# Random-Field Spin Models beyond 1 Loop: A Mechanism for Decreasing the Lower Critical Dimension

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The functional renormalization group for the random-field and random-anisotropy $O(N)$ sigma models is studied to 2 loop. The ferromagnetic-disordered (F-D) transition fixed point is found to next order in $d = 4 + \epsilon$ for $N > N_c$ ($N_c = 2.834\,740\,8$ for random field, $N_c = 9.441\,21$ for random anisotropy). For $N < N_c$ the lower critical dimension $d = d_{\rm lc}$ plunges below $d_{\rm lc} = 4$: we find *two* fixed points, one describing the quasiordered phase, the other is novel and describes the F-D transition. $d_{\rm lc}$ can be obtained in an $(N_c - N)$ expansion. The theory is also analyzed at large $N$ and a glassy regime is found.

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It is important for numerous experiments to understand how the spontaneous ordering in a pure system is changed by quenched substrate impurities. One class of systems is modeled by elastic objects in random potentials (so-called random manifolds, RMs). Another class is $O(N)$ classical spin models with ferromagnetic couplings in the presence of random fields (RFs) or anisotropies (RAs). The latter describe amorphous magnets [1]. Examples of RFs are liquid crystals in porous media, He-3 in aerogels, nematic elastomers, and ferroelectrics [2]. The $XY$ random-field case $N = 2$ is common to both classes and describes periodic RMs such as charge density waves, Wigner crystals, and vortex lattices [3]. Larkin showed [4] that the pure fixed points (FPs) of both classes are perturbatively unstable to weak disorder for $d < d_c$ ($d_c = 4$ in the generic case). For a continuous symmetry (i.e., the RF Heisenberg model) it was proven [5] that order is destroyed below $d = 4$. This does not settle the difficult question of the lower critical dimension $d_{\rm lc}$ as a weak-disorder phase can survive below $d_c$, if associated with a nontrivial FP, as predicted in $d = 3$ for the Bragg-glass phase with quasi-long-range order (QLRO), i.e., power law decay of spin correlations [6]. For the random-field Ising model (RFIM) $N = 1$, it was argued [7], then proven [8] that the ferromagnetic phase survives in $d = 3$. Developing a field theory to predict $d_{\rm lc}$, and the exponents of the weak-disorder phase and the ferromagnetic-disordered (F-D) transition, has been a long-standing challenge. Both extensive numerics and experiments have not yet produced an unambiguous picture. Among the debated issues are the critical region of the 3D RFIM [9] and the possibility of a QLRO phase in amorphous magnets [2,10].

A peculiar property shared by both classes is that observables are identical to all orders to the corresponding ones in a $d-2$ thermal model [11]. This dimensional reduction (DR) naively predicts $d_{\rm lc} = 4$ for the weak-disorder phase in a RF with a continuum symmetry [12] and no ferromagnetic order for the $d = 3$ RFIM, which is proven wrong [8]. It also predicts $d_{\rm uc} = 6$ for the F-D *transition* FP. While there is agreement that multiple local minima are responsible for DR failure, constructing the field theory beyond DR is a formidable challenge. Recent attempts include a reexamination of $\phi^4$ theory (i.e., soft spins) for the F-D transition near $d = 6$ [13], and large-$N$ studies [14], using replica-symmetry breaking.

As for the pure $O(N)$ model, an alternative to the soft-spin version (near $d = 6$) is the $\sigma$ model near the lower critical dimension (here presumed to be $d = 4$). In 1985, Fisher [15] noticed that an infinite set of operators become relevant near $d = 4$ in the RF $O(N)$ model. These were encoded in a single function $R(\phi)$ for which functional renormalization group (FRG) equations were derived to 1 loop, but no new FP was found. For a RM problem [16] it was found that a cusp develops in the function $R(\phi)$ (the disorder correlator), a crucial feature which allows one to obtain nontrivial exponents and evade DR. A fixed point for the RF model was later found [6] in $d = 4 - \epsilon$ for $N = 2$. It was noticed only very recently [17] that the 1-loop FRG equations of Ref. [15] possess fixed points in $d = 4 + \epsilon$ for $N \geq 3$, providing a description of the long-sought critical exponents of the F-D transition.

In spite of these advances, many questions remain. Constructing FRG beyond 1 loop (and checking its internal consistency) is highly nontrivial. Progress was made for RMs [18,19], and one hopes for extension to RFs. Some questions necessitate a 2-loop treatment, e.g., for the depinning transition, as shown in [20]. In RF and RA models the 1-loop analysis predicted some repulsive FPs in $d = 4 + \epsilon$ (for larger values of $N$), and some attractive ones [6,21] in $d = 4 - \epsilon$. The overall picture thus suggests a lowering of the critical dimension, but how it occurs remains unclear. Finally, the situation at large $N$ is also puzzling. Recently, via a truncation of exact RG [22] it was claimed that DR is recovered for large $N$.

Our aim in this Letter is twofold. We reexamine the overall scenario for the fixed points and phases of the $O(N)$ model using FRG. This requires the FRG to 2 loop. Here we present selected results; details are presented elsewhere [23]. We find a novel mechanism for how the lower critical dimension is decreased below $d = 4$ for

 

$N < N_c$ at some critical value $N_c$. We obtain a description of the bifurcation which occurs at $N_c$, and below $N_c$ we find *two* perturbative FPs. Thanks to 2-loop terms, $d_{\rm lc}$ can be computed in an expansion in $N_c - N$, and the F-D transition below $d = 4$ is found. A study of large $N$ indicates that some glassy behavior survives there.

Let us consider $O(N)$ classical spins $\vec{n}(x)$ of unit norm $\vec{n}^2 = 1$. To describe disorder-averaged correlations one introduces replicas $\vec{n}_a(x)$, $a = 1, \ldots, k$, the limit $k = 0$ being implicit everywhere. The starting model is a nonlinear $\sigma$ model of partition function $\mathcal{Z} = \int \mathcal{D}[\pi] e^{-\mathcal{S}[\pi]}$:

$$\mathcal{S}[\pi] = \int d^d x \bigg[ \frac{1}{2T_0} \sum_a [(\nabla \vec{\pi}_a)^2 + (\nabla \sigma_a)^2] - \frac{1}{T_0} \sum_a M_0 \sigma_a - \frac{1}{2T_0^2} \sum_{ab} \hat{R}_0(\vec{n}_a \vec{n}_b) \bigg], \quad (1)$$

where $\vec{n}_a = (\sigma_a, \vec{\pi}_a)$ with $\sigma_a(x) = \sqrt{1 - \vec{\pi}_a(x)^2}$. A small uniform external field $\sim M_0(1, \vec{0})$ acts as an infrared cutoff. Fluctuations around its direction are parametrized by $(N-1)$ $\pi$ modes. The ferromagnetic exchange produces the 1-replica part, while the random field yields the 2-replica term $\hat{R}_0(z) = z$ for a bare Gaussian RF. RA corresponds to $\hat{R}_0(z) = z^2$. As shown in [15], a full function $\hat{R}(z)$ is generated under RG and marginal in $d = 4$.

To obtain physics at large scales, one computes perturbatively the effective action $\Gamma[n_a(x)]$. It can be expanded in gradients near a uniform background configuration $n_a^0$, and split in 1-, 2- and higher-replica terms. From rotational invariance it is natural to look for $\Gamma$ in the form (1) with $\vec{n}_a \to \vec{n}_a^R = (\sigma_a^R, \vec{\pi}_a^R)$, $\sigma_a \to \sigma_a^R = \sqrt{1 - (\pi_a^R)^2}$, $\pi_a \to \pi_a^R = Z^{-1/2} \pi_a$, $T_0 \to T_R = T_0/Z_T$, $M_0 \to M_R = M_0 \sqrt{Z}/Z_T$, $m = \sqrt{M_R}$ the renormalized mass of the $\vec{\pi}_a$ modes, and $\hat{R}_0(\vec{n}_a \vec{n}_b) \to m^\epsilon \hat{R}(\vec{n}_a^R \vec{n}_b^R)$. Higher vertices generated under RG are irrelevant by power counting, and hence discarded. Renormalization of $T$ contributes to the flow of $\hat{R}$, and one sets $T = 0$ at the end.

One computes $Z$, $Z_T$, and $\hat{R}$ perturbatively in $\hat{R}_0$ and extracts $\beta$ and $\gamma$ functions $\beta[\hat{R}](z) = -m\partial_m \hat{R}(z)$, $\gamma = -m\partial_m \ln Z$, and $\gamma_{\rm T} = -m\partial_m \ln Z_T$, derivatives taken at fixed $\hat{R}_0, T_0, M_0$. Although calculation of the $Z$ factors is simplified due to DR, anomalous contributions appear from the nonanalyticity of $\hat{R}(z)$. To compute $\hat{R}(z)$, one chooses a pair of uniform background fields $(n_a^0, n_b^0)$ for each $(a, b)$. We use a basis for the fluctuating fields (to be integrated over) such that $\vec{n}_a = (\sigma_a, \eta_a, \vec{\rho}_a)$, $\vec{n}_b = (\sigma_b, \eta_b, \vec{\rho}_b)$, where $\eta$ lies in the plane common to $(\vec{n}_a^0, \vec{n}_b^0)$, and $\vec{\rho}_a$ along the perpendicular $N - 2$ directions; both have diagonal propagators. Denoting $\vec{n}_a^0 \vec{n}_b^0 = \cos\phi_{ab}$, one has $\vec{n}_a \vec{n}_b = \cos\phi_{ab}(\sigma_a \sigma_b + \eta_a \eta_b) + \sin\phi_{ab}(\sigma_a \eta_b - \sigma_b \eta_a) + \vec{\rho}_a \vec{\rho}_b$. One gets factors of $(N-2)$ from the contraction of $\vec{\rho}$. Our calculation to 2 loops results in the flow equation for the function $R(\phi) = \hat{R}(z = \cos\phi)$, and $\epsilon = 4 - d$:

$$\begin{aligned}\partial_\ell R(\phi) &= \epsilon R(\phi) + \frac{1}{2} R''(\phi)^2 - R''(0) R''(\phi) + (N-2)\bigg[\frac{1}{2}\frac{R'(\phi)^2}{\sin^2\phi} - \cot\phi R'(\phi) R''(0)\bigg] + \frac{1}{2}[R''(\phi) - R''(0)] R'''(\phi)^2 \\ &\quad + (N-2)\bigg[\frac{\cot\phi}{\sin^4\phi} R'(\phi)^3 - \frac{5 + \cos 2\phi}{4\sin^4\phi} R'(\phi)^2 R''(\phi) + \frac{1}{2\sin^2\phi} R''(\phi)^3 - \frac{1}{4\sin^4\phi} R''(0)[2(2 + \cos 2\phi) R'(\phi)^2 \\ &\quad - 6\sin 2\phi R'(\phi) R''(\phi) + (5 + \cos 2\phi)\sin^2\phi R''(\phi)^2]\bigg] - \frac{N+2}{8} R'''(0^+)^2 R''(\phi) - \frac{N-2}{4}\cot\phi R'''(0^+)^2 R'(\phi) \\ &\quad - 2(N-2)[R''(0) - R''(0)^2 + \gamma_a R'''(0^+)^2] R(\phi),\end{aligned} \quad (2)$$

with $\partial_l := -m\partial_m$, and the last factor proportional to $R(\phi)$ is $-2\gamma_{\rm T}$ and it takes into account the renormalization of temperature. Thanks to the anomalous terms, arising from a nonanalytic $R(\phi)$, this $\beta$ function preserves (at most) a linear cusp [i.e., finite $R'''(0^+)$], and reproduces for $N = 2$ the previous 2-loop results for the periodic RM [18]. For $N > 2$, anomalous contributions are determined following [24]. $\gamma$ is found as

$$\gamma = (N-1) R''(0) + \frac{3N-2}{8} R'''(0^+)^2, \quad (3)$$

either via a calculation of $\langle \sigma_a \rangle$ [25] or of the mass corrections, a result consistent with the $\beta$ function (2) [26]. The determination of $\gamma_{\rm T}$ is more delicate, and we have allowed for an anomalous contribution $\gamma_a$, whose effect is minor and discussed below [27]. The correlation exponents (standard definition [17]) are obtained as $\bar{\eta} = \epsilon - \gamma$, $\eta = \gamma_{\rm T} - \gamma$ at the FP. (2) has the form

$$\partial_\ell R = \epsilon R + {\rm B}(R, R) + {\rm C}(R, R, R) + O(R^4). \quad (4)$$

We now discuss its solution, first in the RF case, and setting $\gamma_a = 0$. The 1-loop flow-equation (setting $C = 0$) admits, in dimensions larger than 4, a fixed point $R^*_{\rm F\text{-}D}$ with a single repulsive direction, argued by Feldman to describe the F-D zero temperature transition. This is true only for $N > N_c$. For $N < N_c$ this fixed point *disappears* and instead an *attractive fixed point* $R^*_{\rm QLRO}$ appears which describes the Bragg glass for $N = 2$. We have determined $N_c = 2.834\,740\,8$ and the solution $R_c(u)$ which satisfies ${\rm B}(R_c, R_c)|_{N=N_c} = 0$. It is formally the solution at $\epsilon = 0$. Since the FRG flow vanishes to 1 loop along the direction of $R_c$, examination of the 2-loop terms is needed to understand what happens at $N = N_c$. In particular, the F-D transition should still exist for $N < N_c$, though it cannot be found at 1 loop. It is not even clear *a priori* whether it remains perturbative.

The scenario found is perturbative, accessible within a double expansion in $\sqrt{|\epsilon|}$ and $N - N_c$. To this aim, we write the leading terms in $N - N_c$ and $\epsilon$ of (4), namely,

$$\partial_\ell R = \epsilon R + \mathrm{B}_c(R,R) + \mathrm{C}_c(R,R,R) + (N - N_c)\mathrm{B}_N(R,R)\mathrm{B}_c(\cdots) = \mathrm{B}(\cdots)|_{N=N_c},$$
$$\mathrm{C}_c(\cdots) = \mathrm{C}(\cdots)|_{N=N_c}. \quad (5)$$

One looks for a fixed point of (5) of the form $R(u) = gR_c(u) + g^2\delta R(u)$, with $g>0$, $R_c''(0) = -1$, and its flow. Surprisingly, close to $N = N_c$, the functional flow for the disorder $R$ is captured by an equation for its strength $g$:

$$\partial_l g = \epsilon g + 1.092(N - N_c)g^2 + 2.352g^3. \quad (6)$$

The solution is shown schematically in Fig. 1. Setting $g = (N_c - N)f$, there are three FPs (for exponents see [23]):

$$\frac{\epsilon}{(N - N_c)^2} - 1.092f + 2.352f^2 = 0 \quad \text{or} \quad f = 0. \quad (7)$$

For $N > N_c$ the physical branch is $f < 0$. As seen in Fig. 1, for $d > 4$ there is a ferromagnetic phase (i.e., $f = 0$ is attractive) and an unstable FP describing the F-D transition, given by the negative branch of (7). At $N = N_c$ one sees from (6) that the F-D fixed point is still perturbative but in a $\sqrt{\epsilon}$ expansion for $g$ (and for the critical exponents). For $N < N_c$ the physical side is $f > 0$ and there are two branches in Fig. 2 corresponding to two nontrivial fixed points. One is the infrared attractive FP for weak disorder which describes the quasiordered ferromagnetic phase; the second one is unstable and describes the transition to the disordered phase with a flow to strong coupling. These two fixed points exist only for $\epsilon < \epsilon_c$ and annihilate at $\epsilon_c$. The lower critical dimension of the RF model for $N < N_c$ is lowered from $d = 4$ to

$$d_{\mathrm{lc}}^{\mathrm{RF}} = 4 - \epsilon_c \approx 4 - 0.1268(N - N_c)^2 + O((N - N_c)^3). \quad (8)$$

Note that the mechanism is different from the more conventional criterion $d - 4 + \eta(d) = 0$ at $d = d_{\mathrm{lc}}$.

The same analysis for the random anisotropy class yields $N_c = 9.441\,21$. The equivalent of (6) becomes $\partial_l g = \epsilon g + 0.549(N - N_c)g^2 + 47.6g^3$, leading to $d_{\mathrm{lc}}^{\mathrm{RA}} \approx 4 - 0.001\,58(N - N_c)^2$. Although it yields $d_{\mathrm{lc}}(N = 3) \approx 3.93$ and no QLRO phase in $d = 3$, naive extrapolation should be taken with caution given the high value of $N_c$. Numerical values for $d_{\mathrm{lc}}$ are changed for $\gamma_a \neq 0$, but the scenario is robust as long as $\gamma_a$ is smaller than some critical value $\gamma_c$ [28].

We now discuss the FRG flow equations for $N$ large. From a truncated exact RG, Tarjus and Tissier (TT) [22] found that the linear cusp of the F-D fixed point for $d > 4$ vanishes for $N > N^*(d)$, i.e., $R''''(0^+) = 0$, and that the nonanalyticity becomes weaker as $N$ increases (as $|\phi|^n$ with $n \sim N$). Analytical study of the derivatives of (2) confirms the existence of this peculiar FP to 2 loop and predicts $N^*(d, 2p)$, beyond which the set of $\{R^{(2k)}(0)\}$ for $k \le p$ admits a stable FP, with $R^{(2k-1)}(0^+) = 0$ for $k \le p$ and $R^{(2k-1)}(0^+) \neq 0$ for $k > p$. We find

$$N^*(d) = N^*(d, 4) = 18 + 49\epsilon/5 + \cdots, \quad (9)$$

which yields a slope roughly twice the one of Fig. 1 of [22]. This remarkable FP raises some puzzles. Although weaker than a cusp its nonanalyticity should imply some (weaker) metastability in the system. It is thus unclear whether DR is fully restored: to prove it one should rule out feedback from anomalous higher-loop terms in exponents or the $\beta$ function. Finally, one also wonders about its basin of attraction. As shown in Fig. 3, the FRG flow for $R''''(0)$ is still to large values if its bare value is large enough, indicating some tendency to glassy behavior.

To explore these effects we now study the F-D phase transition at large $N$ and $d > 4$. We obtain, at both large $N$ and fixed $d$ (extending Ref. [19]), and to 1 loop, the flow equation for the rescaled $\tilde{R}(z = \cos\phi) = NR(\phi)/|\epsilon|$:

$$\partial_l \tilde{R} = -\tilde{R} + 2\tilde{R}_1'\tilde{R} - \tilde{R}_1'\tilde{R}'z + \frac{1}{2}\tilde{R}'^2 = 0. \quad (10)$$

We denote $y(z) = \tilde{R}'(z)$, $y_0 = \tilde{R}'(1) = -NR''(0)/|\epsilon|$, and $r_4 = NR''''(0)/|\epsilon|$. There are two analytic FPs $\tilde{R}(z) = z - 1/2$ and $\tilde{R}(z) = z^2/2$, corresponding both to $y_0 = 1$ and to $r_4 = 1$ and $r_4 = 4$, respectively. This agrees with the flow of the derivatives for analytic $R(\phi)$: $\partial_l y_0 = y_0(y_0 - 1)$, and at $y_0 = 1$: $\partial_l r_4 = \frac{1}{3}(r_4 - 1)(r_4 - 4)$. The first FP is the large-$N$ limit of the TT fixed point; the second FP is repulsive and divides the region where $r_4 \to \infty$ [nonanalytic $R(\phi)$] in a finite RG time $l_c$ (Larkin scale). For $y_0 > 1$, we find a family of nonanalytic fixed points with a linear cusp, parametrized by an integer $n \ge 2$, such that $y_0 = n/(n - 1)$, $z = y - (y_0 - 1)(y/y_0)^n$. The solutions with $n$

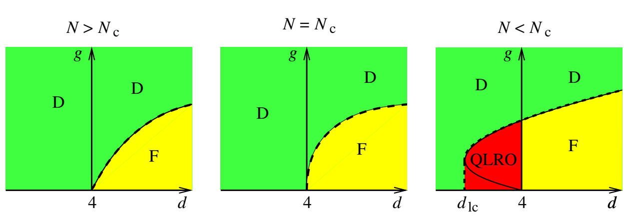


FIG. 1 (color online). Phase diagram. D = disordered; F = ferromagnetic; QLRO = quasi-long-range order.

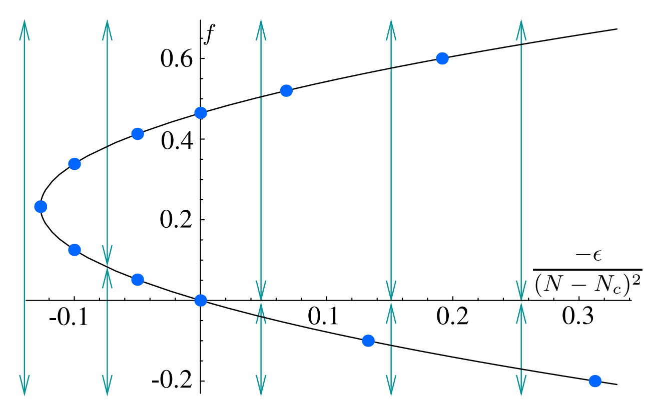


FIG. 2 (color online). Parametric plot for solutions of (5) for $N < N_c$ (solid circles) for RFs, equivalent to (6) (solid line, parabola) and flow (arrows). $f$ parametrizes disorder, and only $f \ge 0$ is physical. Compare with Fig. 1, right panel.

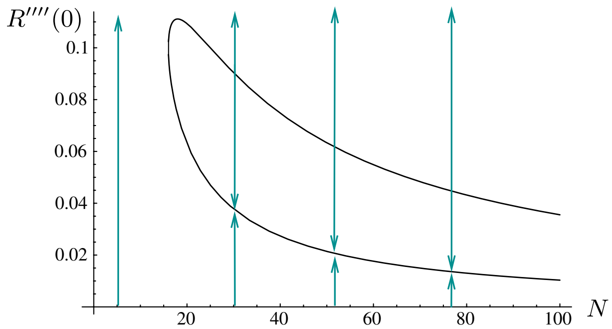


FIG. 3 (color online). Flow for $R''''(0)/|\epsilon|$ for $d = 4 - \epsilon > 4$ as a function of $N$. The two branches behave as $1/N$ and $4/N$ at large $N$.

[i.e., $z(y)$] odd correspond to random anisotropy ($R'(\phi) = R'(\phi + \pi)$). The $n = 2$ RF fixed point is $R(\phi) = 2\cos(\phi) + \frac{8\sqrt{2}}{3}\sin^3(\phi/2) - \frac{4}{3}$. To elucidate their role, we obtained the exact solution for the flow both below $l_c$, i.e., $z = \frac{y}{y_0} + (y_0 - 1)\Phi(\frac{y}{y_0})$ [$\Phi(x)$ parametrizes the bare disorder, $\Phi(1) = 0$], and above $l_c$, with an anomalous flow for $y_0$. Matching at $l_c$ yields the critical manifold for RF disorder, defined from the conditions that $\Phi'(w) = \Phi(w)/w = 1$ has a root $0 \leq w \leq 1$. It is different from the naive DR condition $y_0 = 1$, valid for small $r_4$. The $n = 2$ FP corresponds to bare disorder such that the root $w = 0$. Hence it is multicritical. Generic initial conditions within the critical F-D manifold flow back to the TT FP; i.e., the linear cusp decreases to zero [29], however, only at an infinite scale. Hence we expect a long crossover within a glassy region, characterized by a cusp, and metastability on finite scales. (The physics associated with a similar reentrant crossover for RM for $d > 4$ is discussed in Appendix H of [30].) The large-$N$ limit is subtle. Taking $N \to \infty$ at fixed volume on a bare model with $\hat{R}_0(z) = z$ yields only the analytic FP, equivalent to a replica-symmetric saddle point. Higher monomials $z^p$ are generated in perturbation theory, at higher order in $1/N$. Thus, for $N$ large but fixed and an infinite size, one must first coarse grain to generate a nontrivial function $\hat{R}_0(z)$, before taking $N \to \infty$.

In conclusion, we obtained the 2-loop FRG functions for the random field and anisotropy $\sigma$ models. We found a new fixed point and a scenario for the decrease of the lower critical dimension. This rules out the scenario left open at 1 loop that the bifurcation close to $d = 4$ simply occurs within the (quasi)ordered phase.

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