Field theory conjecture for loop-erased random walks

Andrei A. Fedorenko, Pierre Le Doussal, and Kay J. Wiese
CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France
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We give evidence that the functional renormalization group (FRG), developed to study disordered systems, may provide a field theoretic description for the loop-erased random walk (LERW), allowing to compute its fractal dimension in a systematic expansion in $\varepsilon = 4 - d$. Up to two loop, the FRG agrees with rigorous bounds, correctly reproduces the leading logarithmic corrections at the upper critical dimension $d = 4$, and compares well with numerical studies. We obtain the universal subleading logarithmic correction in $d = 4$, which can be used as a further test of the conjecture.

The loop-erased random walk (LERW) was introduced by Lawler [1] as an alternative to the self-avoiding walk (SAW), which is relevant in polymer physics. On the lattice, the LERW is defined as the trajectory of a random walk in which any loop is erased as soon as it is formed. As the SAW, the LERW has no self-intersections, but is more tractable mathematically. It has been proven that the LERW has a scaling limit in all dimensions [2, 3, 4, 5, 6, 7]. The number of steps (or time) t it takes to reach the distance L scales as $t_L \sim L^z$, where z is the fractal dimension of LERW. In $d = 2$ this scaling limit is conformally invariant and described [8, 9] by the stochastic Loewner evolution SLE_2. Though the LERW and SAW belong to different universality classes, the LERW itself has received significant attention due to applications in combinatorics, self-organized criticality (SOC), and, more recently, conformal field theory and SLE. The LERW can be viewed as a special case of the Laplacian random walk [10] and can be mapped [4] to the problem of uniform spanning trees (UST): Chemical (i.e. shortest) paths on UST obey LERW statistics. It is proven that the upper critical dimension is $d_{\mathrm{uc}} = 4$, the same as for the SAW, since for $d > 4$ the traces of two random walks do almost surely not intersect. Hence for $d > 4$ the fractal dimension of the LERW is that of a simple random walk, $z = 2$. It was proven a while ago [11] that for $d < 4$ it is bounded from above by the Flory estimate for the SAW exponent:

$$R^2 \sim t^{2/z} \quad , \quad z < \frac{d+2}{3} = 2 - \frac{\varepsilon}{3}, \tag{1}$$

where $R = \langle R(t)^2 \rangle^{1/2}$ is the radius of gyration, and we have introduced $\varepsilon = 4 - d$. The mapping to UST and equivalently to the q-state Potts model at $q \to 0$ was used in $d = 2$ to predict [4] $z_{\mathrm{LERW}}(d = 2) = \frac{5}{4}$, later proved in Ref. 12. It is a particular case, for $\kappa = 2$, of the fractal dimension $d_{\mathrm{f}} = 1 + \frac{\kappa}{8}$ of the trace of SLE_κ (a simple curve for $\kappa < 4$). Another connection in $d = 2$ is to the $O(N)$ loop model with $N = -2$, both corresponding to a conformal field theory (CFT) with central charge $c = -2$. The leading logarithmic corrections at the upper critical dimension $d = 4$ where obtained by Lawler [5] who proved that:

$$R^2 \sim t(\ln t)^{1/3}. \tag{2}$$

In $d = 3$ the value of z_{LERW} is known only from numerics. The most precise estimate was obtained by Agrawal and Dhar [13],

$$z_{\mathrm{LERW}}(d = 3) = 1.6183 \pm 0.0004\,, \tag{3}$$

improving on the previous numerical result [14] $z = 1.623 \pm 0.011$.

Contrary to the SAW, which is described by the $O(N)$ model at $N = 0$, there seems to be at present no field-theoretic approach to compute the LERW exponent in a dimensional expansion around $d = 4$. This is surprising, especially when compared with the recent progress in CFT descriptions in $d = 2$. In this short paper we propose a field theoretic description for the LERW, based on the Functional RG, a method developed to study disordered systems. We build on a connection proposed a while ago between the depinning transition of periodic elastic systems, also called charge density waves (CDW) in random media and sandpile models. The correspondence, on which we detail below, is indirect, through a chain of related models: From LERW to UST to sandpiles to CDW-depinning and finally to functional renormalization group (FRG) field theory. Some of the connections are not rigorous. At the end we show that the FRG passes all the tests of presently known results for LERW. In particular, it reproduces the correct leading logarithmic corrections in $d = 4$ given by Eq. (2), and makes a prediction for the subleading logarithmic correction which we hope will be tested in the near future.

We now present briefly the intermediate models. The Bak-Tang-Wiesenfeld (BTW) sandpile model was proposed as a prototype for driven dissipative systems exhibiting SOC [15]. It is defined on the d-dimensional hyper-cubic lattice with L^d sites. The configuration at time t is given by the integer number of grains $h(x, t)$ at site x. The site x is unstable if $h(x, t) > 2d$ in which case it relaxes according to the toppling rule

$$\begin{aligned} h(x, t+1) &= h(x, t) - 2d\,, \\ h(y, t+1) &= h(y, t) + 1\,, \end{aligned} \tag{4}$$

where y denotes all the $2d$ nearest neighbors of site x. The neighbor sites may then become unstable and the toppling continue. The order of topplings is irrelevant, thus the notion of "Abelian" sandpile, which allows to use e.g. parallel dynamics. The process continues until no unstable site remains, i.e. the avalanche ends. This is achieved through grains that leave the system at the boundary. To ensure a steady state one drives the system by adding a grain to a randomly chosen site x after each avalanche. Stable configurations are either transient, or recurrent in which case they appear in the stationary state with equal probability [16]. The recurrent configurations are shown to be in one-to-one correspondence with spanning

TABLE I: Padé approximants for z in $d = 3$.

$[n/m]$	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	2	1.71429	1.6	1.61074
$n=1$	1.66667	1.5	1.61194	
$n=2$	1.55556	1.63158		
$n=3$	1.60069			

trees [17] on the same lattice plus a sink (into which fall all grains leaving the system through the boundary). These spanning trees are connected sets of edges touching all lattice sites with no loops, and chosen with uniform probability (UST). Thus the time t and length L in the sandpile dynamics are related by the fractal dimension of the chemical path along a spanning tree, which is nothing but a LERW.

Narayan and Middleton proposed that the charge density wave (CDW) near the depinning transition can be viewed to some extent as a BTW sandpile model [18, 19]. The configuration of a d-dimensional elastic object, such as a CDW moving in a disordered medium is parameterized by a displacement field u_{xt}, $x \in \mathbb{R}^d$. In the continuous limit, the dynamics is described by the over-damped equation of motion

$$\eta \partial_t u_{xt} = \nabla^2 u_{xt} + F(x, u_{xt}) + f, \tag{5}$$

where η is the friction, f the driving force, and F the random pinning force. The latter is taken to be Gaussian with zero mean and correlator

$$\overline{F(x,u)F(x',u')} \;=\; \Delta(u-u')\delta^d(x-x'). \tag{6}$$

where $\Delta(u)$ is an even periodic function (chosen here with period 1). Discretizing space and time, one can rewrite Eq. (5) near the depinning transition [32] in the form of the automaton model (4) with $h(x,t)$ playing the role of a coarse-grained curvature of the elastic object. Both models, the original BTW model and the discretized version of Eq. (5), are driven by adding grains, $h(x,t+1) = h(x,t)+1$. In the CDW model grains are added with a cycle restriction [33]. While this difference may seem inconsequential, it illustrates that the mapping is presently not rigorous; moreover an ad-hoc discretization must be used. Nevertheless it is supported by numerics [18]. It strongly suggests that the dynamic exponent z describing the depinning transition of a d-dimensional CDW coincides with the fractal dimension of LERW in d dimensions.

The field theory which describes the depinning transition for system (5) is based on the Functional RG [20]. Recent progress has shown that full consistency requires a 2-loop study. The coupling constant of the theory is *a function*. The corresponding flow equations to 2-loop order read [21]

$$\begin{aligned}\partial_l \Delta(u) &= \varepsilon\Delta(u) - \frac{1}{2}[(\Delta(u)-\Delta(0))^2]'' \\ &+\frac{1}{2}\left[(\Delta(u)-\Delta(0))\,\Delta'(u)^2\right]'' + \frac{1}{2}[\Delta'(0^+)]^2\Delta''(u), \end{aligned} \tag{7}$$

$$\partial_l \ln\eta = -\Delta''(0) + \Delta''(0)^2 + \Delta'''(0^+)\Delta'(0^+)\left[\frac{3}{2} - \ln 2\right], \tag{8}$$

where $l = \ln L$, and L the infrared cutoff, e.g. the size of the system. Below $d = 4$ the flow equation (7) has a fixed point (FP) solution $\Delta^*(u)$ with a cusp at the origin: $\Delta^{*\prime}(0^+) \neq 0$. Taking into account that $t \sim \eta_l L^2$ and that in the vicinity of the FP η_l scales with L according to Eq. (8), one finds that $t \sim L^z$, and to 2-loop order the exponent z is given by [21]:

$$z = 2 - \frac{\varepsilon}{3} - \frac{\varepsilon^2}{9} + O(\varepsilon^3). \tag{9}$$

Comparison with the exact bound (1) is encouraging since the 2-loop correction has the correct sign. To estimate values of z in $d = 2, 3$ we compute different Padé approximants $[n/m]$ to (9). In $d = 2$ we obtain $z(d=2) = 1.23 \pm 0.22$, consistent with the presumed exact value $\frac{5}{4}$ but of poor accuracy [34] due to the large expansion parameter $\varepsilon = 2$. Table I contains the approximants $[n/m]$ in $d = 3$. To improve the accuracy we construct the higher order Padé approximants with $n+m = 3$ imposing that $z(d=2) = \frac{5}{4}$ for $\varepsilon = 2$. The average and root-mean-square of all approximants with $n+m=3$ yields

$$z = 1.614 \pm 0.011\ . \tag{10}$$

The same procedure based on one loop only produces $z = 1.638 \pm 0.012$. The value (10) is our best 2-loop FRG prediction for $d = 3$ and is in fairly good agreement with the numerical result (3).

The field theory is most predictive at the upper critical dimension, where it yields exact results. For FRG some were obtained previously (see e.g. Refs. 22, 23, 24). Here we compute the logarithmic corrections to the dynamics which yield a prediction for the LERW in $d = 4$. For $\varepsilon = 0$ one shows from (7) that the (periodic) disorder correlator approaches the FP solution $\Delta^*(u) = 0$ as follows (for $0 \leq u \leq 1$):

$$\Delta_l(u) = \left[\frac{1}{6l} + \frac{\ln l}{9l^2}\right]\left[\frac{1}{6} - u(1-u)\right] + O\!\left(\frac{1}{l^2}\right). \tag{11}$$

Substituting Eq. (11) into Eq. (8) we obtain

$$\ln\frac{\eta_l}{\eta_0} = -\frac{1}{3}\ln l + \frac{2\ln l}{9l} + O\!\left(\frac{1}{l}\right). \tag{12}$$

Renormalizing the relation $t \sim \eta_l L^2$ up to scale L we arrive at

$$t \sim L^2(\ln L)^{-1/3}\left[1 + \frac{2\ln\ln L}{9\ln L} + O\!\left(\frac{1}{\ln L}\right)\right], \tag{13}$$

which can be rewritten as

$$L^2 \sim t(\ln t)^{1/3}\left[1 - \frac{\ln\ln t}{3\ln t} + O\!\left(\frac{1}{\ln t}\right)\right]. \tag{14}$$

The scale L^2 can be taken as R^2, the radius of gyration. We note that the leading order of Eq. (14) coincides with the result (2) of Lawler. Here we obtain the universal subleading correction.

The prediction (14) could be tested numerically, very much as for the corresponding prediction [25] for the SAW in $d = 4$, checked in [26]; there subleading corrections are necessary to properly fit the numerical data at any feasible chain length. To this purpose, it is useful to note that, as for the SAW, there is only a single non-universal constant in the correction term in the parenthesis in (14), $c/\ln t$. It can e.g. be put to zero by a proper choice of t_0, setting $t \to t/t_0$ (in which case further corrections are $1/\ln^2 t$ and determined by 3- and higher-loop terms, not considered here). For comparison, one similarly finds, from the 2-loop β-function [27] of the $O(N)$ model in $d=4$ that $L^2 \sim t(\ln t)^{a_N}(1-b_N \frac{\ln\ln t}{\ln t})$ with $a_N=(N+2)/(N+8)$ and $b_N=(N+2)(68+8N-N^2)/(N+8)^3$ from which the Duplantier values for the SAW [25], $a_0=1/4$ and $b_0=17/64$ are retrieved at $N=0$. Note that no value of N can account for (14), thus a representation of LERW via the $O(N)$ field theory, if feasible at all, would at least require a more complicated operator correspondance.

Note finally that the exponent τ for the avalanche-size distribution [35] was recently computed to one loop within FRG [28, 29]. In the sandpile literature no controlled calculation exists for the corresponding exponent, usually called τ_s, but the formula $\tau_s(d)=2-2/d$ leading to $\tau_s=4/3$ in $d=3$ has been conjectured [13] from scaling arguments. The FRG was found to agree to $O(\epsilon)$ with this formula, assuming $\tau=\tau_s$. Evaluation of 2-loop corrections is in progress as a further test of the conjecture and of the relations between sandpile models and depinning.

To conclude, it may appear surprising that LERW be related to a field theory based on functional RG, where one would expect a “simpler”, more conventional, field theory based on a single relevant coupling constant, as for the SAW. One clue may be that for periodic systems (CDW) the FRG posesses a stable submanifold with only two coupling constants, $\Delta(u)=a+bu(1-u)$ (for $0\le u\le 1$), which contains the leading critical behavior. From (7) one obtains [21, 29] the spectrum of convergence to the fixed point [36] as $1/(\ln L)^{1+\alpha_n}$ in $d=4$ and $L^{-\omega_n}$ with $\omega_n=\alpha_n\epsilon+\beta_n\epsilon^2+O(\epsilon^2)$ for $d<4$; $\alpha_n=\frac{1}{3}(3+n)(1+2n)$, $\beta_n=-\frac{1}{9}(n+2)(2n+1)(2n+3)$. The values $n=1,2,\ldots$ correspond to the convergence to the submanifold, and $n=0$ to the convergence inside it. Convergence to the submanifold is fast, with leading eigenvalue $1+\alpha_1=5$. Fast convergence was also found in numerics [30] where the FRG function $\Delta(u)$ in (7) was directly *measured* in $d=1,2,3$. The conjecture raises many other interesting issues to be explored, such as the possibility to predict other LERW observables and corrections to scaling, the role of other universality classes for CDW and interface depinning, the connections to fermionic field theory (known for UST, see e.g. [31], and conjectured for the FRG [24]). Work is in progress in these directions.

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[32] The system driven by a homogenous force f is said to be at the depinning transition, when an increase in f would lead to a never-ending motion.
[33] A second grain can be added to a particular site only if all other sites have received a grain.
[34] We used the average and root mean squared of the set $z_{[1/0]}=4/3$, $z_{[0/1]}=3/2$, $z_{[2/0]}=8/9$, $z_{[0/2]}=6/5$. We excluded approximant $[1/1]$ which accidentally vanishes.
[35] The probability to have an avalanche of size s is proportional to $s^{-\tau}$, with a cutoff for large and small avalanches.
[36] Apart from the trivial uniform mode of eigenvalue ϵ.