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Finite temperature free fermions and the Kardar-Parisi-Zhang equation at finite time

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We consider the system of N one-dimensional free fermions confined by a harmonic well $V(x) = m\omega^2 x^2/2$ at finite inverse temperature $\beta = 1/T$. The average density of fermions $\rho_N(x,T)$ at position x is derived. For $N \gg 1$ and $\beta \sim \mathcal{O}(1/N)$, $\rho_N(x,T)$ is described by a scaling function interpolating between a Gaussian at high temperature, for $\beta \ll 1/N$, and the Wigner semi-circle law at low temperature, for $\beta \gg N^{-1}$. In the latter regime, we unveil a scaling limit, for $\beta\hbar\omega = bN^{-1/3}$, where the fluctuations close to the edge of the support, at $x \sim \pm\sqrt{2\hbar N/(m\omega)}$, are described by a limiting kernel $K_b^{\rm ff}(s,s')$ that depends continuously on b and is a generalization of the Airy kernel, found in the Gaussian Unitary Ensemble of random matrices. Remarkably, exactly the same kernel $K_b^{\rm ff}(s,s')$ arises in the exact solution of the Kardar-Parisi-Zhang (KPZ) equation in 1+1 dimensions at finite time t, with the correspondence $t = b^3$.

There is currently intense activity in the field of low dimensional quantum systems. This is largely motivated by new experimental developments for manipulating fundamental quantum systems, notably ultra-cold atoms [1, 2], where the confining potentials are optically generated.

One of the most fundamental quantum systems is that of N non-interacting spinless fermions in one dimension, confined in a harmonic trap $V(x) = \frac{1}{2}m\omega^2x^2$. Recently, the zero temperature ($T = 0$) properties of this system have been extensively studied [3–10], and a deep connection between this free fermion problem and random matrix theory (RMT) has been established. Specifically, the probability density function (PDF) of the positions x_i's of the N fermions, given by the modulus squared of the ground state wave function $\Psi_0(x_1,\cdots,x_N)$, can be written as

$$|\Psi_0(x_1,\cdots,x_N)|^2 = \frac{1}{z_N(\alpha)}\prod_{i<j}(x_i-x_j)^2 e^{-\alpha^2\sum_{i=1}^N x_i^2} \quad (1)$$

with $\alpha = \sqrt{m\omega/\hbar}$, and where $z_N(\alpha)$ is a normalization constant. Eq. (1) shows that the rescaled positions αx_i's behave statistically as the eigenvalues of random $N\times N$ matrices of the Gaussian Unitary Ensemble (GUE) of random matrix theory (RMT) [11, 12]. From Eq. (1), the properties of the ground state can be rapidly deduced using RMT.

In particular, the average density of free fermions $\rho_N(x, T=0)$ in the ground state is given, in the large N limit, by the Wigner semi-circle law [11, 12],

$$\rho_N(x,T=0) \sim \frac{\alpha}{\sqrt{N}}\rho_{\rm sc}\left(\frac{\alpha\, x}{\sqrt{N}}\right) \;, \quad (2)$$

where $\rho_{\rm sc}(x) = \frac{1}{\pi}\sqrt{2-x^2}$, on the finite support $[-\sqrt{2N}/\alpha, \sqrt{2N}/\alpha]$. An important property of free fermions at $T = 0$, inherited from their connection with the eigenvalues of random matrices (1), is that they constitute a determinantal point process, for any finite N. This means that their statistical properties are fully encoded in a two-point kernel $K_N(x,y)$, which, roughly speaking, is the *Green's function* of the model. Remarkably, any k-point correlation function can be written as a $k\times k$ determinant built from $K_N(x,y)$ [see Eqs. (24, 25) below].

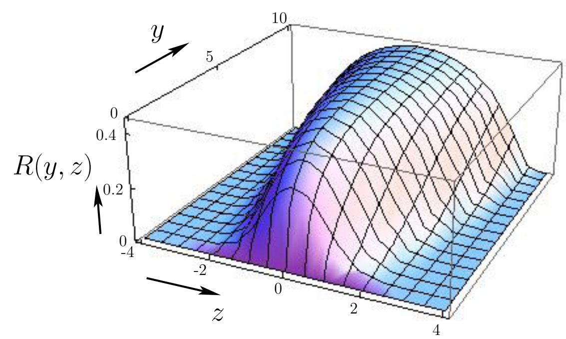

FIG. 1. (Color online) Plot of the scaling function $R(y,z)$ associated with the density $\rho_N(x,T)$ (8), given in Eq. (9).

Early developments in RMT focused on the *bulk regime*, where both x and y are in the middle of the *Wigner sea*, say close to the origin where both $x \propto u/\sqrt{N}$ and $y \propto v/\sqrt{N}$ (*i.e.*, of the order of the typical inter-particle spacing $\sim 1/(\rho_{\rm sc}(0)\sqrt{N})$). In this region the statistics of eigenvalues, and consequently the positions of the free fermions at $T = 0$ (1), are described by the so called sine-kernel $K_N(x,y) \propto K_{\rm Sine}(u-v)$ where $K_{\rm Sine}(w) = \sin(\pi w)/(\pi w)$. More recently, there has been a huge interest in the statistics of eigenvalues at the edge of the Wigner sea which, for fermions, corresponds to the fluctuations close to $x = \pm\sqrt{2N}/\alpha$. To probe these fluctuations at the edge, a natural observable is the position of the rightmost fermion, $x_{\max}(T=0) = \max_{1\le i\le N} x_i$ in the ground state. From Eq. (1) we can immediately infer that the typical (quantum) fluctuations of $x_{\max}(T=0)$, correctly shifted and scaled, are described by the Tracy-Widom (TW) distribution $F_2(\xi)$ associated with the fluctuations of the largest eigenvalue $\lambda_{\max}$ of GUE random matrices [13]. Namely, one has

$$\alpha\, x_{\max}(T=0) = \sqrt{2N} + \frac{1}{\sqrt{2}N^{1/6}}\chi_2 \;, \qquad (3)$$

where χ_2 is a random variable whose cumulative distribution $F_2(\xi)$ can be written as a Fredholm determinant [14]

$$F_2(\xi) = \Pr[\chi_2 \le \xi] = \det(I - P_\xi K_{\rm Ai} P_\xi) \;, \qquad (4)$$

where $K_{\rm Ai}(x,y)$ is the Airy kernel [13, 27]

$$K_{\rm Ai}(x,y) = \frac{{\rm Ai}(x){\rm Ai}'(y) - {\rm Ai}'(y){\rm Ai}(x)}{x-y} \;. \qquad (5)$$

Here ${\rm Ai}(x)$ is the Airy function and P_ξ is the projector on the interval $[\xi,+\infty)$. Intriguingly, it was found [8] that for a generic confining potential of the form $V(x) \sim x^p/p$, the local fluctuations in the fermion problem at $T=0$ are universal both in the bulk, where they are given by the Sine-kernel, and at the edge, where they are described by the Airy kernel in Eq. (5).

The beautiful connection embodied in Eq. (1) means that any question about free fermions in a harmonic trap at $T=0$ can be translated into a RMT problem. It is thus very natural to study the effect of non-zero temperature in the free fermion system, for which much less is known (see however [15, 16]). In this Letter, we analyze this system at finite T and find a very rich behavior for the density of fermions, as well as for the fluctuations of the right most fermion. In particular, we find a fascinating link between free fermions at *finite* T and the Kardar-Parisi-Zhang (KPZ) equation in $1+1$ dimensions at *finite time*.

Let us summarize our main results. First we compute the average density $\rho_N(x,T)$, for large N, in the two following limits: (i) low temperature limit $T=0$ discussed above (2), which holds in the scaling limit, $N\to\infty$, $x\to\infty$ keeping $x/\sqrt{N}$ fixed and (ii) $T\to\infty$ (equivalently $\beta\to 0$ limit). In case (ii), the average density has the classical Gibbs-Boltzman distribution

$$\rho_N(x, T\to\infty) \sim \sqrt{\frac{\beta\, m\, \omega^2}{2\pi}}\, \exp\left[-\frac{\beta}{2}\, m\, \omega^2\, x^2\right] \;, \qquad (6)$$

which holds also in the scaling limit $\beta\to 0$, $x\to\infty$ but keeping $x\sqrt{\beta}$ fixed (with the limit $N\to\infty$ already taken). Note that $\hbar$ has disappeared from Eq. (6) as expected. We address the question: how does $\rho_N(x,T)$ interpolate between the two limits $T\to 0$ in (2) and $T\to\infty$ in (6)? There are two natural dimensionless scaling variables in this problem

$$y = \beta\, N\, \hbar\, \omega \;,\; z = x\sqrt{\frac{\beta}{2}\, m\, \omega^2} \;, \qquad (7)$$

in terms of which we find that the density takes the scaling form

$$\rho_N(x,T) \sim \frac{\alpha}{\sqrt{N}} R\left(\beta\, N\, \hbar\, \omega = y, x\sqrt{\frac{\beta}{2}\, m\omega^2} = z\right), \qquad (8)$$

which holds in the scaling limit: (a) $\beta\to 0$, $N\to\infty$ but keeping $y=\beta\, N\, \hbar\, \omega$ fixed (i.e., $T\sim N$) and (b) $\beta\to 0$, $x\to\infty$ but keeping $z = x\sqrt{\frac{\beta}{2}\, m\, \omega^2}$ fixed (i.e., $x\sim\sqrt{T}$). We find that the scaling function $R(y,z)$, for all y and z, is given by

$$R(y,z) = -\frac{1}{\sqrt{2\pi\, y}}\, {\rm Li}_{1/2}\left(-\left(e^y - 1\right)\, e^{-z^2}\right) \;, \qquad (9)$$

where ${\rm Li}_n(x) = \sum_{k=1}^\infty x^k/k^n$ is the polylogarithm function. In Fig. 1 we show a 3d-plot of $R(y,z)$. One can check, from an asymptotic analysis of $R(y,z)$, that one recovers the two limiting behaviors of Eqs. (2) and (6).

We now consider a different *low temperature* scaling limit where $T = b^{-1}N^{1/3}\hbar\omega$, corresponding to the case $y\to 0$ in Eq. (8). In this scaling limit $\rho_N(x, T = b^{-1}N^{1/3}\hbar\omega)$ is thus given by the Wigner semicircle (2), which has a finite support $[-\sqrt{2N}/\alpha, \sqrt{2N}/\alpha]$. We show that for $N\gg 1$, N free fermions at finite temperature in the canonical ensemble behave asymptotically as a determinantal point-process (which is not true for finite N). Close to the edge where $x\approx\sqrt{2N}/\alpha$, and for $\beta\hbar\omega = bN^{-1/3}$ we show that this determinantal point process is characterized by a limiting kernel $K_b^{\rm ff}(s,s')$ given by

$$K_b^{\rm ff}(s,s') = \int_{-\infty}^{\infty} \frac{[{\rm Ai}(s+u){\rm Ai}(s'+u)]}{e^{-b\,u}+1} du \;, \qquad (10)$$

which is a generalization of the Airy-kernel (5). As a consequence, we find that the cumulative distribution of the position of the rightmost fermion $x_{\max}(T)$ is given by the following Fredholm determinant [14]

$$\Pr\left(x_{\max}(T) \le \frac{\sqrt{2N}}{\alpha} + \frac{N^{-\frac{1}{6}}}{\alpha\sqrt{2}}\xi\right) \underset{N\to\infty}{\to} \det(I - P_\xi K_b^{\rm ff} P_\xi). \qquad (11)$$

Note that, using that $\lim_{b\to\infty} K_b^{\rm ff}(s,s') = K_{\rm Ai}(s,s')$ in Eq. (11), we recover the TW distribution of Eq. (4) in the limit $b\to\infty$, as one should.

Remarkably, exactly the same expression as Eq. (11) was recently found in the study of the (1+1)-d KPZ equation in curved geometry. The KPZ equation describes the time evolution of a height field $h(x,t)$ at point x and time t as follows

$$\partial_t h = \nu\partial_x^2 h + \frac{\lambda_0}{2}\left(\partial_x h\right)^2 + \sqrt{D}\eta(x,t) \;, \qquad (12)$$

where $\eta(x,t)$ is a Gaussian white noise with zero mean and correlator $\langle\eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$. We start from the narrow wedge initial condition, $h(x,0) = -|x|/\delta$, with $\delta\ll 1$, which gives rise to a curved (or *droplet*) mean profile as time evolves [17]. Defining the natural time unit $t^* = 2(2\nu)^5/(D^2\lambda_0^4)$ and $\gamma_t = (t/t^*)^{1/3}$ [18], the time-dependent generating function

$$g_t(\zeta) = \langle \exp(-e^{\gamma_t \tilde{h}(0,t)-\zeta})\rangle \qquad (13)$$

of $\tilde{h}(0,t) = (\frac{\lambda_0 h(0,t)}{2\nu} + \frac{t}{12t^*})/\gamma_t$, the rescaled height at $x=0$, is expressed as a Fredholm determinant [17, 19–22]:

$$g_t(\zeta) = \det(I - P_\zeta K_t^{\rm KPZ} P_\zeta) \tag{14}$$

$$K_t^{\rm KPZ}(x,y) = \int_{-\infty}^{\infty} \frac{{\rm Ai}(z+x){\rm Ai}(z+y)}{e^{-\gamma_t z}+1}\, dz \,. \tag{15}$$

Comparing Eqs. (10) for the fermions and (15) for KPZ, we see that the two kernels are the same $K_b^{\rm ff} = K_{t=b^3}^{\rm KPZ}$ in time unit t^*. While comparison of Eqs. (11) and (14) show that the cumulative distribution of $x_{\max}(T)$ for the free fermion problem is the same as the generating function $g_{t=b^3}(\zeta)$ in the KPZ equation.

Average density of fermions. The joint probability density function (PDF) of the positions of the N fermions is constructed from the single particle wave functions

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x) \,, \tag{16}$$

where H_k is the Hermite polynomial of degree k. In the canonical ensemble it is given by the Boltzmann weighted sum of slater determinants:

$$P_{\rm joint}(x_1, \cdots x_N) = \frac{1}{Z_N(\beta)} \sum_{k_1<\cdots<k_N} \left[\det_{1\leq i,j \leq N}(\varphi_{k_i}(x_j))\right]^2 \times e^{-\beta(\epsilon_{k_1}+\cdots+\epsilon_{k_N})} \,, \tag{17}$$

where $Z_N(\beta)$ is a normalization constant and where $\epsilon_k = \hbar\omega(k+1/2)$ are the single particle energy levels (in Eq. (17) all the k_i's range from 0 to ∞). We first compute the mean density of free fermions $\rho_N(x,T) = N^{-1}\sum_{i=1}^N \langle \delta(x-x_i)\rangle$, where $\langle \ldots \rangle$ means an average computed with (17). This amounts, up to a multiplicative constant, to integrating the joint PDF $P_{\rm joint}(x, x_2, \cdots x_N)$ over the last $N-1$ variables, yielding a rather complicated expression which, however, simplifies in the large N limit where the canonical ensemble and the grand-canonical ensemble become equivalent. Hence, for large N one obtains (see also [15, 16])

$$\rho_N(x,T) \approx \frac{1}{N}\sum_{k=0}^{\infty} \frac{[\varphi_k(x)]^2}{e^{\beta(\epsilon_k-\mu)}+1} \,, \tag{18}$$

where $1/(e^{\beta(\epsilon_k-\mu)}+1)$ is the Fermi factor and the chemical potential μ is fixed by imposing that mean number of fermions is N:

$$N = \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\epsilon_k-\mu)}+1} \,. \tag{19}$$

We first analyze Eqs. (18) and (19) in the scaling limit where, $N\to\infty$, with $\beta\to 0$ and $x\to\infty$ but keeping y and z defined in (7) fixed (implying in particular $\beta \sim 1/N$). In this limit, the sums in (18) and (19) can be replaced by integrals and, from (19), we find $e^{\beta\mu} = e^y - 1$. Then using the asymptotic behavior of the Hermite polynomials $H_n(x)$ for large degree n [23], one obtains the scaling form given by Eq. (8) together with the explicit expression of the scaling function given in (9).

We turn to Eqs. (18) and (19) in the scaling limit where $\beta\hbar\omega = bN^{-1/3}$, thus $y\to\infty$. Hence the average density is given by its, Wigner-semi-circle, $T=0$ limit (2). In this regime, an interesting scaling limit emerges close to the edges, for $x = \pm\sqrt{2N}/\alpha$. To analyze the behavior of $\rho_N(x,T)$ close to $x = \sqrt{2N}/\alpha$ [24] we insert the expression of $\beta\mu \sim bN^{2/3}$, obtained from (19), into (18) and perform a change of variable in the sum, by setting $k = N+m$, to obtain:

$$\rho_N(x,T) \sim \frac{1}{N}\sum_{m=-N}^{\infty} \frac{[\varphi_{N+m}(x)]^2}{\exp{(bm/N^{1/3})}+1} \,. \tag{20}$$

Using the Plancherel-Rotach formula for Hermite polynomials at the edge (see for instance Ref. [25]) yields:

$$\varphi_{N+m}\left(\frac{\sqrt{2N}}{\alpha} + \frac{s}{\sqrt{2}\alpha}N^{-\frac{1}{6}}\right) \sim \sqrt{\alpha}\frac{2^{\frac14}}{N^{\frac{1}{12}}}{\rm Ai}\left(s - \frac{m}{N^{\frac13}}\right) \,, \tag{21}$$

up to terms of order $\mathcal{O}(N^{-2/3})$. Hence by inserting this asymptotic formula (21) into Eq. (20) and replacing the discrete sum over m by an integral we obtain [23]:

$$\rho_N\left(\frac{\sqrt{2N}}{\alpha} + \frac{s}{\sqrt{2}\alpha}N^{-\frac16}, b^{-1}N^{\frac13}\hbar\omega\right) \sim \alpha N^{-\frac56}\tilde{\rho}_{\rm edge}(s) \,, \tag{22}$$

where $\tilde{\rho}_{\rm edge}(s)$ is given by

$$\tilde{\rho}_{\rm edge}(s) = \sqrt{2}K_b^{\rm ff}(s,s) \,, \tag{23}$$

and the kernel $K_b^{\rm ff}(s,s')$ is given in (10). In the zero-temperature limit $b\to\infty$, we recover $\tilde{\rho}_{\rm edge}(s) \sim \sqrt{2}\left[({\rm Ai}'(s))^2 - s{\rm Ai}^2(s)\right]$, the standard result for the mean density of eigenvalues at the edge of the spectrum of GUE random matrices [26, 27]. In Fig. 2, we show how $\tilde{\rho}_{\rm edge}(s)$ behaves for different values of the reduced inverse temperature b.

Kernel and higher order correlation functions. More generally, one can study the n-point correlation function for N free fermions at finite temperature. We define $R_n(x_1, \cdots x_n)$ as

$$R_n(x_1,\cdots,x_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty} dx_{n+1}\cdots\int_{-\infty}^{\infty} dx_N \times P_{\rm joint}(x_1,\cdots,x_n,x_{n+1},\cdots,x_N) \tag{24}$$

where $P_{\rm joint}(x_1,\cdots,x_N)$ is the joint PDF of the N fermions at finite temperature of Eq. (17) [28]. Using the equivalence, in the large N limit, between the canonical and grand-canonical ensembles, one can show that

$$R_n(x_1,\cdots,x_n) \approx \det_{1\leq i,j\leq n} K_N(x_i,x_j) \tag{25}$$

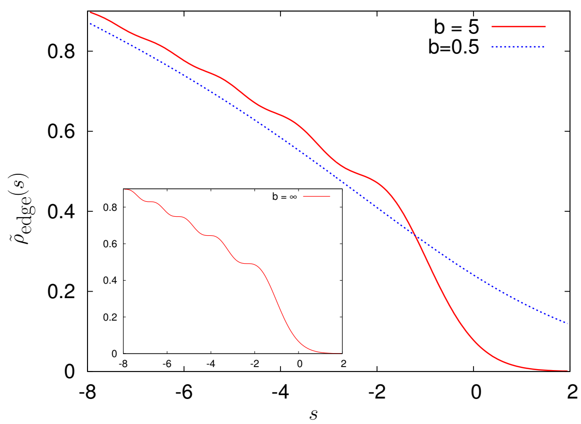

FIG. 2. (Color online) Plot of $\tilde{\rho}_{\rm edge}(s)$ given in Eq. (23) corresponding to two different (scaled) temperatures $b = 0.5$ (dotted line) and $b = 5$ (solid line). **Inset:** plot of $\tilde{\rho}_{\rm edge}(s)$ corresponding to $b \to \infty$ given in the text and shown here for comparison with the main plot.

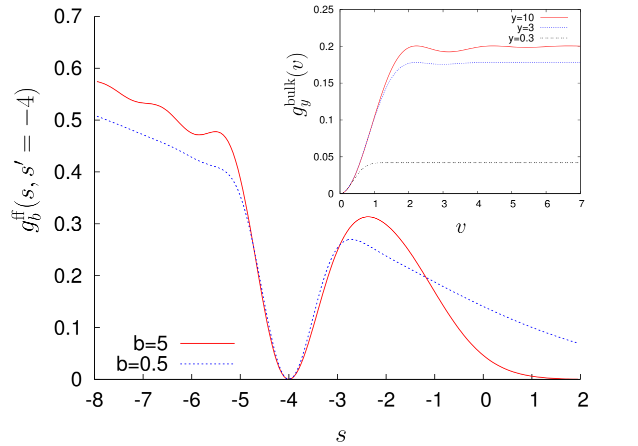

FIG. 3. (Color online) Plot of the 2-point correlation function at the edge $g_b^{\rm ff}(s, s' = -4)$ as a function of s and for scaled inverse temperatures $b = 0.5$ and 5. **Inset:** Plot of the 2-point correlation function in the bulk $g_y^{\rm bulk}(v)$ versus v for scaled inverse temperatures $y = 0.3, 3$ and 10.

where the kernel $K_N(x, x')$ is given by

$$K_N(x, x') = \sum_{k=0}^{\infty} \frac{\varphi_k(x)\varphi_k(x')}{e^{\beta(\epsilon_k - \mu)} + 1} \, . \tag{26}$$

We first analyze the kernel (26) in the scaling limit where $N \to \infty$ and $\beta \to 0$ keeping y in (7) fixed (i.e., $\beta \sim \mathcal{O}(1/N)$). In this limit, if we are interested in the behavior of $K_N(x, x')$ in the *bulk* where both $x \sim (u/\alpha)N^{-1/2}$ and $x' \sim (u'/\alpha)N^{-1/2}$ are close to the origin, one finds (see also [16])

$$\lim_{N\to\infty} \frac{N^{-\frac{1}{2}}}{\alpha} K_N\left(\frac{u}{\alpha}N^{-\frac{1}{2}}, \frac{u'}{\alpha}N^{-1/2}\right) \sim K_y^{\rm bulk}(u - u')$$

$$K_y^{\rm bulk}(v) = \frac{1}{\pi\sqrt{2y}} \int_0^{\infty} \frac{\cos\left(\sqrt{\frac{2p}{y}}v\right)}{(1 + e^p/(e^y - 1))\sqrt{p}} \, dp \, . \tag{27}$$

Note that in the low temperature limit $y \to \infty$, the Fermi factor in Eq. (27) behaves like a theta function, $\propto \theta(y - p)$, implying that $\lim_{y\to\infty} K_y^{\rm bulk}(v)/K_y^{\rm bulk}(0) = [\sin(v\sqrt{2})]/(v\sqrt{2})$, which (up to a scaling factor) is the expected sine-kernel. In the inset of Fig. 3 we show a plot of the pair-correlation function $g_y^{\rm bulk}(s) = [K_y^{\rm bulk}(0)]^2 - [K_y^{\rm bulk}(s)]^2$ for different values of the scaled inverse temperature y.

However, in the low temperature scaling limit where $\beta\hbar\omega = bN^{-1/3}$, the kernel in the bulk is given by the sine-kernel while the interesting behavior occurs at the edge $x \sim \pm\sqrt{2N}/\alpha$. In this limit, performing the same analysis as above (20)-(23), one finds that the kernel $K_N(x, x')$ (26) takes the scaling form:

$$\lim_{N\to\infty} \frac{N^{-\frac{1}{6}}}{\sqrt{2}\alpha} K_N\left(\frac{\sqrt{2N}}{\alpha} + \frac{s}{\sqrt{2}\alpha}N^{-\frac{1}{6}}, \frac{\sqrt{2N}}{\alpha} + \frac{s'}{\sqrt{2}\alpha}N^{-\frac{1}{6}}\right) = K_b^{\rm ff}(s, s') \, , \tag{28}$$

where $K_b^{\rm ff}(s, s')$ is given in Eq. (10). In Fig. 3 we show a plot of the 2-point correlation function at the edge $g_b^{\rm ff}(s, s') = K_b^{\rm ff}(s, s)K_b^{\rm ff}(s', s') - [K_b^{\rm ff}(s, s')]^2$ for different scaled inverse temperatures b. The properties of determinantal point processes [29] then imply that the cumulative distribution of the position of the rightmost fermion $x_{\rm max}(T)$ is given by Eq. (11).

We have analyzed the effect of finite temperature on free spinless fermions in a harmonic trap in one dimension. The scaling function, showing how the average density of the system, in the bulk, crosses over from the Gaussian Gibbs-Boltzmann form at high temperatures (6) to the Wigner semi-circle law (2) at low temperatures, has been computed. For large N, the equivalence between canonical and grand canonical ensembles implies that free fermion statistics can be described as a determinantal process even at finite temperature. We derived the kernel for this process at finite temperature in the bulk, and also close to the edge of the bulk at low temperature. The statistics of the rightmost fermion turns out to be governed by a finite temperature generalization of the TW distribution. The temperature dependent kernel found here also exhibit a tantalizing connection with the one appearing in exact solutions of the KPZ equation [19–22] at finite times t with the correspondence $t = 2b^3$. This connection in fact holds for generic confining potentials such that $V(x) \sim x^p/p$, with $p > 0$, for x large; the kernel scaling is universal, and only the scalings with N are modified [23].

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[29] See e.g. K. Johansson, *Random matrices and determinantal processes*, in Lecture Notes of the Les Houches Summer School 2005 (A. Bovier, F. Dunlop, A. van Enter, F. den Hollander, and J. Dalibard, eds.), Elsevier Science, (2006); arXiv:math-ph/0510038.