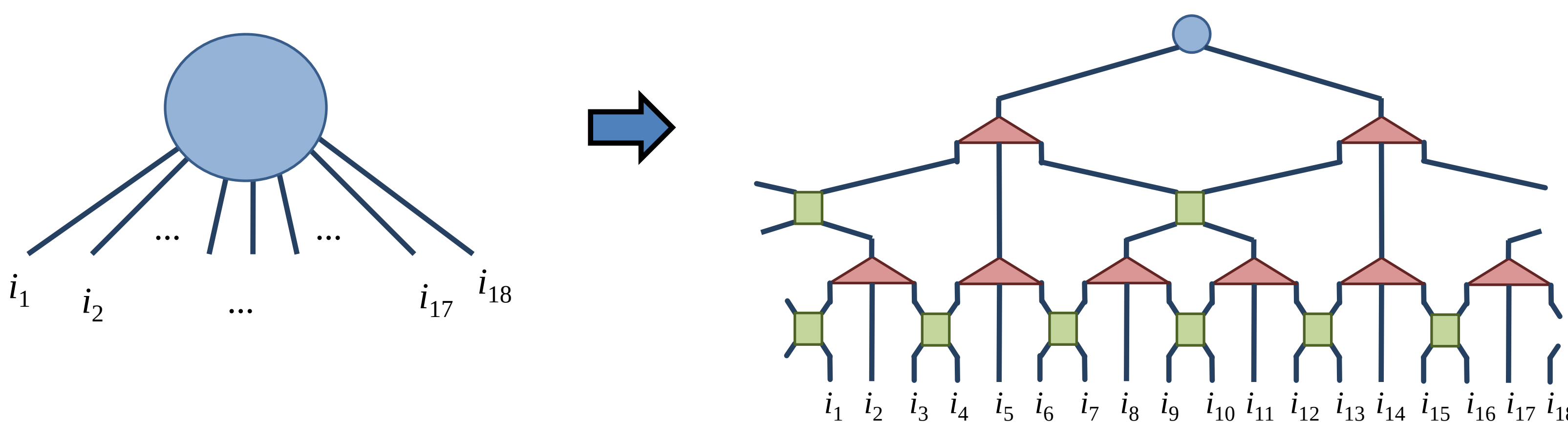
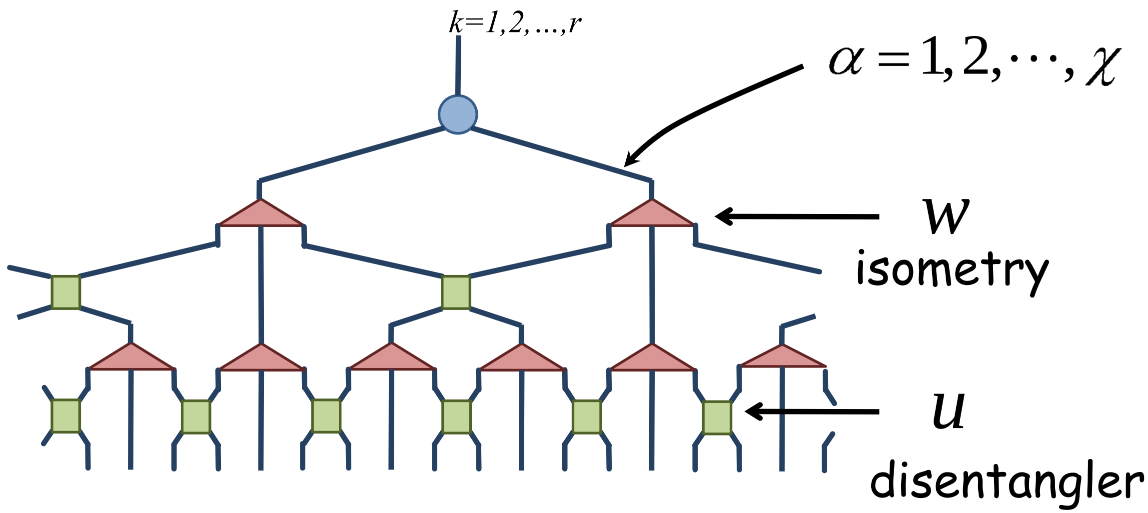
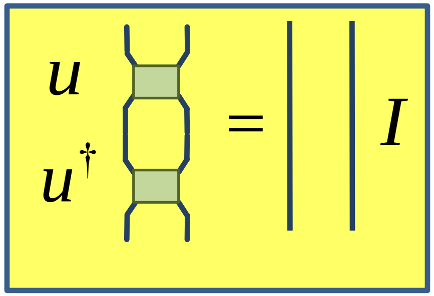
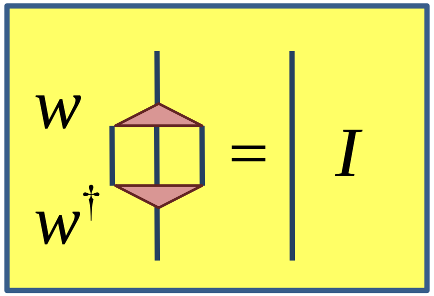


MERA (multi-scale entanglement renormalization ansatz)



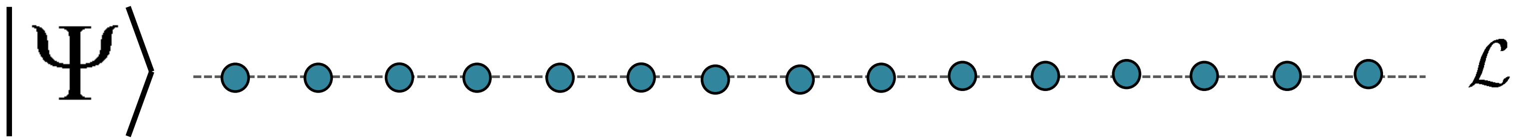


disentangler

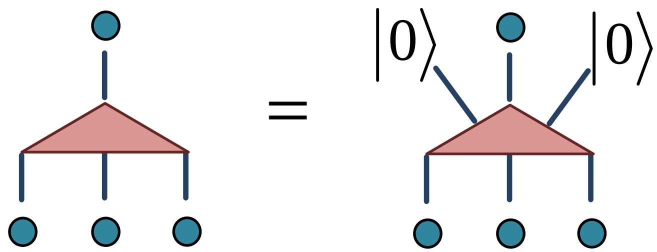


isometry

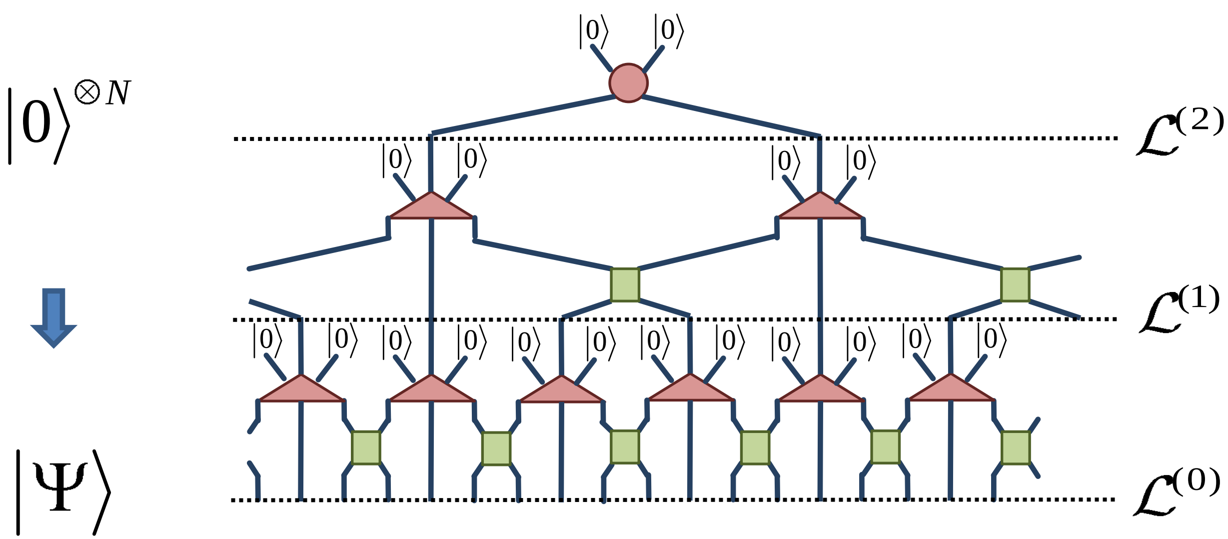
MERA as a quantum circuit



isometry

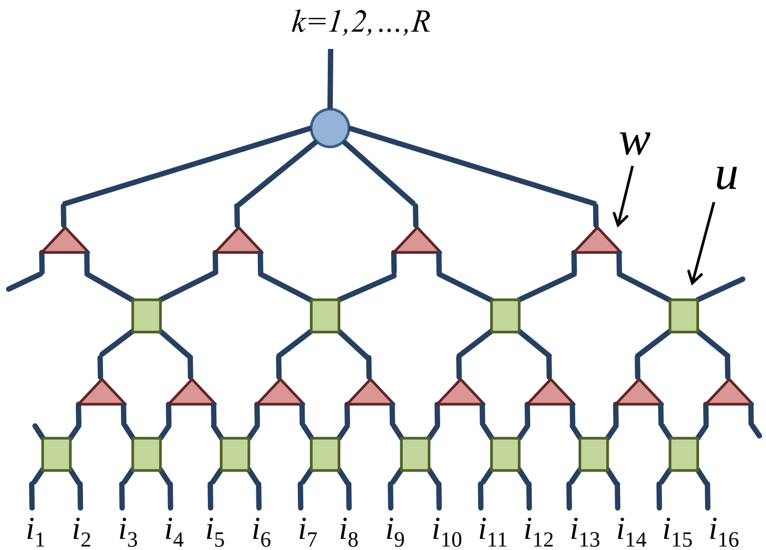


unitary

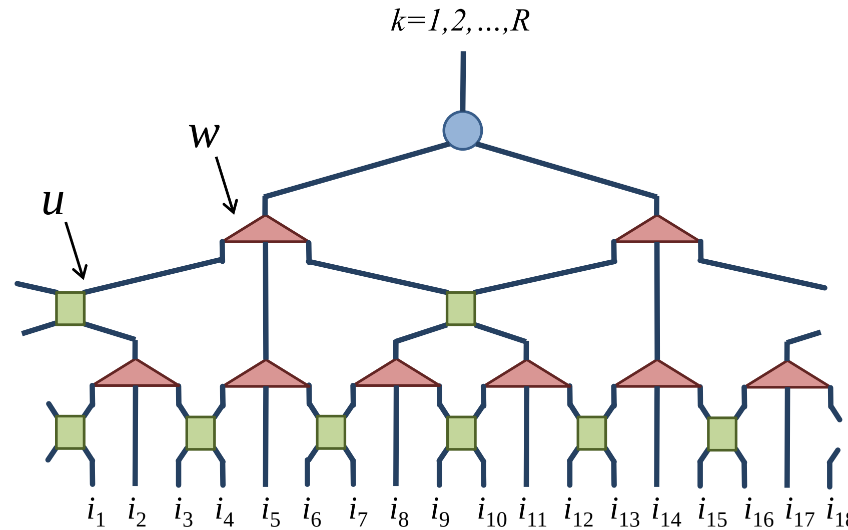


- Many possibilities

examples: binary 1D MERA

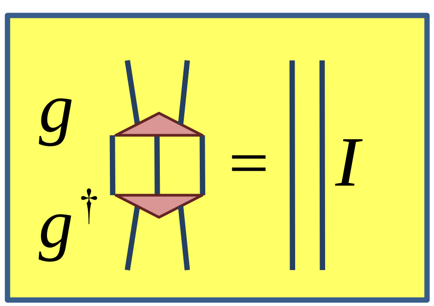


ternary 1D MERA

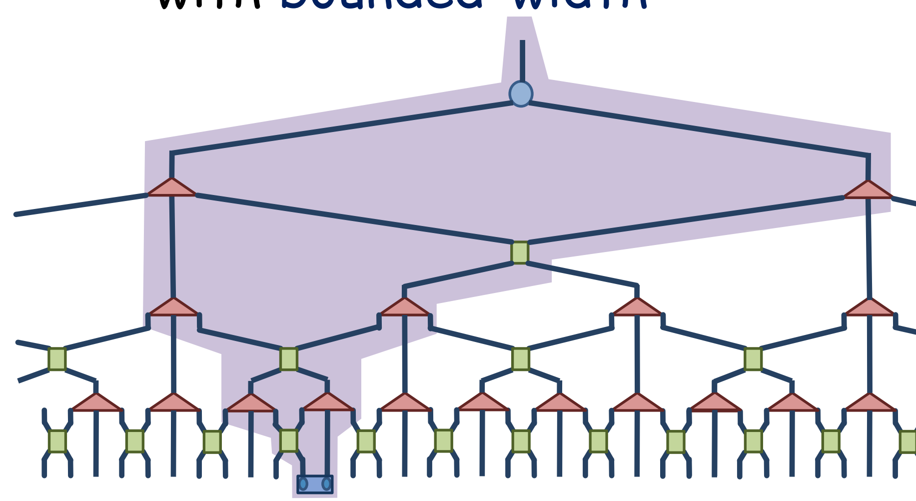


- Defining properties:

1) isometric tensors



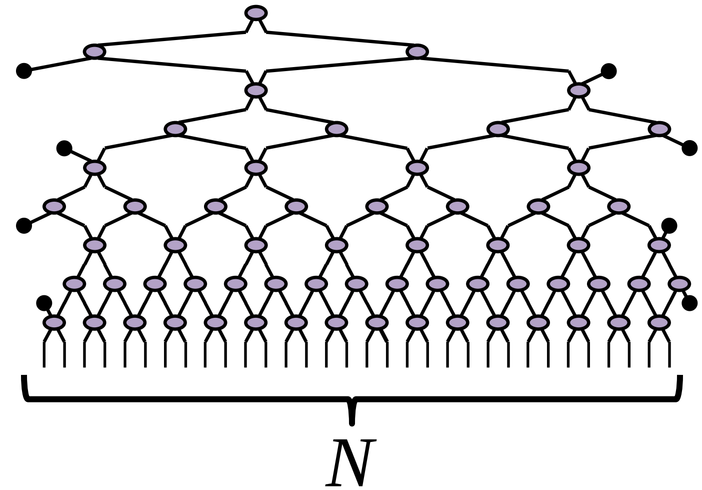
2) past causal cone with bounded 'width'



What is the MERA useful for?

- ground states/low energy subspaces in 1D, 2D lattices

- at criticality: critical exponents/CFT



Simulation costs:

$O(N)$		$O(\log(N))$		$O(1)$
$\chi^p N$ inhomogeneous	→	$\chi^p \log N$ translation invariance	→	χ^p translation & scale invariance

χ^p

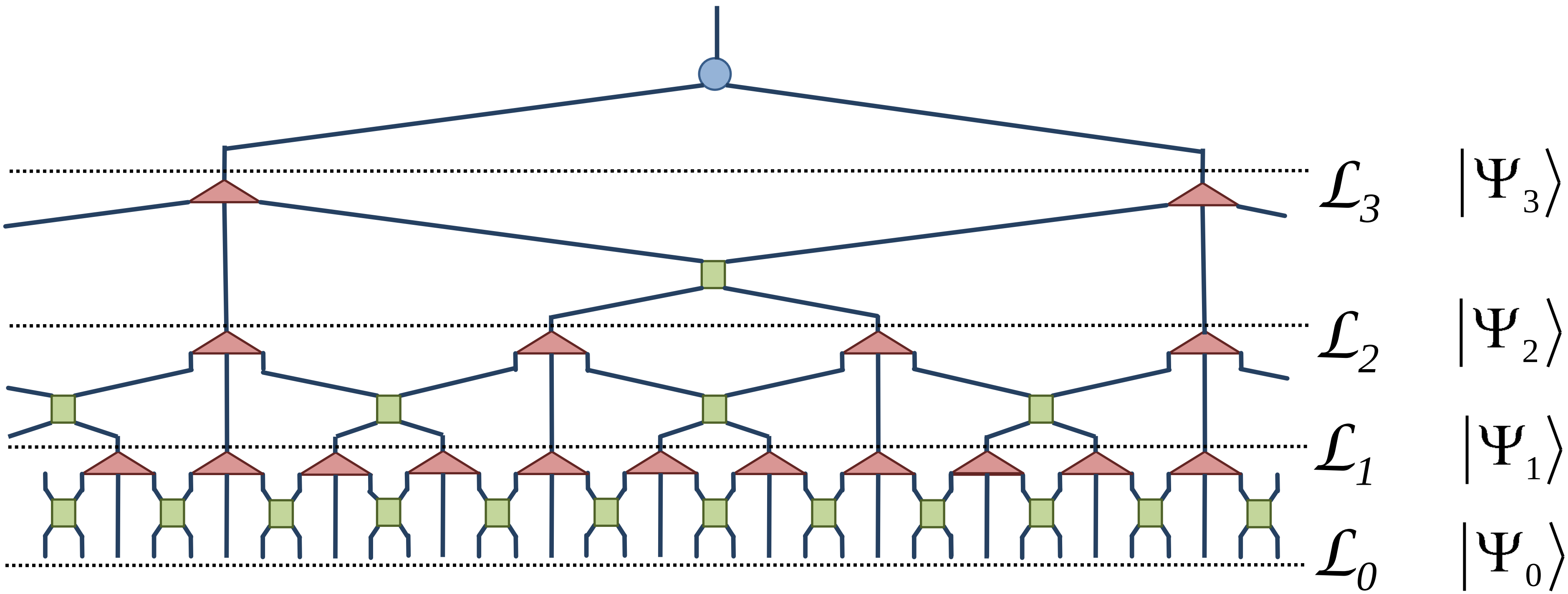
+ (scale invariant) **boundary/defect/ interface**

(no translation invariance !!!)

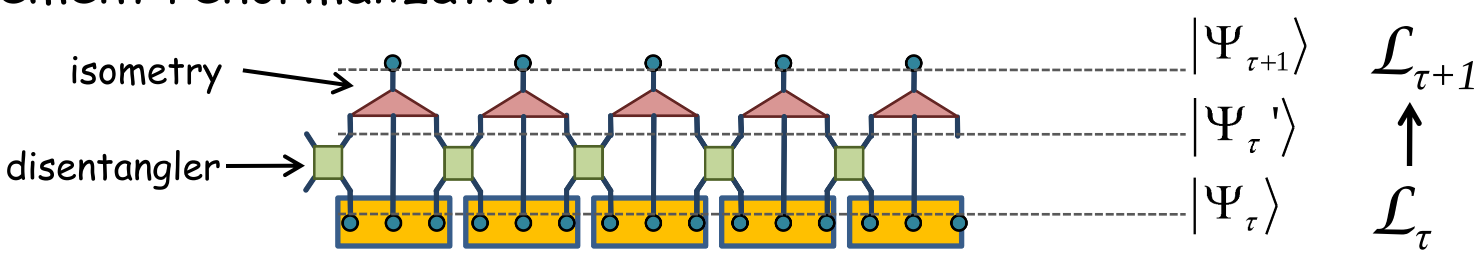
RG transformation

Vidal, Phys. Rev. Lett. 99, 220405 (2007); ibid 101, 110501 (2008)

The MERA defines a coarse-graining transformation

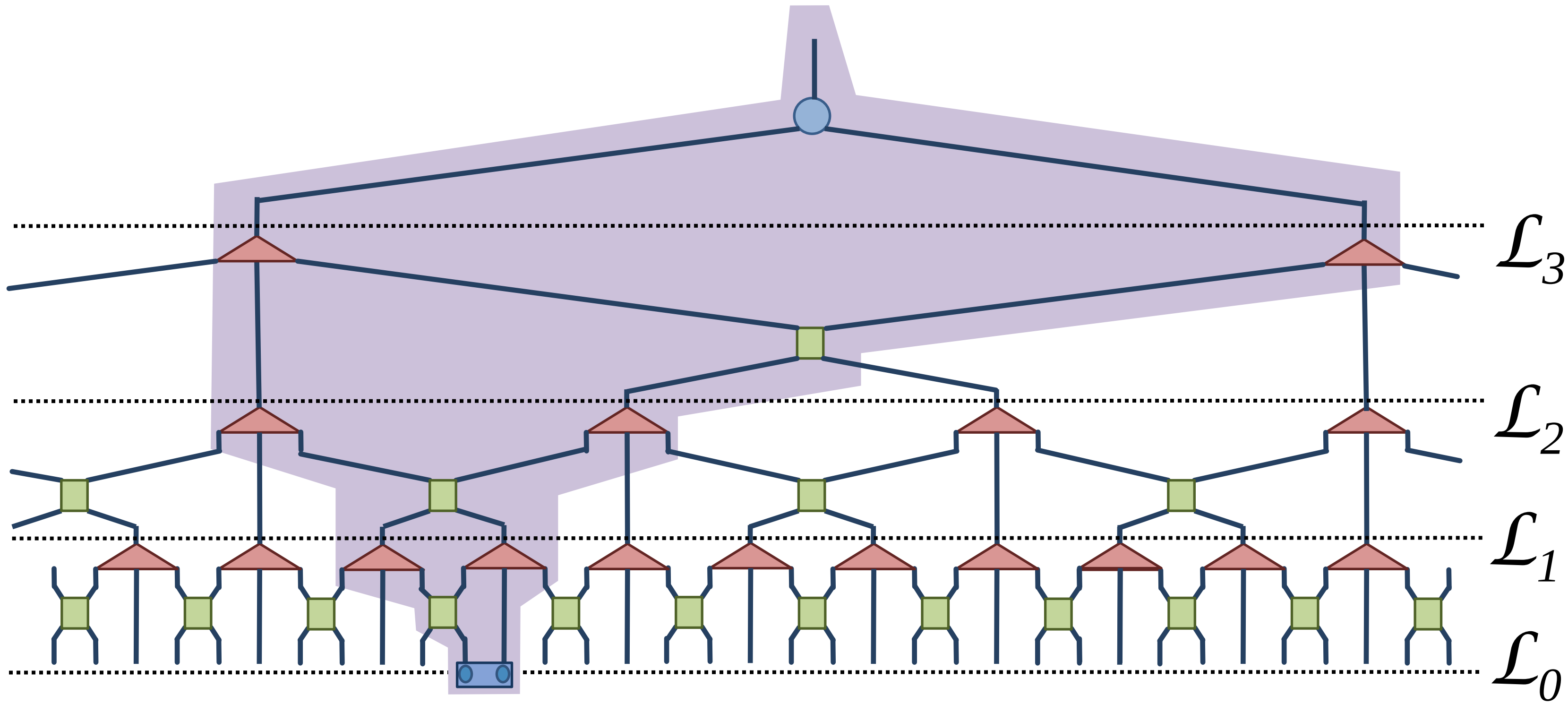


Entanglement renormalization

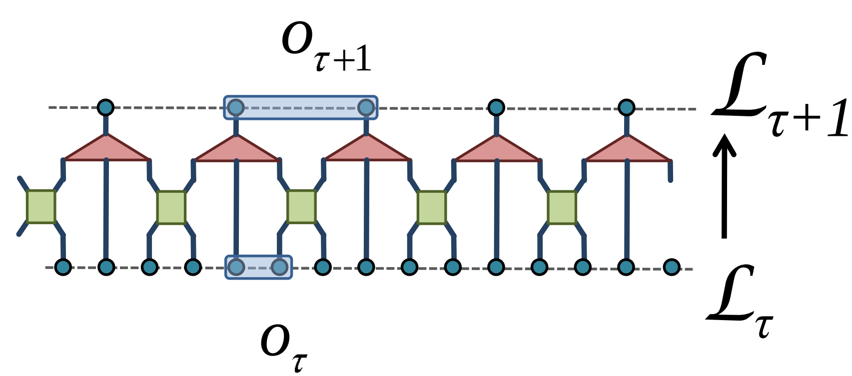


RG transformation

The MERA defines a coarse-graining transformation



- transformation of local operators

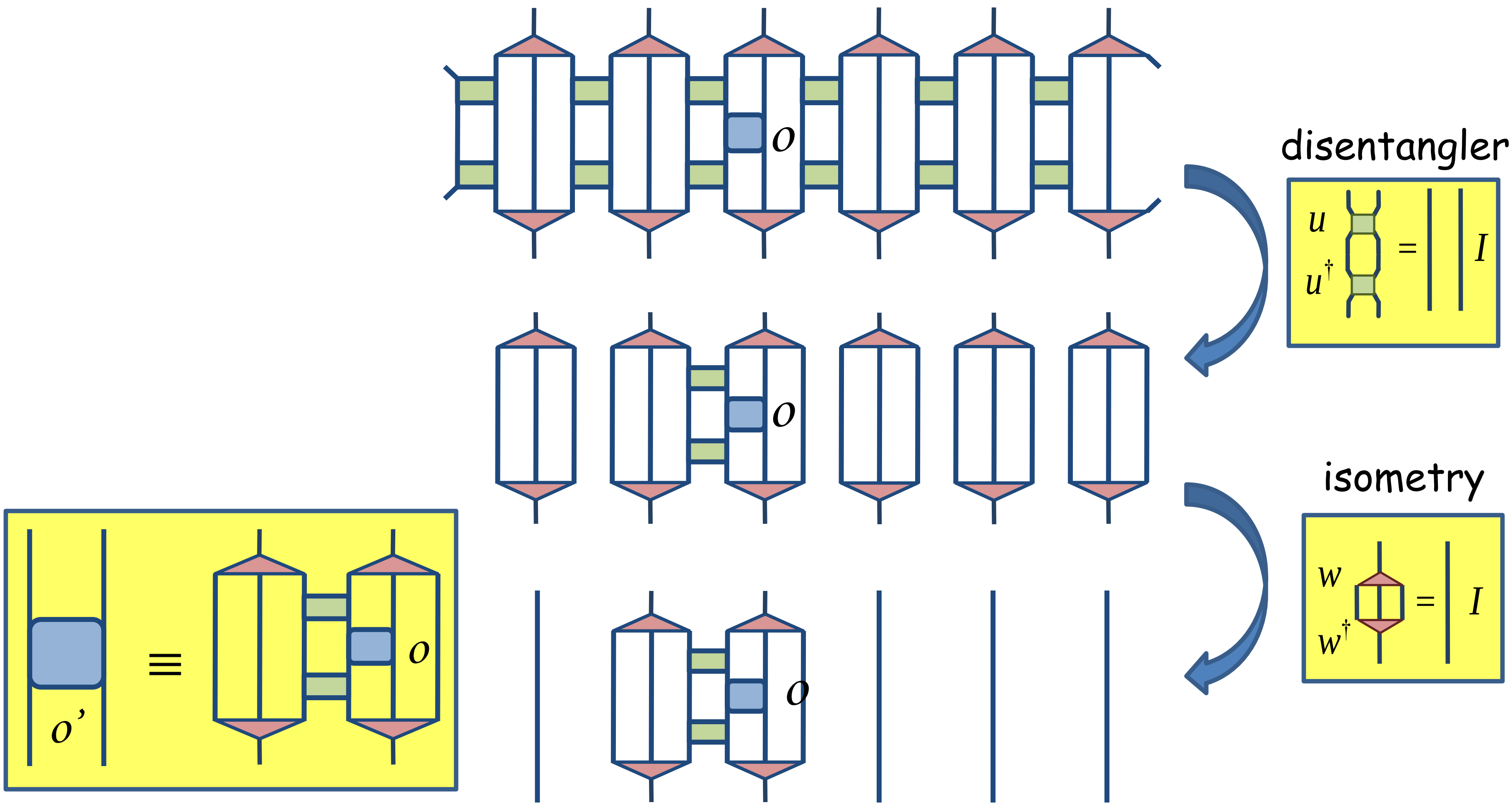


RG transformation

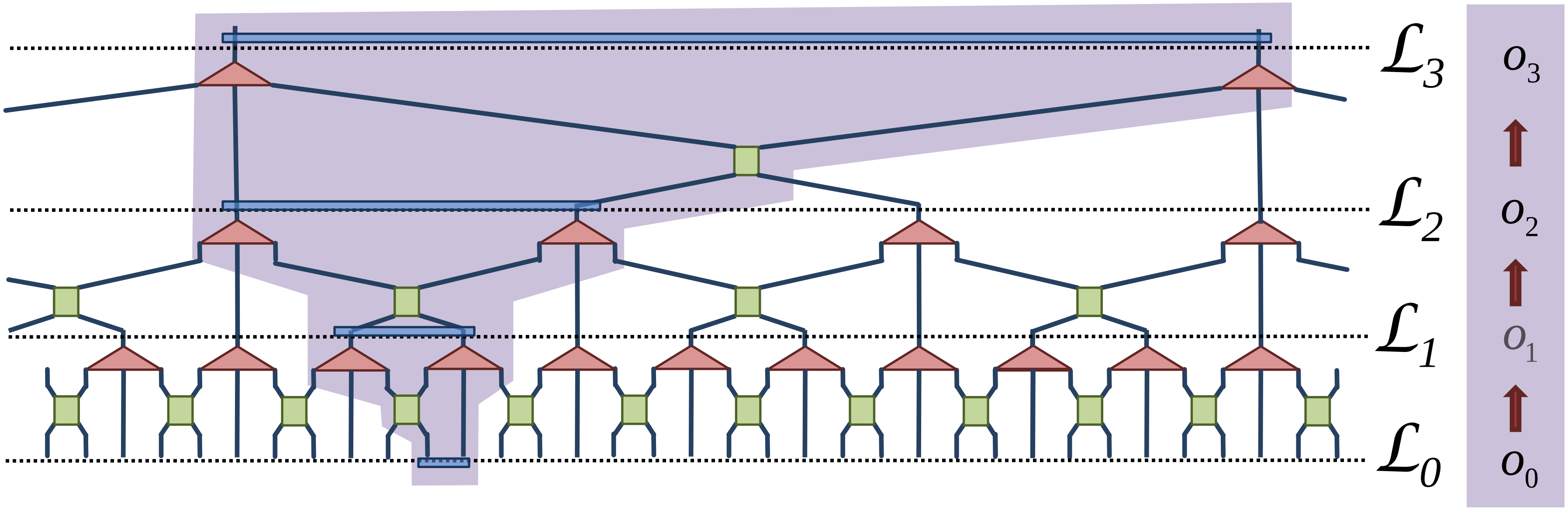
The MERA defines a coarse-graining transformation

$$\mathcal{L} \longrightarrow \mathcal{L}'$$

$$o \rightarrow o'$$

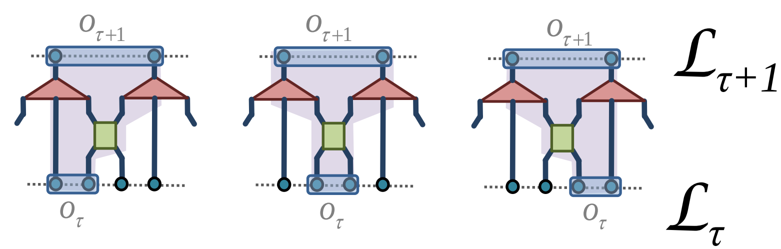


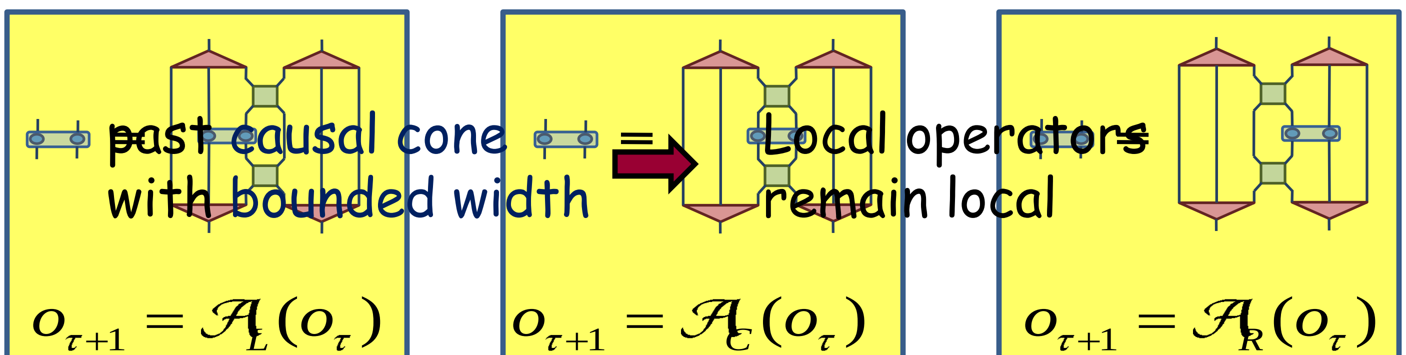
Ascending superoperator



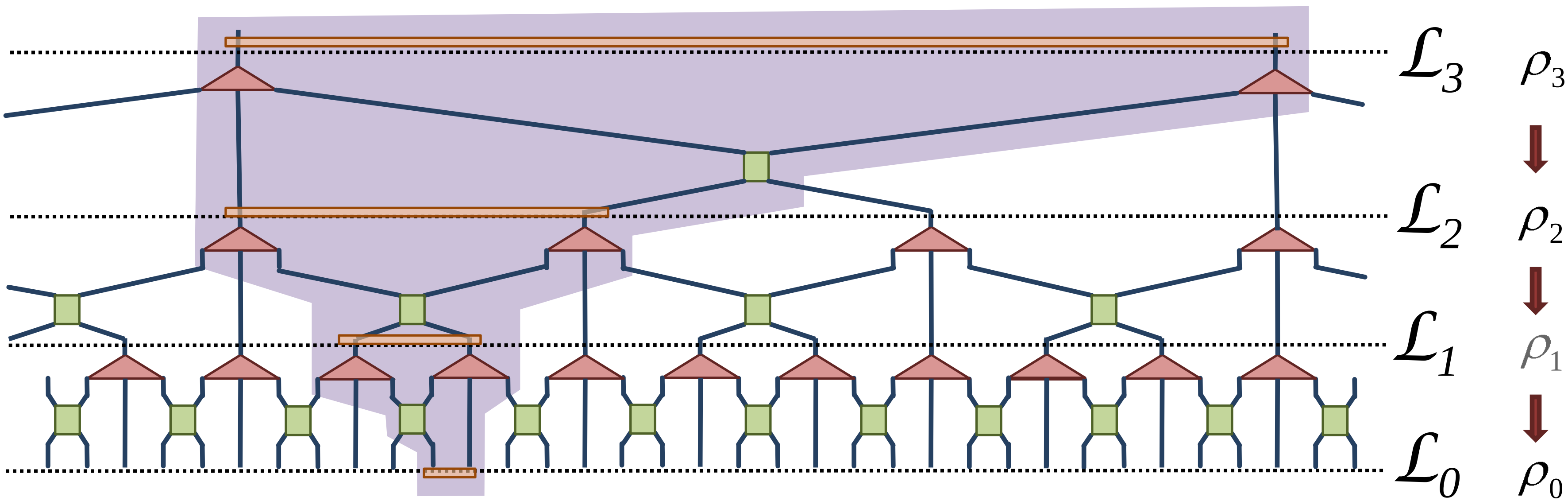
- ascending superoperator $o_{\tau+1} = \mathcal{A}(o_\tau)$

o_0 ➡ o_1 ➡ o_2 ➡ …



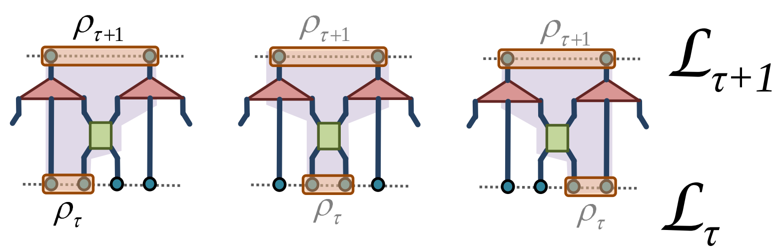


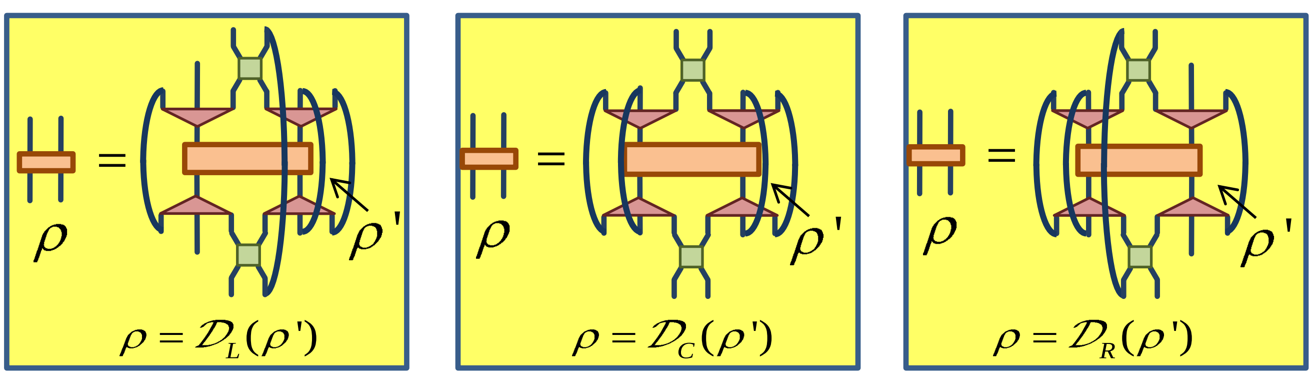
past causal cone with bounded width ➡ Local operators remain local



- descending superoperator $\rho_\tau = \mathcal{D}(\rho_{\tau+1})$

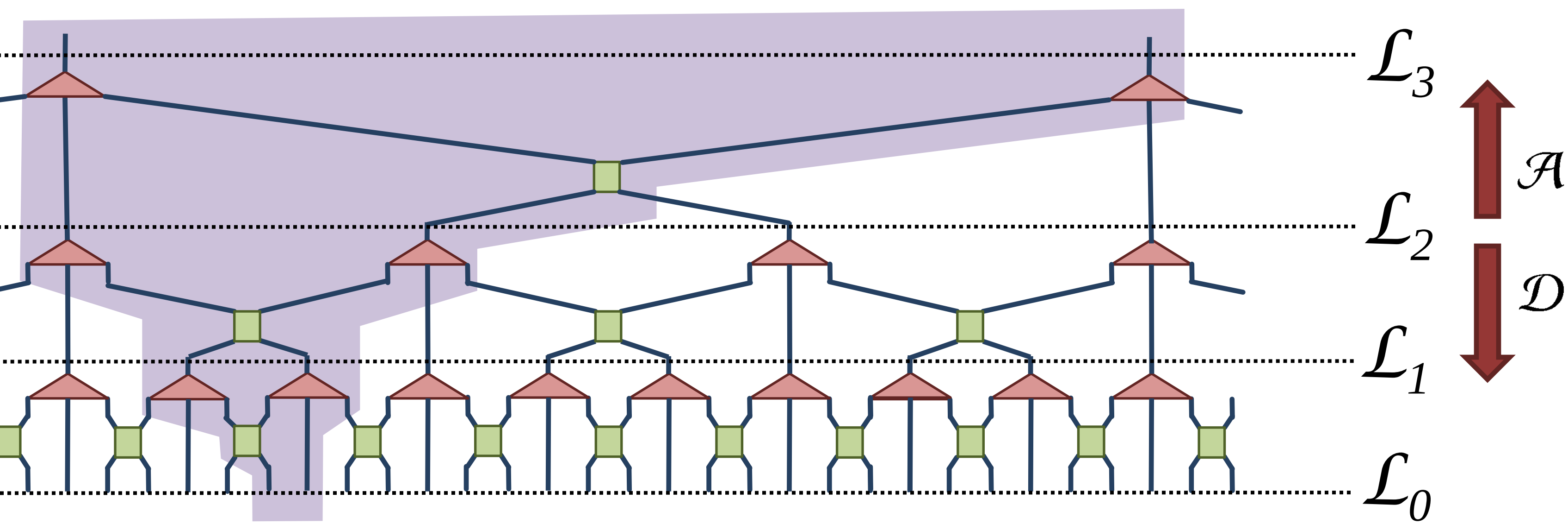
$\rho_0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \dots$



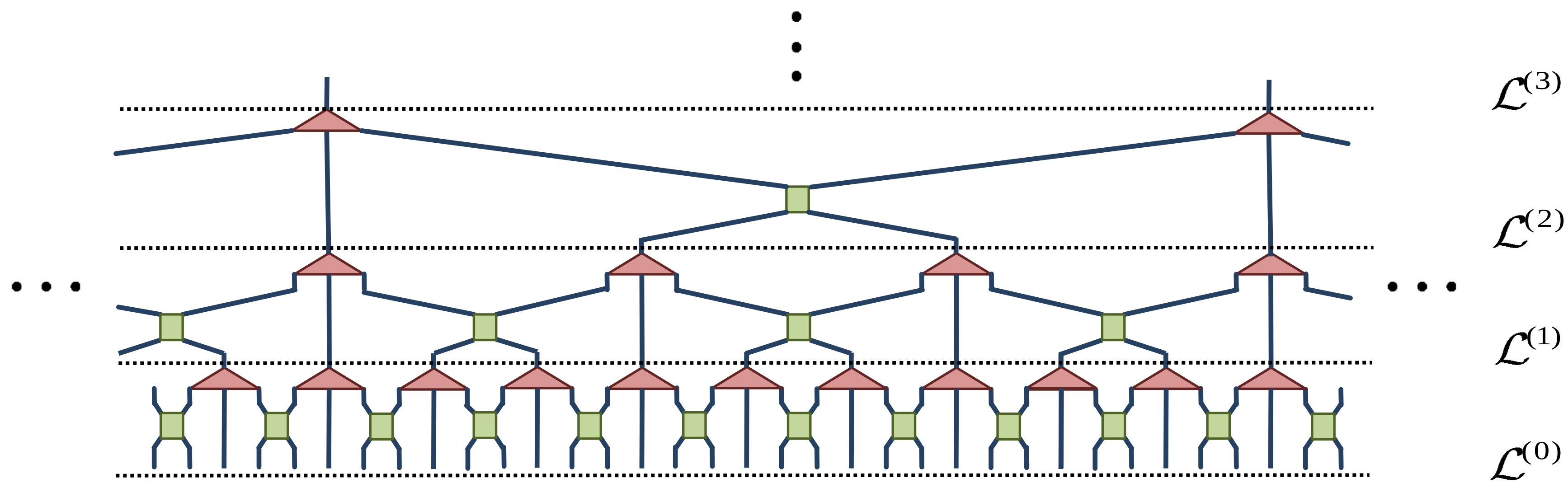


- Description of the system at different length scales

- Ascending and descending super-operators:
  change of length scale (or time in a quantum computation)



Scale invariant MERA



critical systems (1D)

Vidal, Phys. Rev. Lett. 99, 220405 (2007)
Vidal, Phys. Rev. Lett. 101, 110501 (2008)
Evenbly, Vidal, arXiv:0710.0692
Evenbly, Vidal, arXiv:0801.2449

critical exponents → **Giovannetti, Montangero, Fazio**, Phys. Rev. Lett. 101, 180503 (2008)

OPE, CFT → **Pfeifer**, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

Montangero, **Rizzi**, Giovannetti, Fazio, Phys. Rev. B 80, 113103 (2009)
Giovannetti, Montangero, Rizzi, Fazio, Phys. Rev. A 79, 052314(2009)

boundary & defects → Evenbly, Pfeifer, **Pico**, **Iblisdir**, **Tagliacozzo**, **McCulloch**, Vidal, arXiv:0912.1642

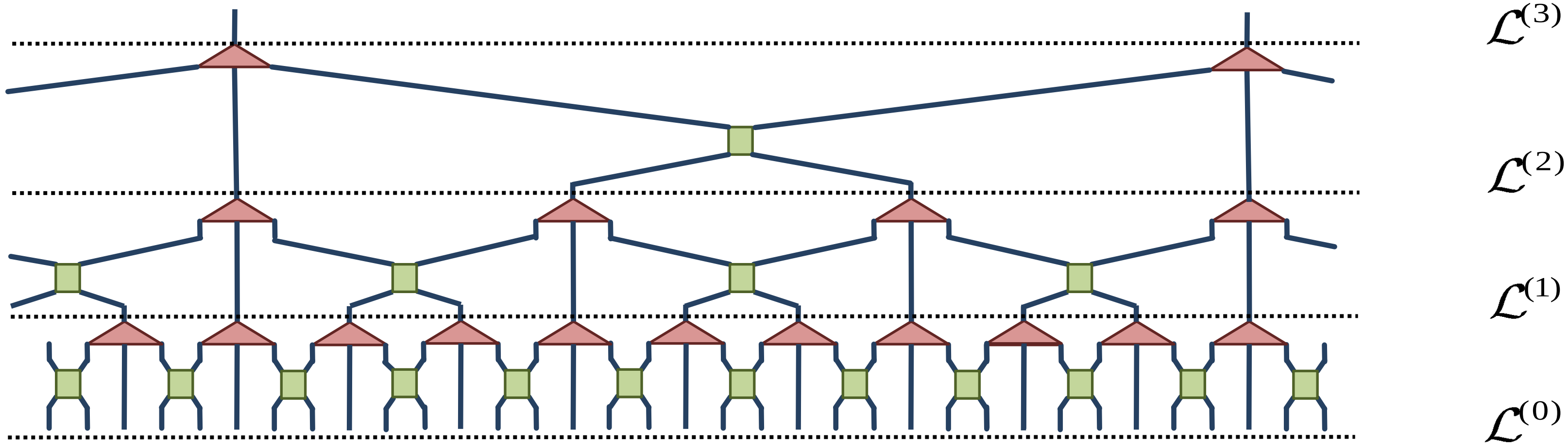
non-local operators → Evenbly, **Corboz**, Vidal, arXiv: 0912.2166

Silvi, Giovannetti, **Calabrese**, **Santoro**, Fazio, arXiv: 0912.2893

topologically ordered systems (2D)

Aguado, Vidal, Phys. Rev. Lett. 100, 070404 (2008)
Koenig, Reichardt, Vidal, Phys. Rev. B 79, 195123 (2009)

Scale invariant MERA



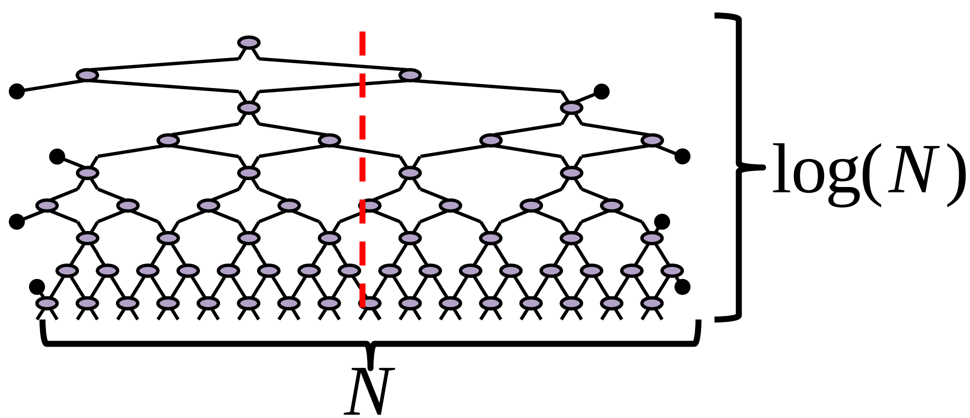
Vidal, Phys. Rev. Lett. 99, 220405 (2007)
Vidal, Phys. Rev. Lett. 101, 110501 (2008)

Evenbly, Vidal, arXiv:0710.0692
Evenbly, Vidal, arXiv:0801.2449

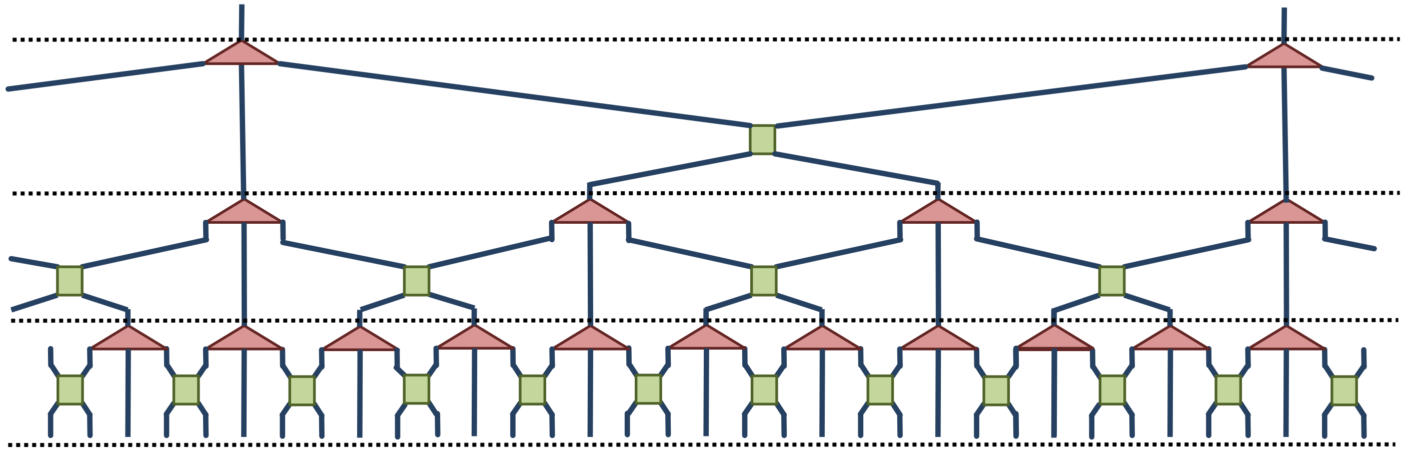
MERA is a good ansatz for ground state at quantum critical point

- it recovers scale invariance! same state and Hamiltonian at each level [shown for Ising model, free fermions, free bosons]
- proper scaling of entanglement entropy

$$S_{N/2} \sim \log(N)$$



Scale invariant MERA



Vidal, Phys. Rev. Lett. 99, 220405 (2007)
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MERA is a good ansatz for ground state at quantum critical point

- it recovers scale invariance! same state and Hamiltonian at each level
  [shown for Ising model, free fermions, free bosons]
- proper scaling of entanglement entropy
- polynomial decay of correlations

constant ascending superoperator $\mathcal{A}$

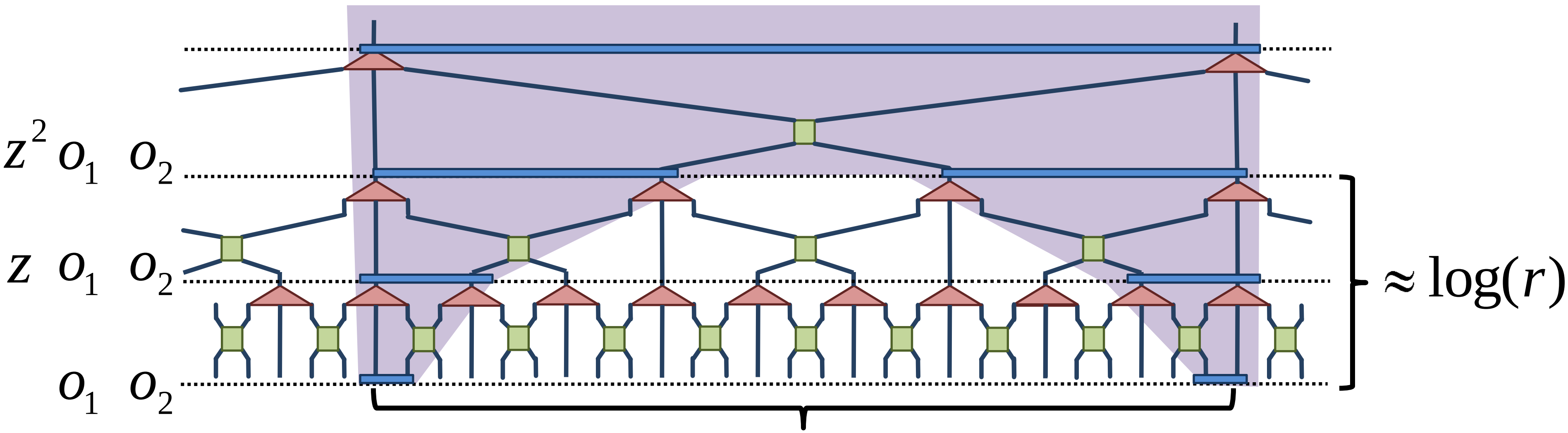
$$o' = \mathcal{A}(o) \equiv \mathcal{S}(o)$$

$o \rightarrow o' \rightarrow o'' \rightarrow \ldots$

scaling superoperator $\mathcal{S}$

Scale invariant MERA

- polynomial decay of correlations Vidal, Phys. Rev. Lett. 101, 110501 (2008)



$$r = |s_1 - s_2|$$

$$C_2(s_1, s_2) \approx z^{\log(r)} = r^{\log(z)} = r^{-q}, \qquad q \equiv -\log(z)$$

$$\mathcal{S}(o_1) = \sqrt{z}\,o_1 \qquad \mathcal{S}(o_2) = \sqrt{z}\,o_2$$

- Eigenvalue decomposition Giovannetti, Montangero, Fazio, Phys. Rev. Lett. 101, 180503 (2008)

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

ϕ_α scaling operator $\qquad \phi_\alpha \rightarrow 3^{-\Delta_\alpha}\phi_\alpha$

Δ_α scaling dimension $\qquad \Delta_\alpha \equiv -\log_3 \lambda_\alpha$

- Eigenvalue decomposition Giovannetti, Montangero, Fazio, Phys. Rev. Lett. 101, 180503 (2008)

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

ϕ_α scaling operator $\phi_\alpha \to 3^{-\Delta_\alpha}\phi_\alpha$

Δ_α scaling dimension $\Delta_\alpha \equiv -\log_3 \lambda_\alpha$

Critical exponents can be extracted from the scaling superoperator
(= quMERA channel, MERA transfer map)

- Connection to Conformal Field Theory Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

spin lattice at quantum critical point → CFT

- central charge c
- primary fields ϕ_α^p

  conformal dimensions $(h_\alpha^p, \bar{h}_\alpha^p)$

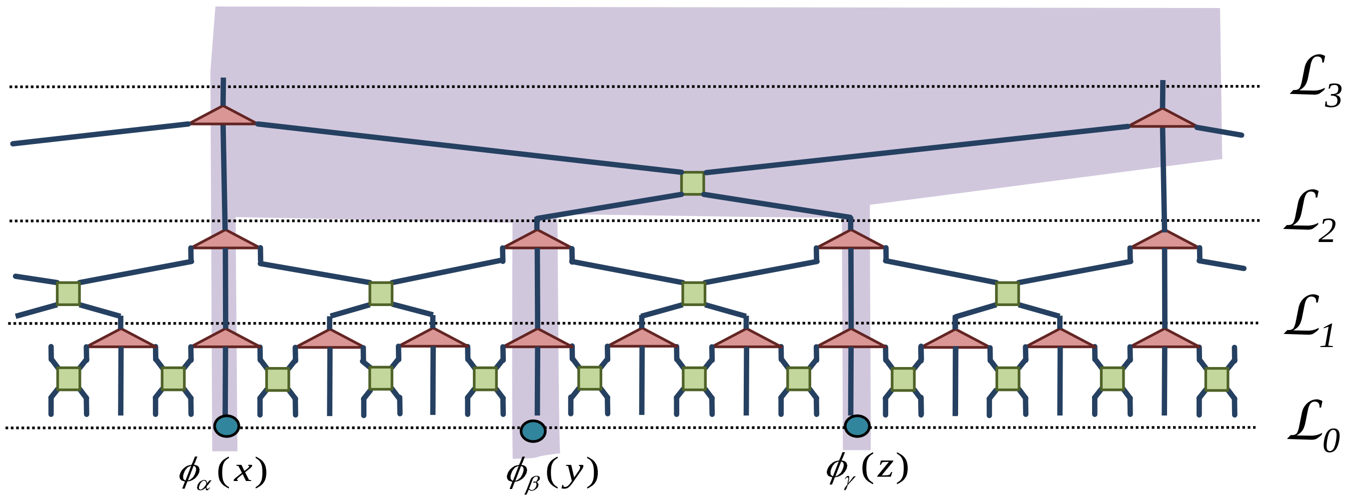
  $\Delta_\alpha^p = h_\alpha^p + \bar{h}_\alpha^p$
- operator product expansion OPE

$$\phi_\alpha^p \times \phi_\beta^p \approx C_{\alpha\beta\gamma}\phi_\gamma^p$$

Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

- operator product expansion OPE from three point correlators

$$\phi_\alpha^p \times \phi_\beta^p \approx C_{\alpha\beta\gamma}\phi_\gamma^p$$



$$\left\langle \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z)\right\rangle = \frac{C_{\alpha\beta\gamma}}{|x-y|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}|y-z|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}|z-x|^{\Delta_\gamma+\Delta_\alpha-\Delta_\beta}}$$

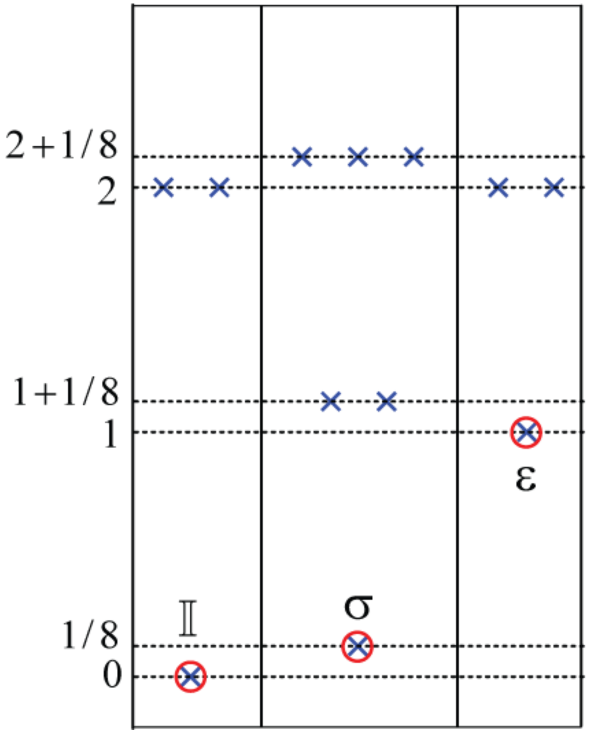
Scale invariant MERA (bulk)

- Example: Ising model

Pfeifer, Evenbly, Vidal, Phys. Rev. A 79(4), 040301(R) (2009)

χ=36 $\tilde{\chi}$=20

scaling operators/dimensions:



identity $\mathbb{I}$ ⇒ ; spin σ ⇒ ; energy ε ⇒

	scaling dimension (exact)	scaling dimension (MERA)	error
identity $\mathbb{I}$	0	0	----
spin σ	0.125	0.124997	0.003%
energy ε	1	0.99993	0.007%
	1.125	1.12495	0.005%
	1.125	1.12499	0.001%
	2	1.99956	0.022%
	2	1.99985	0.007%
	2	1.99994	0.003%
	2	2.00057	0.03%

- Operator product expansion (OPE):

$$C_{\alpha\beta\mathbb{I}} = \delta_{\alpha\beta} \qquad C_{\sigma\sigma\varepsilon} = ½$$

$$C_{\sigma\varepsilon\varepsilon} = C_{\sigma\sigma\sigma} = C_{\varepsilon\varepsilon\varepsilon} = 0$$

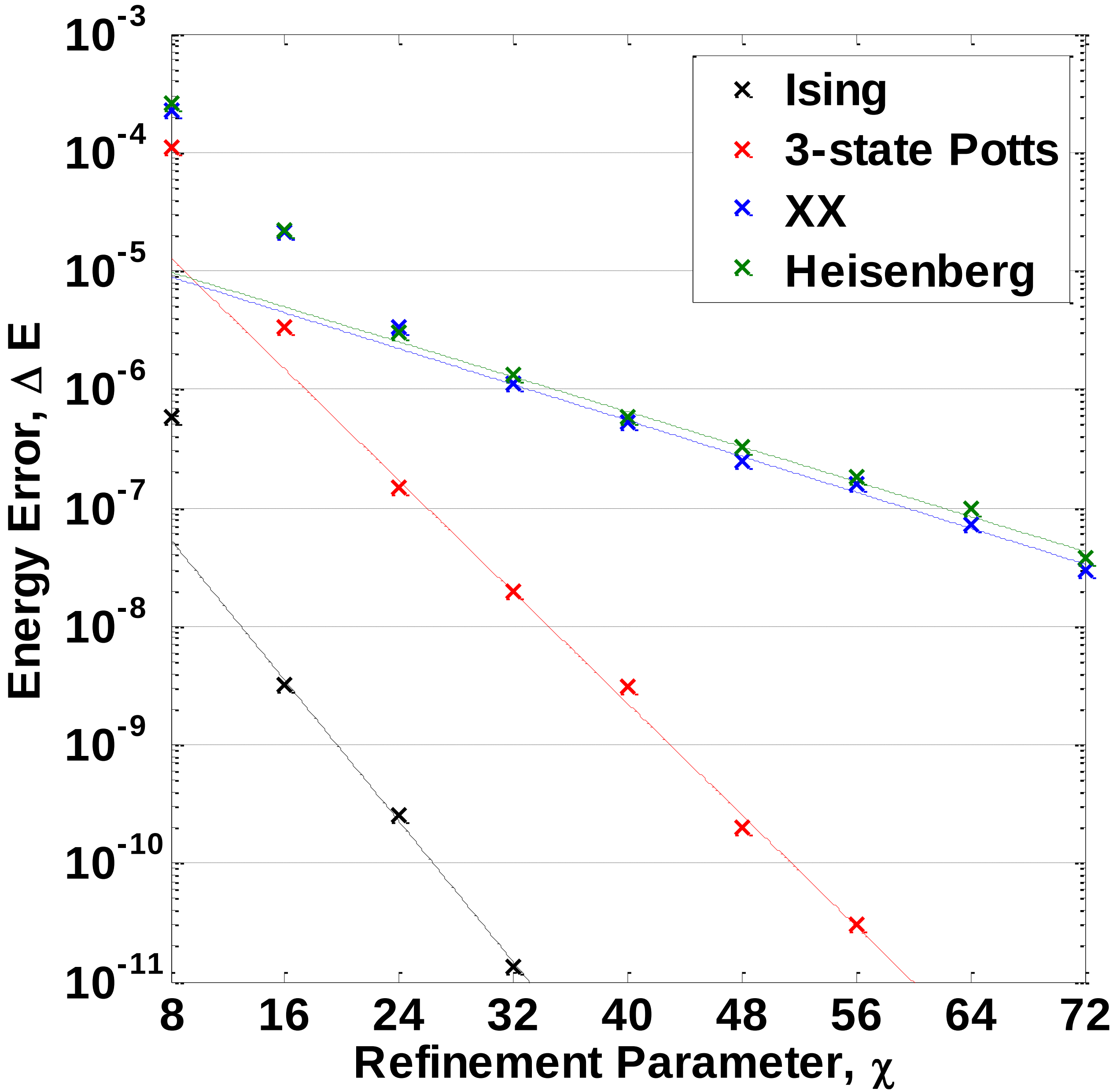
$(\pm 5\times 10^{-4})$

$\Longrightarrow$

fusion rules

$$\varepsilon \times \varepsilon = \mathrm{I}$$
$$\sigma \times \sigma = \mathrm{I} + \varepsilon$$
$$\sigma \times \varepsilon = \sigma$$

• Example: other models



Recent developments:

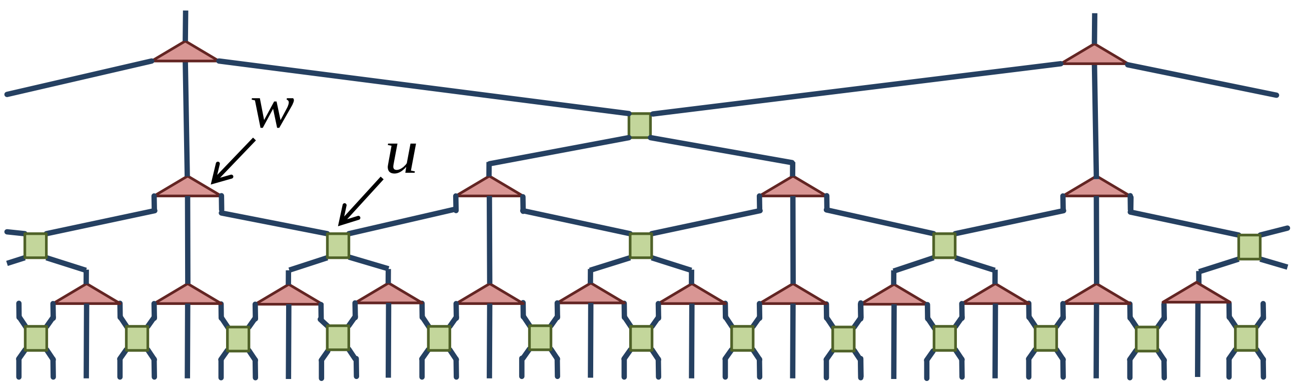
- non-local scaling operators (bulk)
- boundary critical phenomena
- defects (in bulk)

Non-local scaling operators

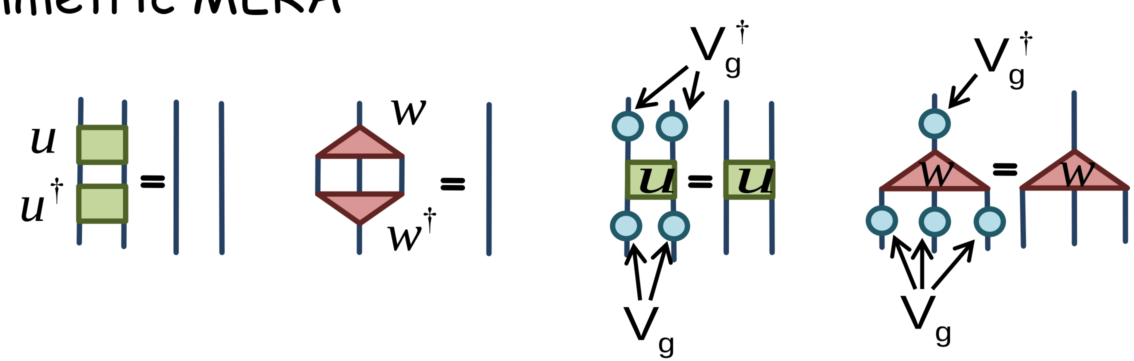
Evenbly, Corboz, Vidal, arXiv: 0912.2166

- global symmetry G $\quad V_g^{\otimes N} \, H \, V_g^{\dagger \, \otimes N} = H \qquad g \in G$

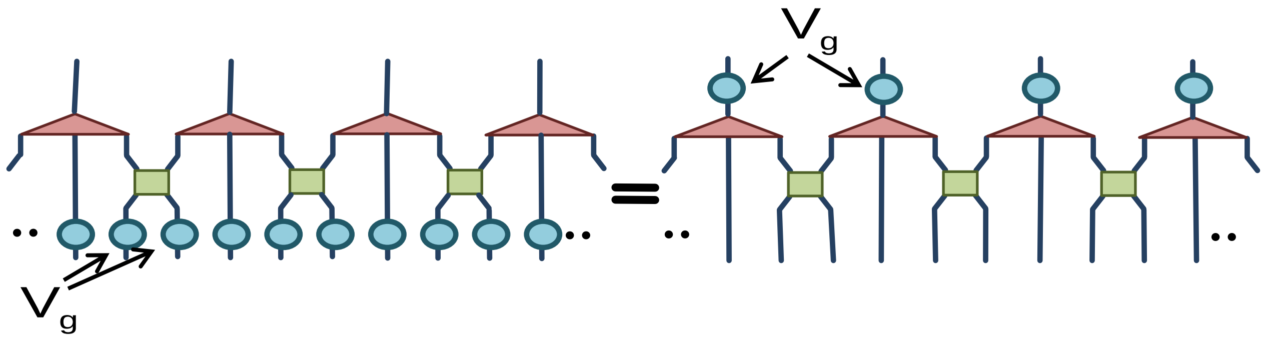
[example $\quad H_{\text{Ising}} \equiv -\sum X_i X_{i+1} - \sum Z_i \qquad Z^{\otimes N} H_{\text{Ising}} \, Z^{\otimes N} = H_{\text{Ising}}$]



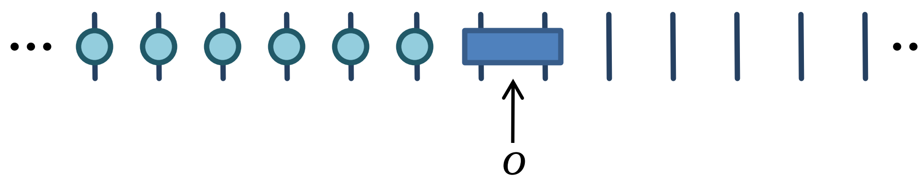
- G-symmetric MERA



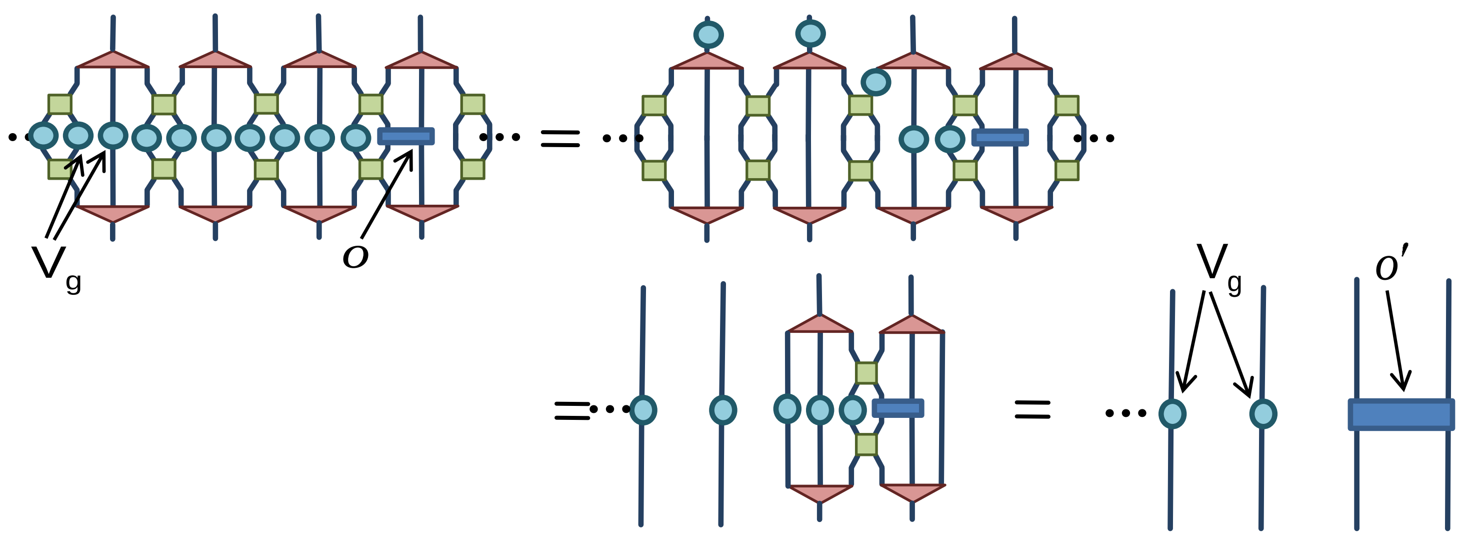
- "the symmetry commutes with the coarse-graining"



- non-local operators of the form



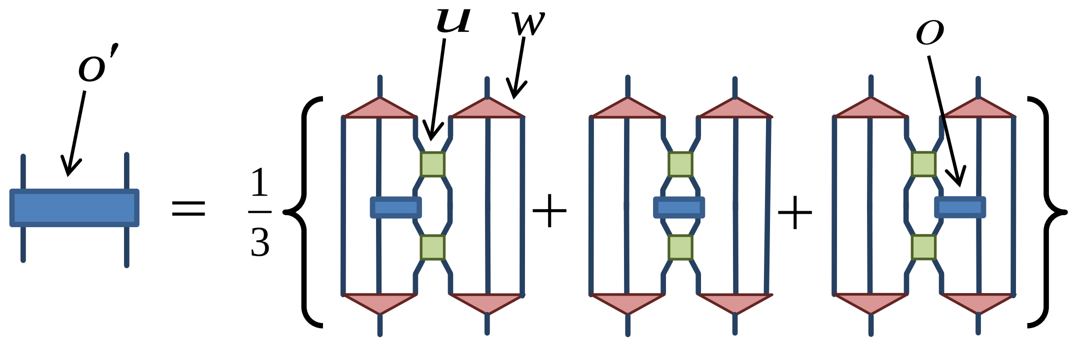
can be "locally" coarse-grained



- local scaling operators

$$o' = \mathcal{S}(o)$$

scaling superoperator



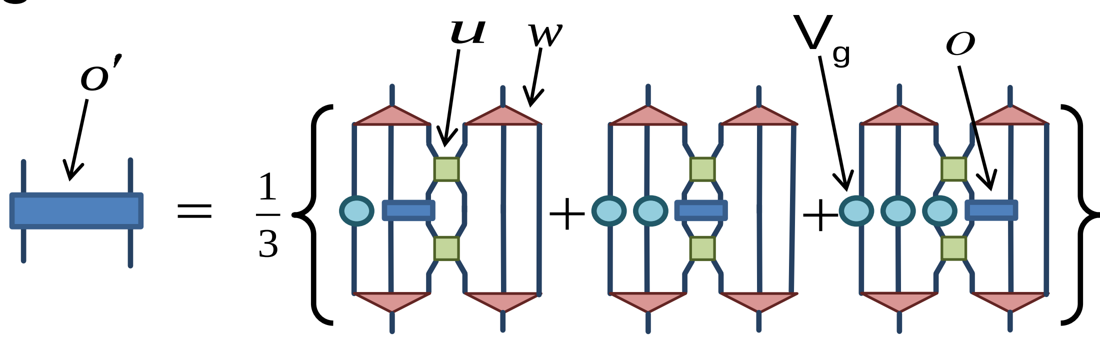
$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

ϕ_α	scaling operator	$\phi_\alpha \to 3^{-\Delta_\alpha} \phi_\alpha$
Δ_α	scaling dimension	$\Delta_\alpha \equiv -\log_3 \lambda_\alpha$

- non-local scaling operators

$$o' = \mathcal{S}_g(o)$$

modified scaling superoperator



$$\mathcal{S}_g(\phi_{g,\alpha}) = \lambda_{g,\alpha} \phi_{g,\alpha}$$

$\phi_{g,\alpha}$	non-local scaling operator	$\phi_{g,\alpha} \to 3^{-\Delta_{g,\alpha}} \phi_{g,\alpha}$
$\Delta_{g,\alpha}$	scaling dimension	$\Delta_{g,\alpha} \equiv -\log_3 \lambda_{g,\alpha}$

Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

$\mathbb{I}$ identity

σ spin

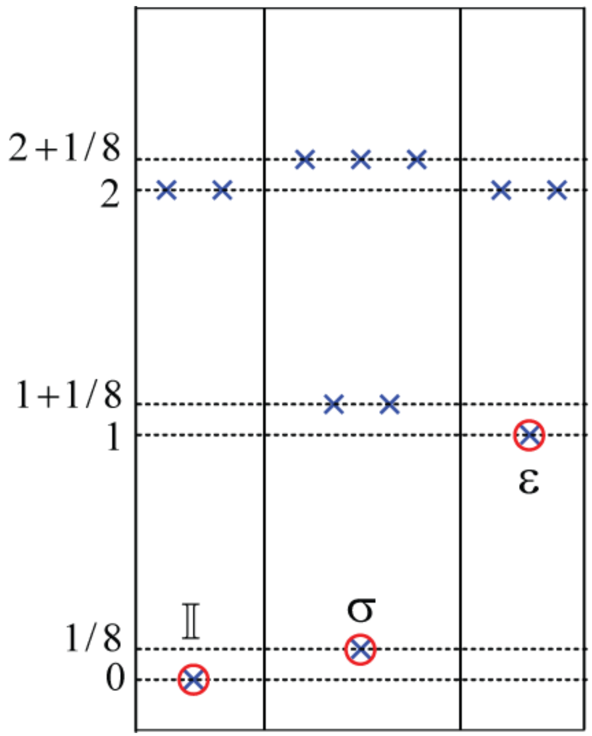
ε energy

μ disorder

ψ fermions

$\bar{\psi}$

local

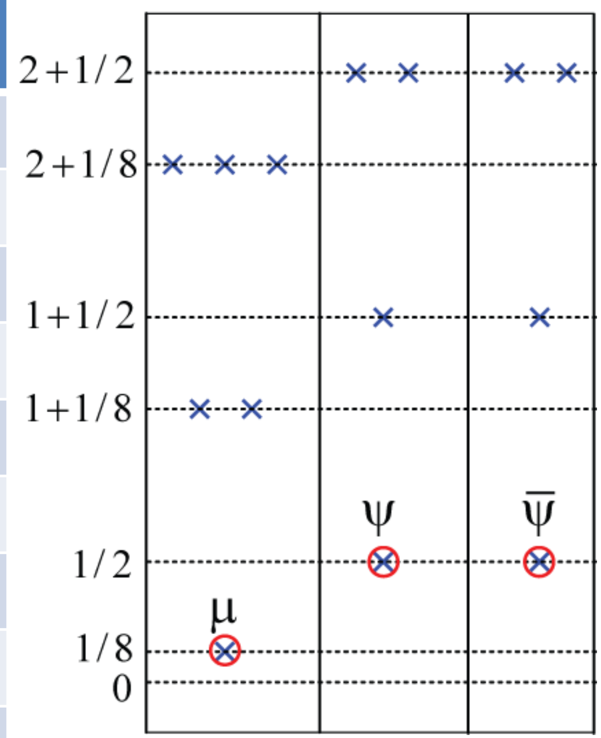


Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

	scaling dimension (exact)	scaling dimension (MERA)	error
disorder μ	1/8	0.1250002	0.0002%
fermions ψ	1/2	0.5	<10−8%
$\bar{\psi}$	1/2	0.5	<10−8%
	1+1/8	1.124937	0.006 %
	1+1/2	1.49999	< 10−5%
	1+1/2	1.49999	< 10−5%
	2+1/8	2.123237	0.083 %
	2+1/8	2.124866	0.006 %
	2+1/8	2.125487	0.023 %

non-local



Non-local scaling operators
Scale invariant MERA (bulk)

- Example: Ising model

OPE for local & non-local primary fields

$$C_{\varepsilon\sigma\sigma} = 1/2 \qquad C_{\varepsilon\psi\bar{\psi}} = i$$

$$C_{\varepsilon\mu\mu} = -1/2 \qquad C_{\varepsilon\bar{\psi}\psi} = -i$$

$$C_{\psi\mu\sigma} = e^{-i\pi/4}/\sqrt{2}$$

$$(\pm 6\times 10^{-4})$$

$$C_{\bar{\psi}\mu\sigma} = e^{i\pi/4}/\sqrt{2}$$

$\Longrightarrow$

fusion rules

$$\varepsilon\times\varepsilon = \mathrm{I}$$
$$\sigma\times\sigma = \mathrm{I}+\varepsilon$$
$$\sigma\times\varepsilon = \sigma$$
$$\mu\times\mu = \mathrm{I}+\varepsilon$$
$$\mu\times\varepsilon = \mu$$
$$\psi\times\psi = \mathrm{I}$$
$$\bar{\psi}\times\bar{\psi} = \mathrm{I}$$
$$\psi\times\bar{\psi} = \varepsilon$$
$$\psi\times\varepsilon = \bar{\psi}$$
$$\bar{\psi}\times\varepsilon = \psi$$
$$\cdots$$

$$\{\mathrm{I},\varepsilon,\sigma,\mu,\psi,\bar{\psi}\}$$

local and semi-local subalgebras

$$\{\mathrm{I},\varepsilon\} \qquad \{\mathrm{I},\varepsilon,\sigma\}$$

$$\{\mathrm{I},\varepsilon,\mu\} \qquad \{\mathrm{I},\varepsilon,\psi,\bar{\psi}\}$$

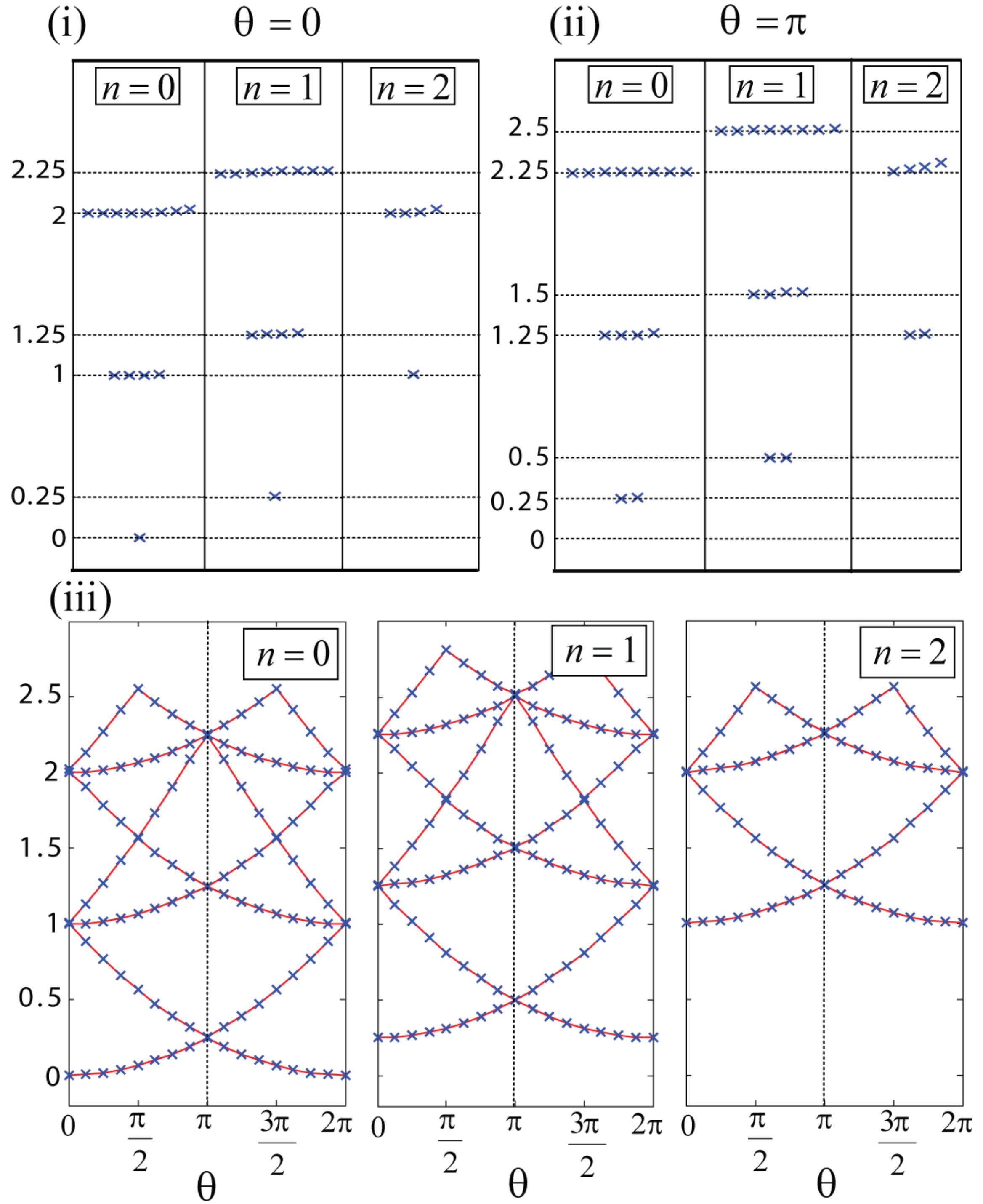
- Example:
  quantum XX model

$$H_{XX} \equiv -\sum (X_i X_{i+1} + Y_i Y_{i+1})$$

G = U(1) symmetry

χ=54 $\tilde{\chi}$=32

(exploiting U(1) symmetry)

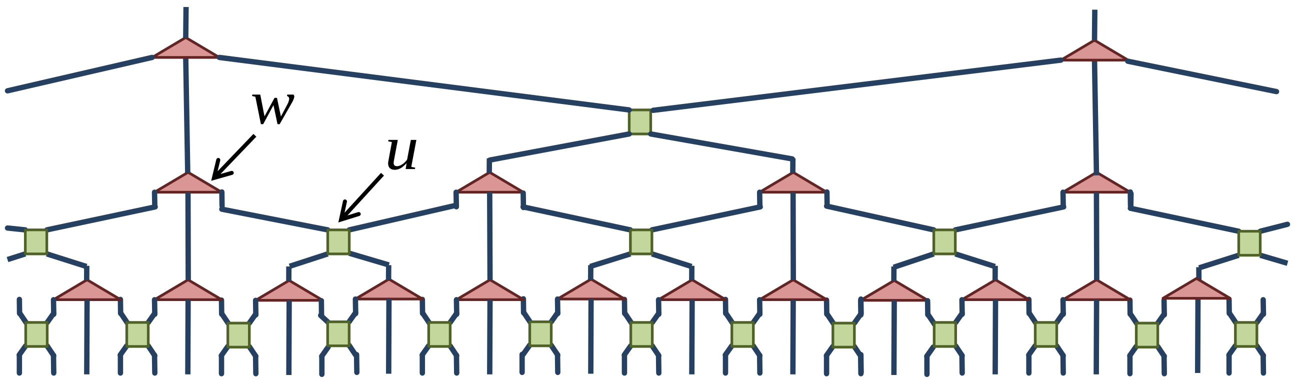


Boundary critical phenomena

Evenbly, Pfeifer, Pico, Iblisdir, Tagliacozzo, McCulloch, Vidal, arXiv:0912.1642

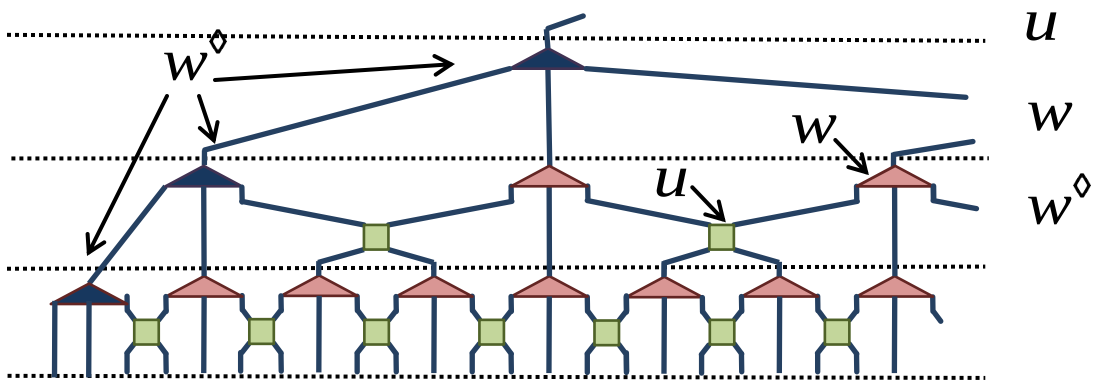
see also Silvi, Giovannetti, Calabrese, Santoro, Fazio, arXiv: 0912.2893

- infinite chain (bulk)



bulk MERA

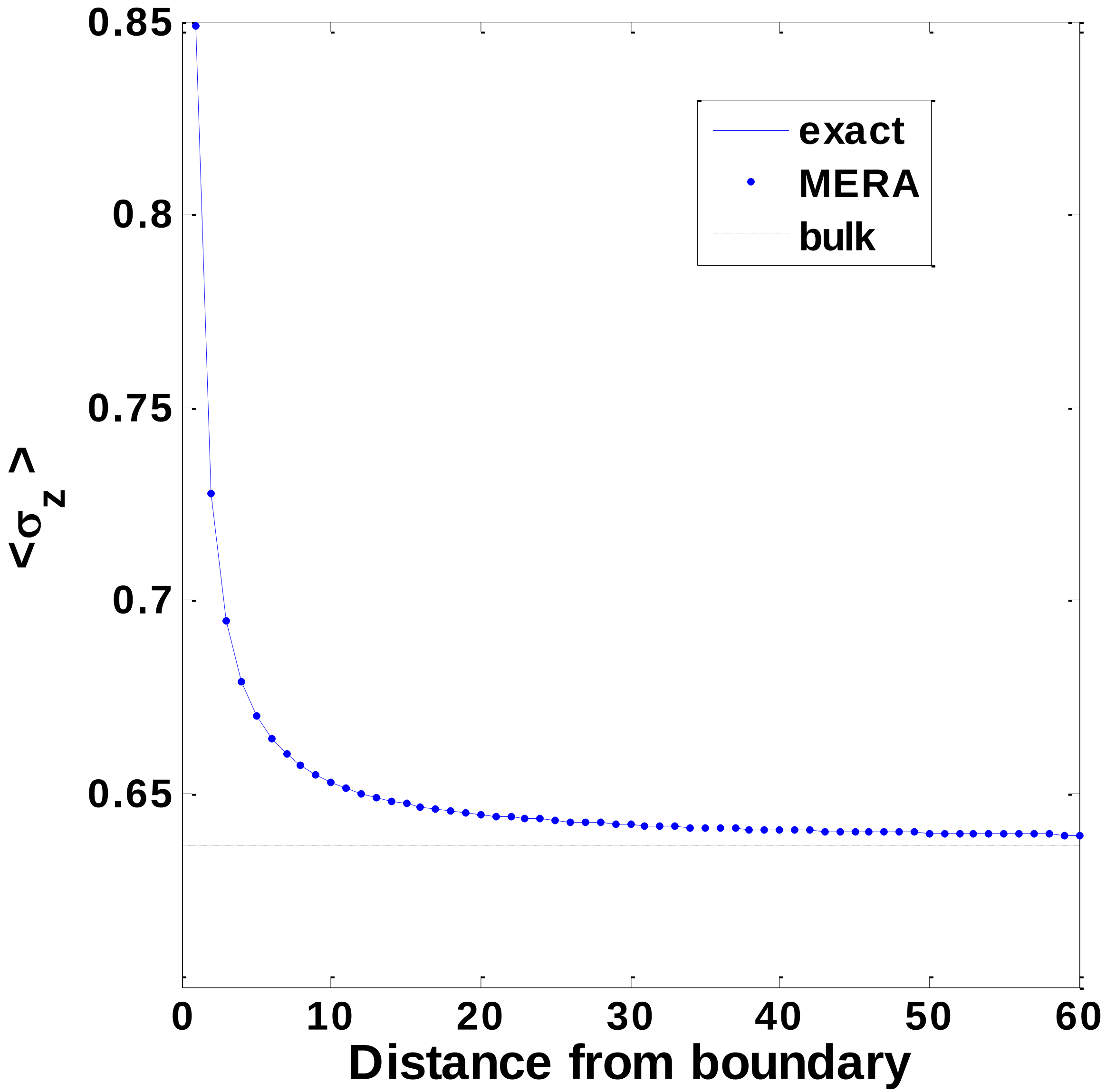
- semi-infinite chain (boundary)

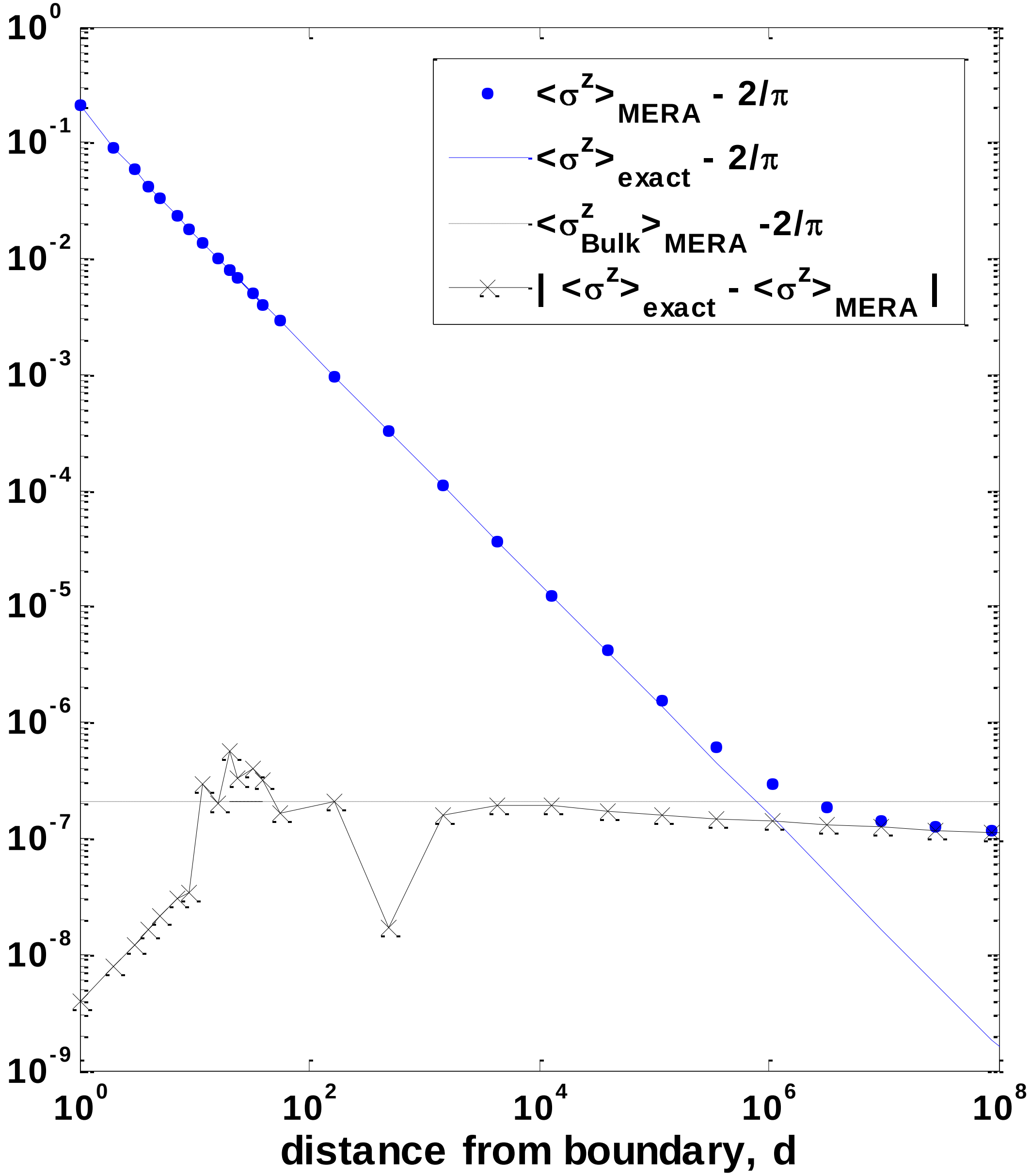


MERA with boundary

• Example: Ising model

Free boundary conditions: local magnetization





- Boundary effects still noticeable far away from the boundary

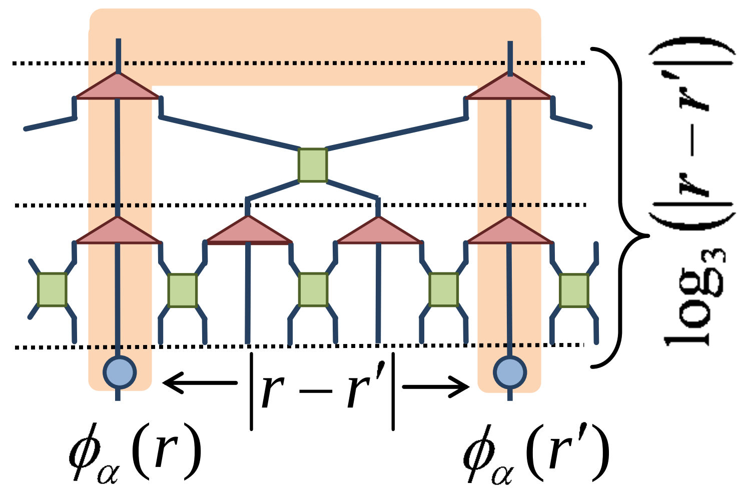
- MERA gets correct magnetization everywhere
(approx. same accuracy as with bulk MERA without boundary)

Bulk expectation values in the presence of a boundary

- bulk

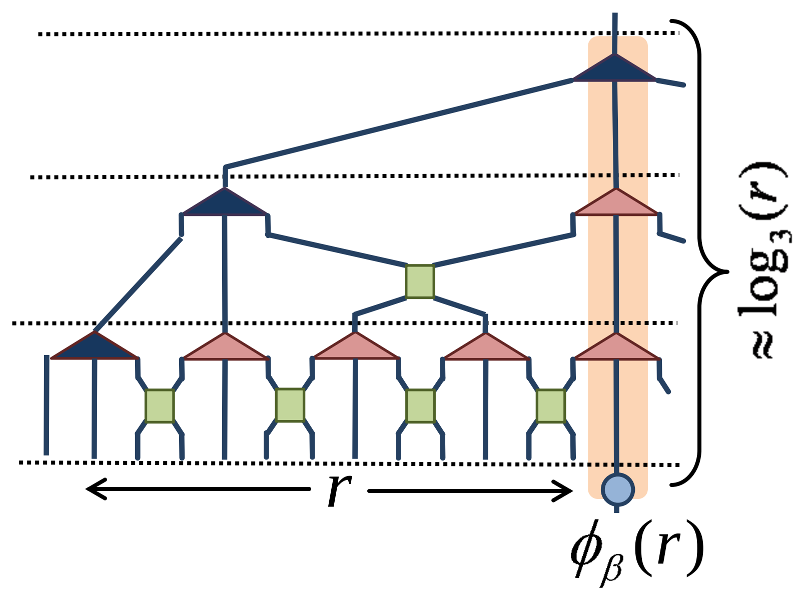
$$\langle \phi_\alpha(r) \rangle = 0$$

$$\langle \phi_\alpha(r)\phi_\alpha(r') \rangle = \frac{1}{|r - r'|^{2\Delta_\alpha}}$$

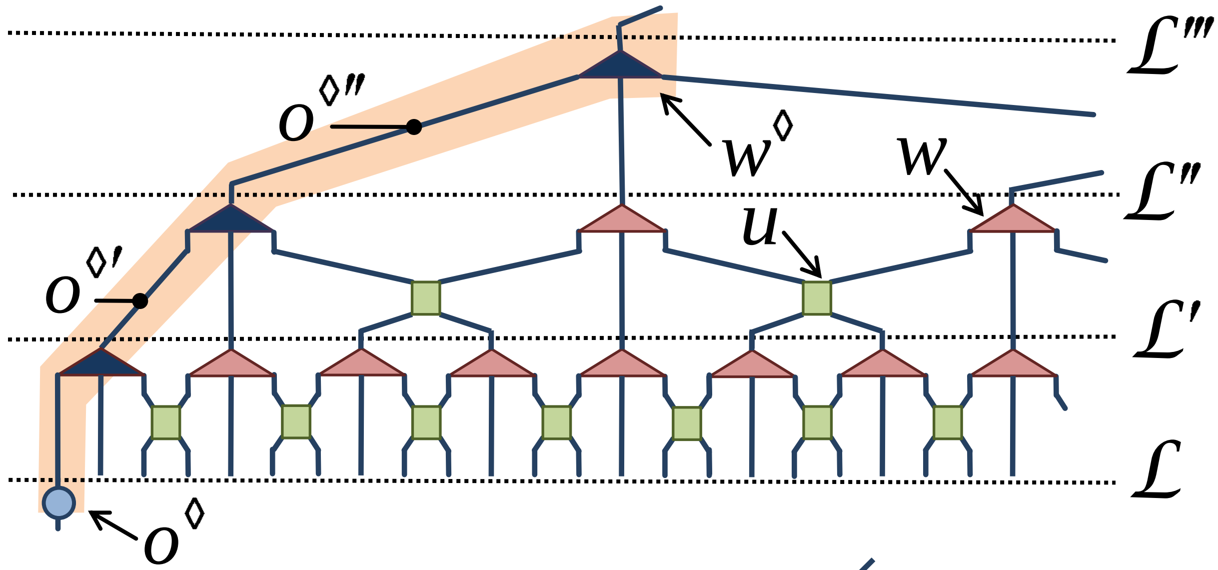


- boundary

$$\langle \phi_\alpha(r) \rangle \approx \frac{1}{|r|^{\Delta_\alpha}}$$

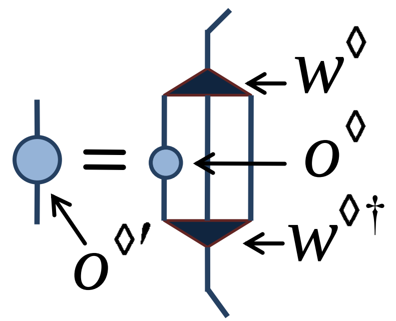


Boundary scaling operators/dimensions



boundary scaling superoperator

$$o^{\lozenge'} = \mathcal{S}^{\lozenge}(o^{\lozenge})$$



$$\mathcal{S}^{\lozenge}(\phi_\alpha^{\lozenge}) = \lambda_\alpha^{\lozenge}\phi_\alpha^{\lozenge}$$

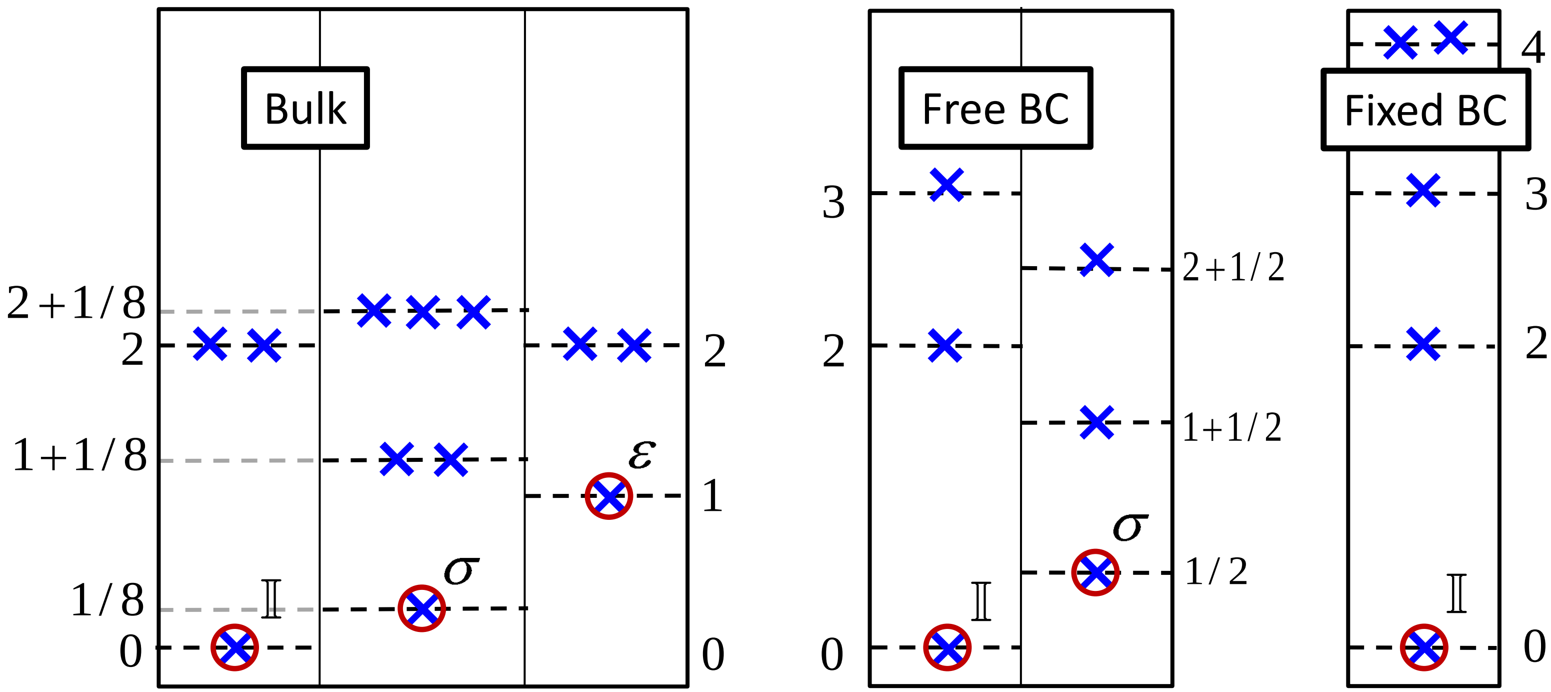
⇒

$\phi_\alpha^{\lozenge}$ boundary scaling operator $\qquad \phi_\alpha^{\lozenge} \to 3^{-\Delta_\alpha^{\lozenge}}\phi_\alpha^{\lozenge}$

$\Delta_\alpha^{\lozenge}$ boundary scaling dimension $\qquad \Delta_\alpha^{\lozenge} \equiv -\log_3 \lambda_\alpha^{\lozenge}$

Boundary scaling operators/dimensions

- Example:
  Ising model



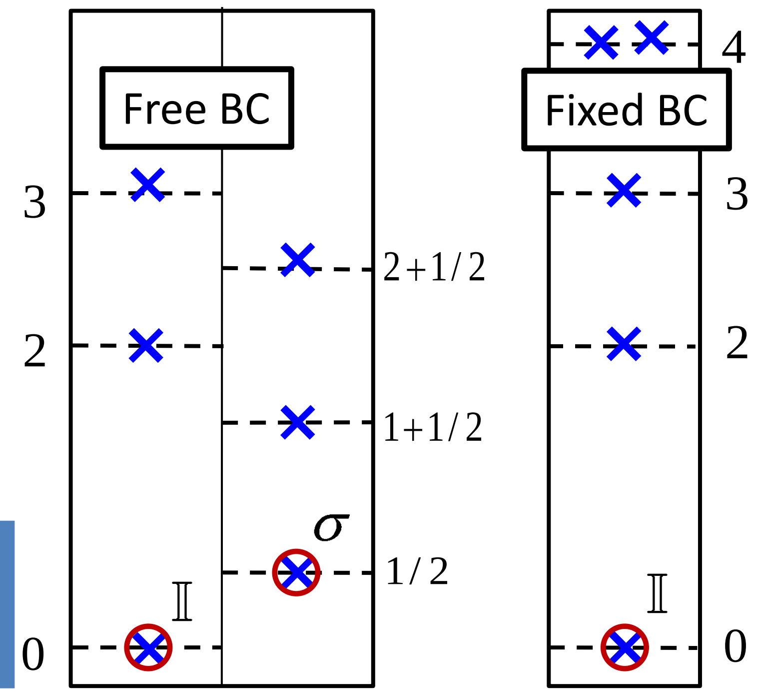
Boundary scaling operators/dimensions

Free BC

	scaling dimension (exact)	scaling dimension (MERA)	error
identity $\mathbb{I}$	0	0	---
spin σ	1/2	0.499	0.2%
	1+1/2	1.503	0.18%
	2	2.001	0.07 %
	2+1/2	2.553	2.1%

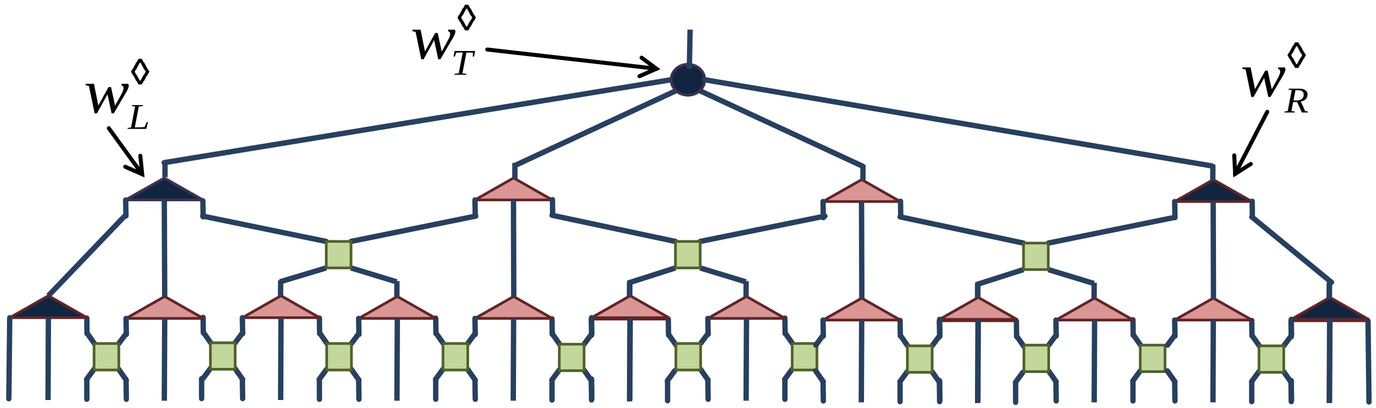
Fixed BC

	scaling dimension (exact)	scaling dimension (MERA)	error
identity $\mathbb{I}$	0	0	---
	2	1.992	0.4%
	3	2.998	0.07%
	4	4.005	0.12 %
	4	4.062	1.5%

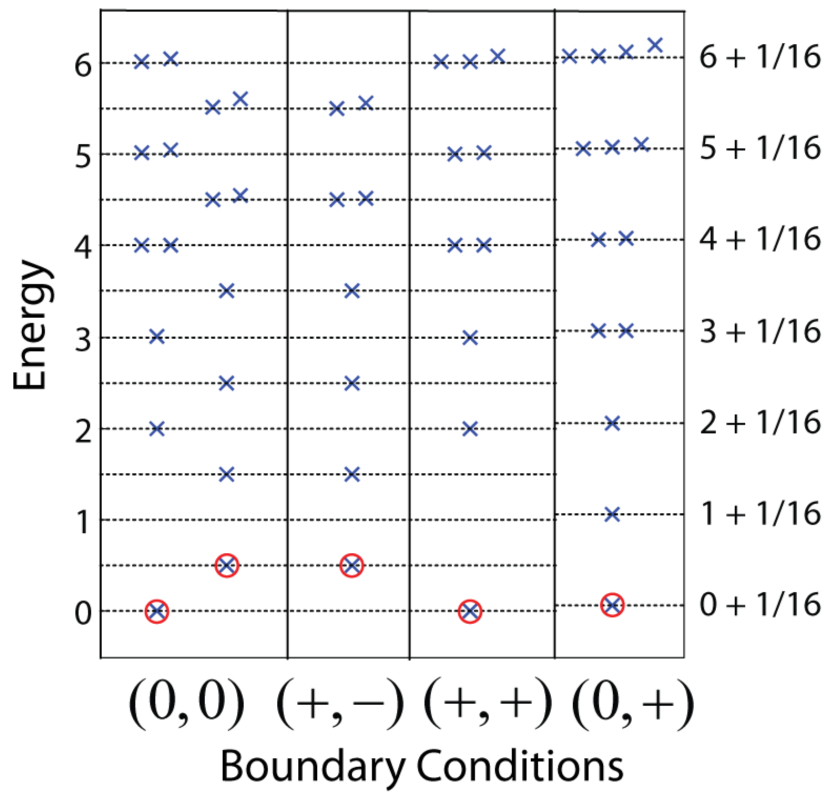


Finite system with two boundaries

- finite MERA with two boundaries

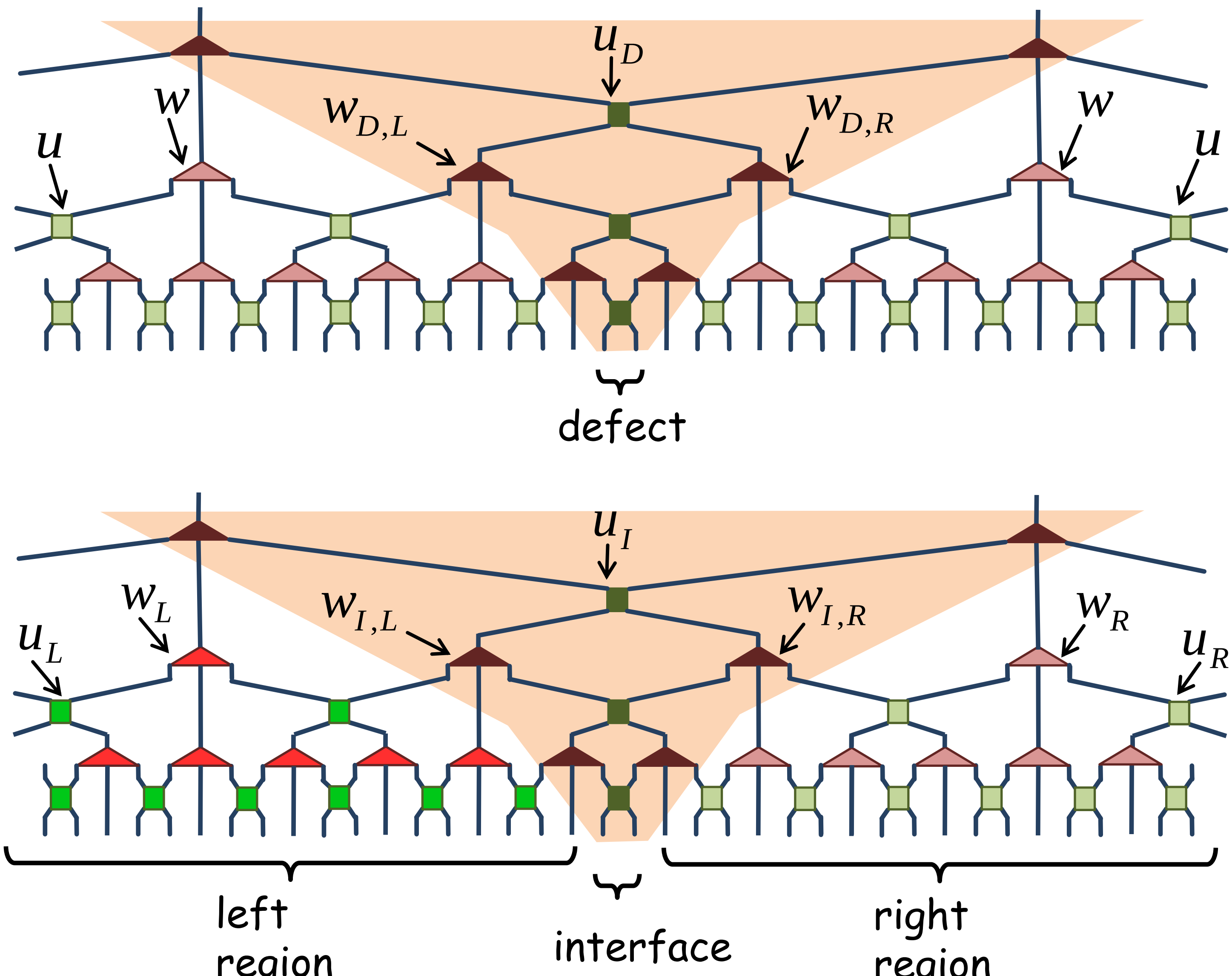


- Energy spectrum

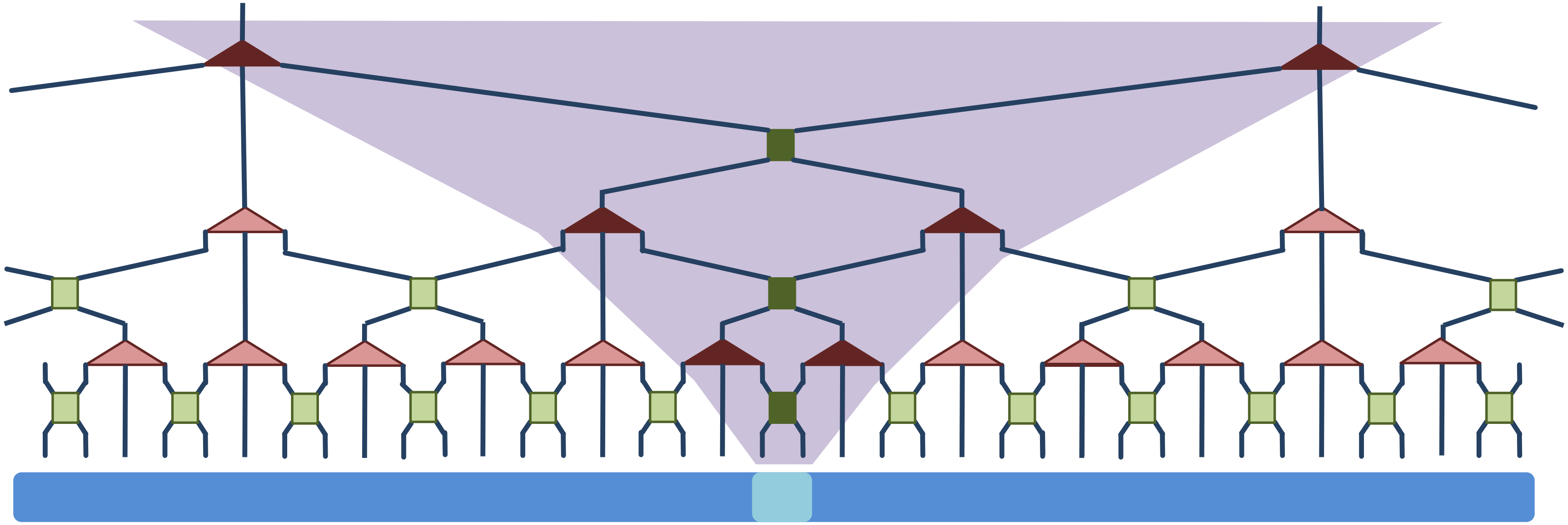


defect / interface

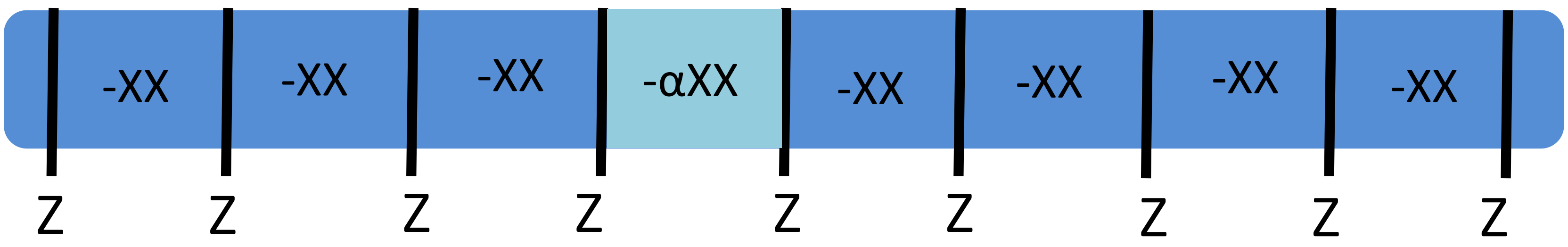
Evenbly, Pfeifer, Pico, Iblisdir, Tagliacozzo, McCulloch, Vidal, arXiv:0912.1642

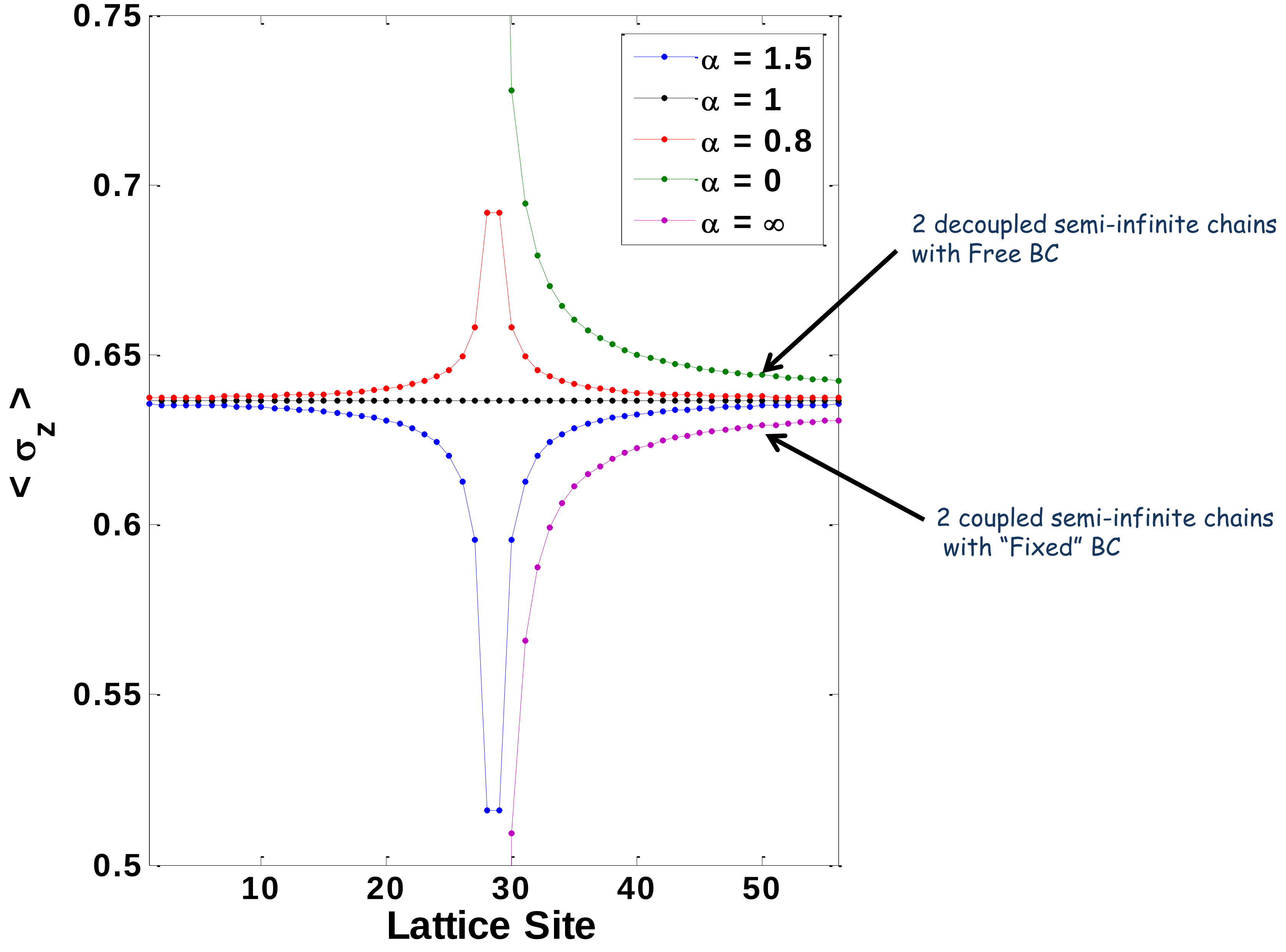


Lattice Defects:



Hamiltonian:

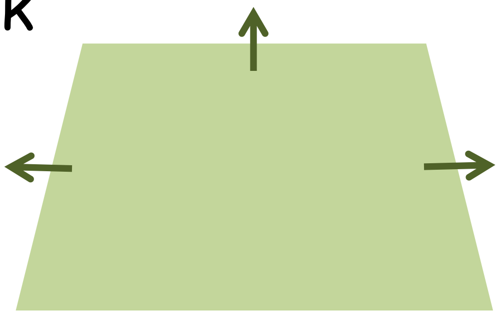




0.75
0.7
0.65
0.6
0.55
0.5
10
20
30
40
50
Lattice Site
< σz >
α = 1.5
α = 1
α = 0.8
α = 0
α = ∞
2 decoupled semi-infinite chains
with Free BC
2 coupled semi-infinite chains
with "Fixed" BC

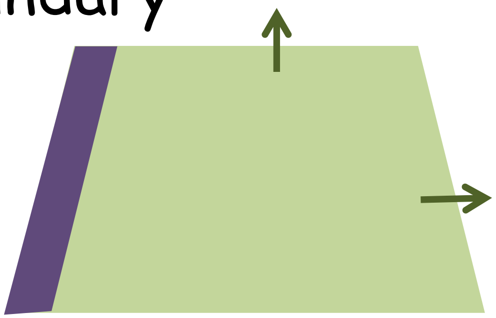
Summary: scale invariant MERA, critical phenomena, and CFT

- Bulk

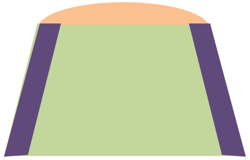


- (local & non-local) scaling operators/dimensions
- CFT: primary fields and OPE

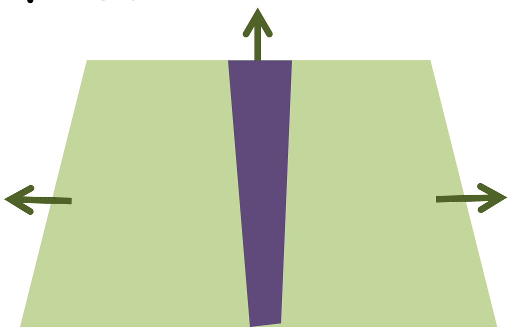
- Boundary



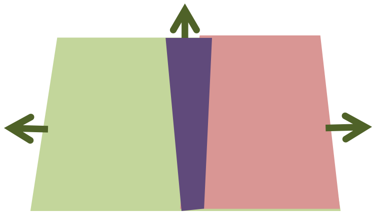
- boundary scaling operators
- BCFT: primary fields and OPE
- finite system with two boundaries



- Defect

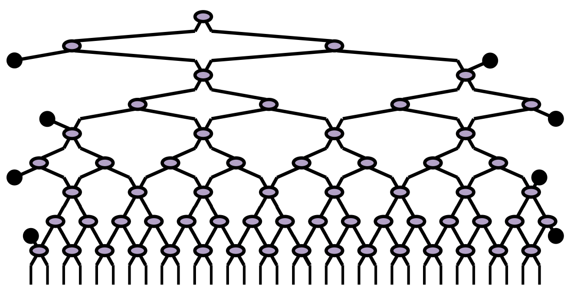


- defect scaling operators/dimensions
- interface



MERA and the AdS/CFT correspondence

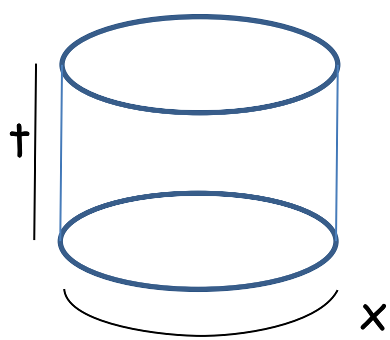
Swingle, arXiv:0905.1317



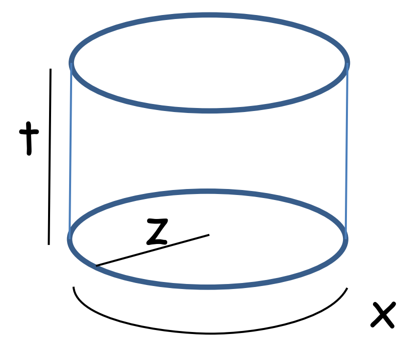
scale invariant MERA

vacuum of CFT

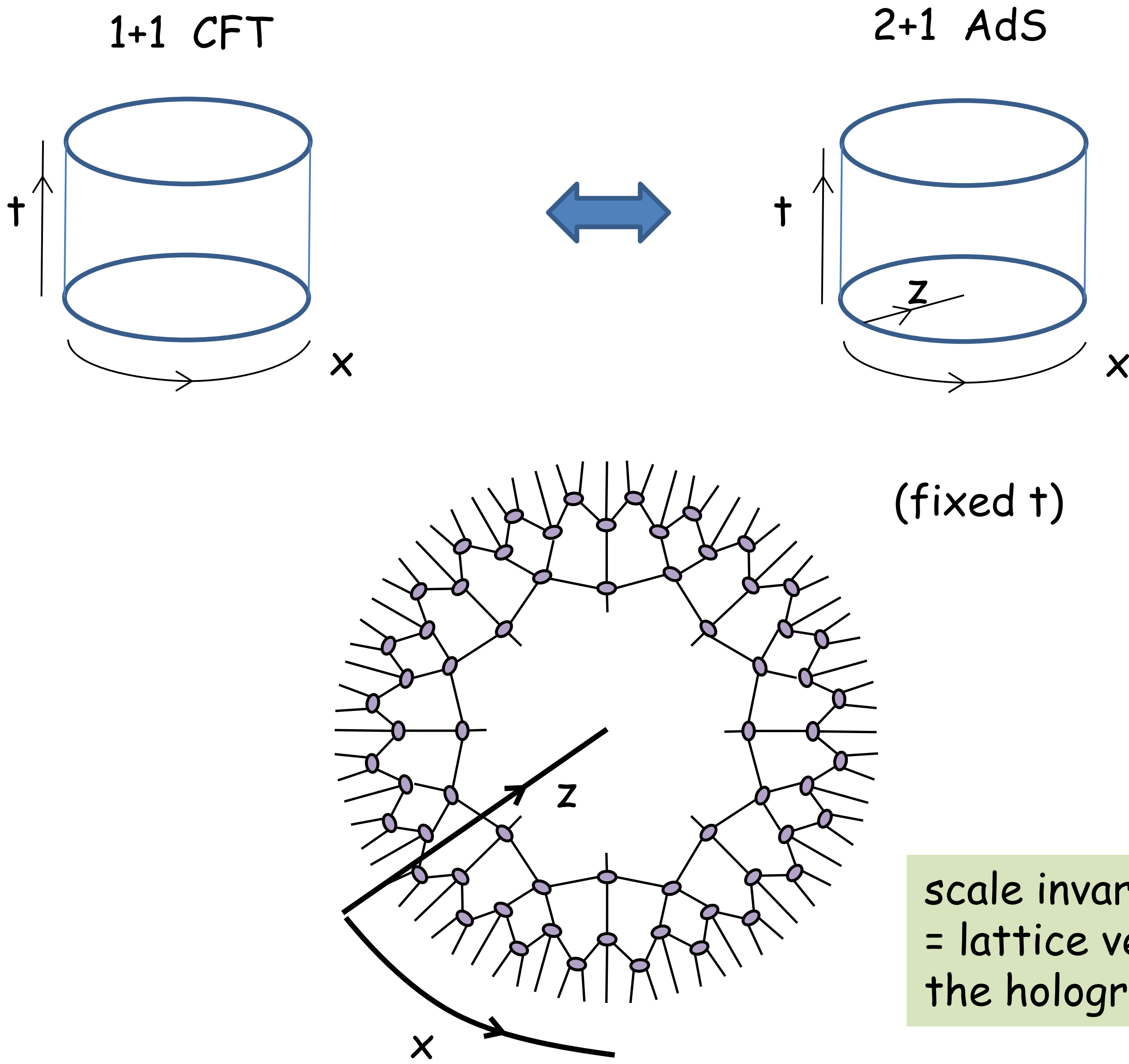
1+1 CFT



2+1 AdS

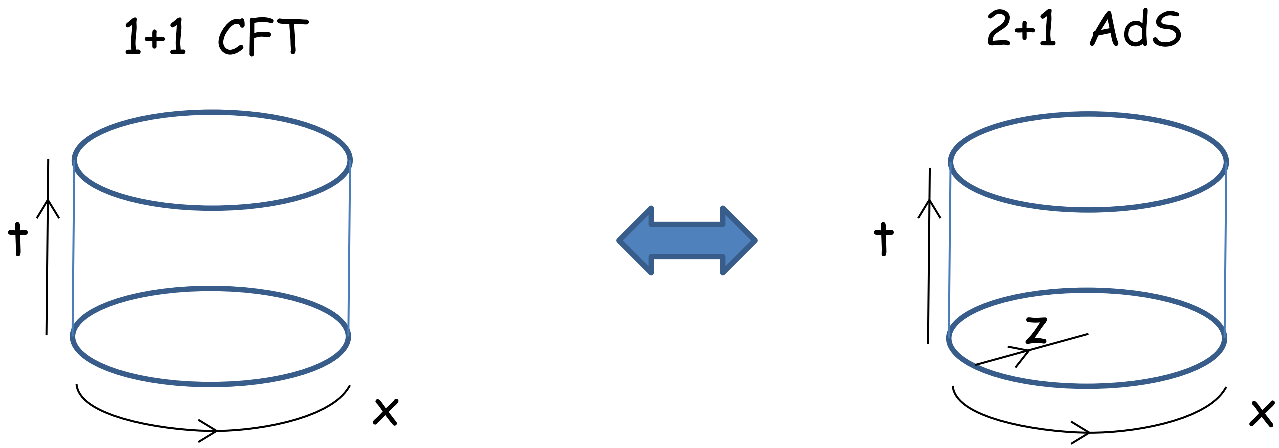


MERA and the AdS/CFT correspondence



scale invariant MERA
= lattice version of
the holographic principle

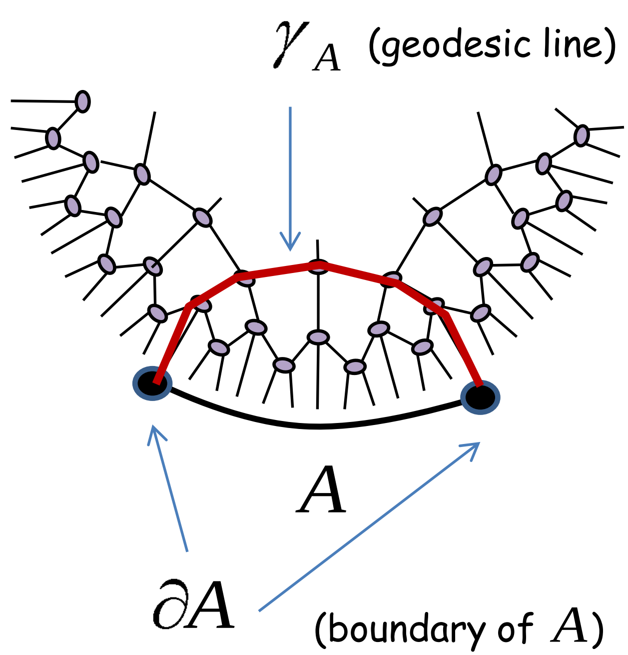
MERA and the AdS/CFT correspondence



Ryu ,Takayanagi, PRL96, 181602 (2006)

Holographic derivation of entanglement entropy

$$S_A \approx |\gamma_A| \approx \log|A|$$



Outline

MERA = **M**ulti-scale **E**ntanglement **R**enormalization **A**nsatz

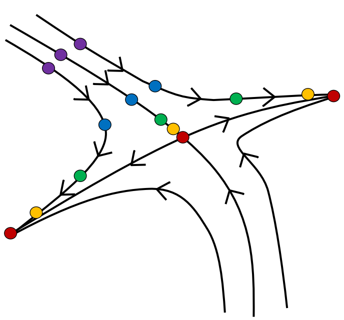
Entanglement Renormalization/MERA

- Renormalization Group (RG) transformation
- Quantum Computation

Scale invariant MERA ⟺ RG fixed point

- critical fixed point ⟺ continuous quantum phase transition ⟺ CFT
  - bulk (local/non-local)
  - boundary
  - defect

Scale invariant MERA ⟺ Holographic principle



collaboration with

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L. Tagliacozzo, I.P. McCulloch

V. Pico (U. Barcelona)
S. Iblisdir (U. Barcelona)