Universal interface width distributions at the depinning threshold

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We compute the probability distribution of the interface width at the depinning threshold, using recent powerful algorithms. It confirms the universality classes found previously. In all cases, the distribution is surprisingly well approximated by a generalized Gaussian theory of independant modes which decay with a characteristic propagator $G(q) = 1/q^{d+2\zeta}$; ζ, the roughness exponent, is computed independently. A functional renormalization analysis explains this result and allows to compute the small deviations, i.e. a universal kurtosis ratio, in agreement with numerics. We stress the importance of the Gaussian theory to interpret numerical data and experiments.

The scaling properties of driven elastic interfaces in random media play an important role in a wide variety of physical situations, ranging from stochastic surface growth to domain walls in disordered magnetic materials, the spreading of fluids on rough substrates, and the dynamics of cracks [1, 2]. These problems share many features with critical phenomena and provide a challenge for theoretical approaches to disordered systems and non-equilibrium phenomena [3–8].

Here we study interfaces described by a scalar height function $h(\mathbf{x})$, where $\mathbf{x}$ is the d-dimensional internal coordinate. We measure the deviation from the mean position as $u(\mathbf{x}) = h(\mathbf{x}) - \langle h \rangle$, where $\langle \ldots \rangle$ stands for the spatial average over all $\mathbf{x}$ of a given interface (cf. Fig. 1). The mean square width of a *single* interface, $w^2(\{u(\mathbf{x})\}) = \langle u^2 \rangle$, can be used to characterize its roughness, and explore universal properties: After averaging over the ensemble of interfaces, w^2 grows with the lateral extension L of the system as

$$\overline{w^2} \propto L^{2\zeta} \quad \text{for} \quad L \to \infty , \tag{1}$$

where ζ is the roughness exponent.

An interesting property is that, for positive ζ, w^2 fluctuates even in the thermodynamic limit [9–11]. This means that the long-range geometric features of the interface are not characterized by the roughness exponent alone, but require the complete probability distribution $P(w^2)$. $P(w^2)$ has been computed for several linear stochastic growth equations without disorder as the Edwards-Wilkinson model, the Mullins-Herring model, and the 1-d KPZ model [10, 11]. In these models, the probability distribution $P(w^2)$ can be rescaled into a form independent of system size and of microscopic details

$$P(w^2) = (1/\overline{w^2})\Phi(w^2/\overline{w^2}) \quad \text{for} \quad L \to \infty. \tag{2}$$

Although $\overline{w^2}$ may contain a non-universal scale, the function $\Phi(z)$ is universal. It has been argued that the shape of $P(w^2)$ can thus be used as a sensitive tool, distinct from ζ, to distinguish between different universality classes [9–12]. Furthermore, $\Phi(z)$ is expected to converge to a δ-function above the upper critical dimension d_{uc}. This has motivated attempts to determine d_{uc} for e.g. the KPZ equation [13]. Probability distributions of order parameters have received much attention

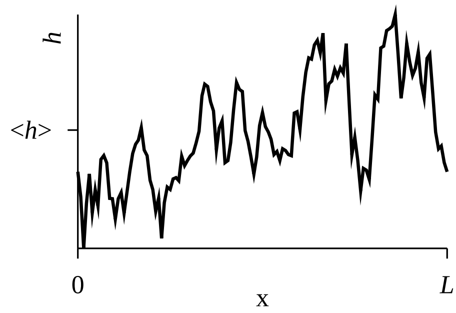

FIG. 1: Example of a $1+1$-dimensional periodic interface $h(x)$ (random walk) with mean value $\langle h \rangle$, and $u(x) = h(x) - \langle h \rangle$.

for related models such as polymers, spin glasses, and random diffusion [14]. The quantity we study here, $P(w^2)$, is the distribution of the lowest order observable which tests the whole function $h(x)$ for $0 < x < L$. It appears as a fundamental quantity in disordered systems.

The aim of this Letter is to compute the width distribution (WD) $\Phi(z)$ for elastic interfaces driven in random media, exactly at the depinning threshold, numerically and from field theory. As in the linear problems treated earlier, we confirm the existence of universal properties in various dimensions d and with several functional forms of the elasticity. The surprising finding is that in all cases $P(w^2)$ (i.e. its shape) is *extremely well* approximated by a simple generalized Gaussian approximation (GA), without any fit parameter, and depends only on ζ, which is determined independently. This suggests that the complicated morphology of interfaces (cracks, domain walls, etc.) may be rendered by a simple ansatz of independent modes with a characteristic decay. This may have important consequences for the analysis of numerical and experimental data. Our numerical results are then understood within a functional renormalization group calculation, detailed in a companion paper [15].

We consider the zero temperature equation of motion of an interface given by

$$\partial_t h(\mathbf{x}, t) = -\frac{\partial E}{\partial h(\mathbf{x}, t)} = f + \eta(\mathbf{x}, h(\mathbf{x}, t)) - \frac{\partial E_{\text{el}}}{\partial h(\mathbf{x}, t)} , \tag{3}$$

where the functional $E(\{h,\mathbf{x}\})$ represents the total energy incorporating potential energy due to the driving force f, the short-range correlated disorder force $\eta(\mathbf{x},h)$, as well as its internal convex elastic energy $E_{\rm el}$. Equation (3) is non-linear, and has not been solved exactly. We are interested in the depinning limit ($f=f_c$) where the velocity of the elastic manifold goes to zero. We use periodic boundary conditions, and recall that the WD $\Phi(z)$, although independent of small scale details, does depend on the boundary condition at large scale.

For our numerical study we use a very efficient algorithm [16–18] which directly determines critical forces f_c as well as the critical interface $h_c(\mathbf{x})$ for a wide range of models. In particular we calculate the WD for interfaces of dimensions $d=1$ and $d=2$, where the elastic energy has the harmonic form $E_{\rm el}(\{h,\mathbf{x}\})\sim(\nabla h)^2$. We have also tested the universality of $\Phi(z)$ in $d=1$, by means of a directed polymer model with an anharmonic quartic elasticity, and for a lattice model with hard local constraint, which have the same $\zeta=0.63$ [16]. As expected, $\Phi(z)$ is always size independent and the WD associated to non-harmonic models can be distinguished from the one resulting from an harmonic elasticity. The harmonic models, in fact, have an exponent $\zeta=1.2$, and thus belong to a different universality class.

For our field theory calculation we use the functional renormalization group method (FRG) originally developped to one loop to describe the model with harmonic elasticity and correct the predictions of dimensional reduction [4, 5]. Recently a renormalized field theory was constructed to 2-loop order [7] which overcomes the deficiencies of the 1-loop analysis; notably to distinguish between statics and driven dynamics, and to account for the large values of the roughness exponent ζ measured e.g. in [17–19] as compared to an earlier conjecture [5] $\zeta=(4-d)/3$. We find that the FRG both suggests the GA as a lowest order approximation in $\epsilon=4-d$ and allows to define and compute universal ratios which probe high cumulants of $P(w^2)$ and deviations from the GA, and are thus more sensitive to details of the universality class. The simplest of them is the generalized kurtosis

$$R=\frac{\int_{x,y}\overline{(u(x)^2u(y)^2)}^{\rm c}}{2\int_{x,y}\left(\overline{u(x)u(y)}\right)^2}\,, \tag{4}$$

where the subscript c indicates the connected expectation value. R is found to be small but non-zero. This directly proves that the correct description of interfaces must go beyond the independent-mode picture.

To introduce the Gaussian approximation in the most elementary way, we first recall [9] the simple periodic random walk of size L, with a Fourier expansion

$$u(x)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi n}{L}x\right)+b_n\sin\left(\frac{2\pi n}{L}x\right)\,. \tag{5}$$

The standard Gaussian probability measure associated with u, i.e. $\mathcal{P}[u]\propto\exp[-\frac{1}{2}\int_0^L{\rm d}x(\frac{\partial u(x)}{\partial x})^2]$ gives

$$\mathcal{P}[u]\propto\exp\left[-\sum_{n=1}^{\infty}\frac{(\pi n)^2}{L}(a_n^2+b_n^2)\right]\,. \tag{6}$$

The probability distribution $P(w^2)$

$$P(w^2)=\int\mathcal{D}[u]\delta(w^2-\langle u^2\rangle)\,\mathcal{P}[u] \tag{7}$$

is obtained from the generating function of its moments

$$W(\lambda)=\int_0^\infty{\rm d}w^2P(w^2){\rm e}^{-\lambda w^2}. \tag{8}$$

Writing Eq. (8) as an integral over a_n and b_n, we obtain

$$Z(\lambda)=\prod_{n=1}^{\infty}\int{\rm d}a_n\,{\rm d}b_n\,{\rm e}^{-\frac{(\pi n)^2}{L}(a_n^2+b_n^2)}{\rm e}^{-\frac{\lambda}{2}(a_n^2+b_n^2)}$$

$$W(\lambda)=\frac{Z(\lambda)}{Z(0)}=\prod_{n=1}^{\infty}\left(1+\frac{\lambda}{2}\frac{L}{(\pi n)^2}\right)^{-1}. \tag{9}$$

For the random walk Eq. (6), $P(w^2)$ can be obtained exactly by inverse Laplace transform of Eq. (9):

$$P(w^2)=\frac{4\pi^2}{L}\sum_{n>0}n^2(-1)^{n+1}\,{\rm e}^{-2w^2(\pi n)^2/L}. \tag{10}$$

Using $\overline{w^2}=-\frac{{\rm d}W}{{\rm d}\lambda}|_{\lambda=0}=\frac{L}{12}$, Eq. (10) can be written in a scaling form

$$\begin{aligned}\Phi(z)&=\overline{w^2}P(w^2)\,,\qquad z=w^2/\overline{w^2}\\&=\frac{\pi^2}{3}\sum_{n>0}n^2(-1)^{n+1}{\rm e}^{-\frac{\pi^2}{6}zn^2}.\end{aligned} \tag{11}$$

The size dependence thus appears only through the average width $\overline{w^2}$. We can generalize Eq. (6), where each mode a_n,b_n has a weight $\propto n^2$, to an arbitrary function of independent Fourier modes

$$\mathcal{P}_{\rm gauss}[u]\propto\exp\left[-\frac{L}{4}\sum_{n>0}(a_n^2+b_n^2)\,G^{-1}\left(\frac{2\pi n}{L}\right)\right]\,. \tag{12}$$

which, in real space, corresponds to

$$\mathcal{P}_{\rm gauss}[u]\propto\exp\left[-\frac{1}{2}\int_0^L\int_0^L{\rm d}x\,{\rm d}y\,u(x)G^{-1}_{xy}\,u(y)\right]\,. \tag{13}$$

The function $G_{xy}=\overline{u(x)u(y)}$ is the exact disorder-averaged 2-point function and can be computed from numerical data. This allows to obtain $\mathcal{P}_{\rm gauss}$ even for a finite system. In the thermodynamic limit, $\mathcal{P}_{\rm gauss}[u]$ is obtained from the behavior of $G_{xy}=G_{x-y}$ for large $|x-y|$ (small q), where $G(q)\sim C/q^{d+2\zeta}$. This means that a single observable, ζ, fixes $\mathcal{P}_{\rm gauss}[u]$ on large scales.

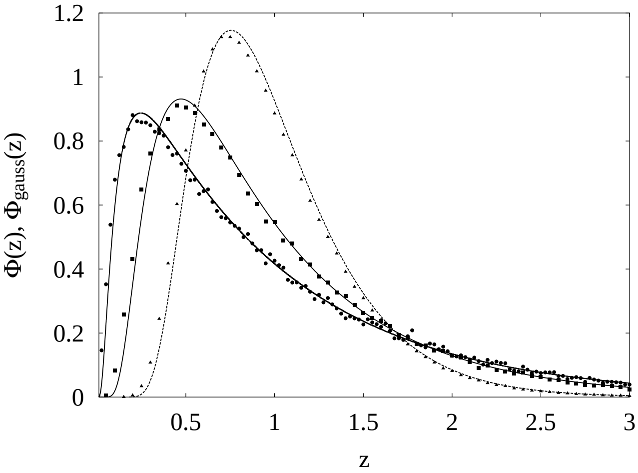

FIG. 2: Scaling function $\Phi(z)$ and $\Phi_{\text{gauss}}(z)$ for: $(1+1)$–d harmonic ($L=256$, 2×10^5 samples, $\zeta=1.25$)—left; $(1+1)$–d an-harmonic ($L=256$, 2×10^5 samples, $\zeta=0.63$)—middle; $(2+1)$–d harmonic ($L=32$, 10^5 samples, $\zeta=0.75$)—right. The scatter in the numerical data is mostly due to binning. Notice that, for $d=1$, the typical value of z is much smaller than its average $\overline{z}=1$.

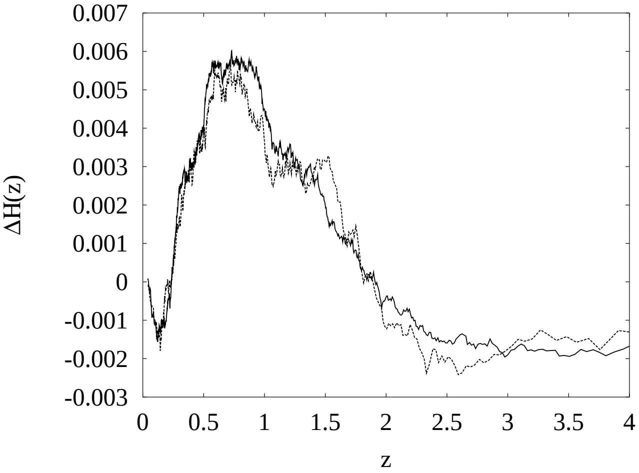

FIG. 3: Difference between the integrated distribution functions of Φ and Φ_{gauss} (with $\zeta=1.25$) obtained from 2×10^5 independent interfaces at $L=256$ (continuum line) and $L=64$ (dashed line), in the $d=1$ harmonic model.

We again determine the generating function for the moments, but for arbitrary ζ and d:

$$W(\lambda)=\prod_{q\neq0}\left(1+2\tilde{\lambda}G(q)\right)^{-1/2}\ , \tag{14}$$

where $\tilde{\lambda}=\lambda/L$, $q=2\pi n/L$, $n\in\mathbb{Z}^d$. Due to the symmetry $q\leftrightarrow -q$, no fractional power appears in Eq. (14), as in Eq. (9), where the exponent -1 stems from the double sum over the a_n and b_n. An explicit sum over poles allows to obtain $\Phi_{\text{gauss}}(z)$ for all ζ and d with excellent precision. All GA interfaces $\{u(x)\}$ can be directly sampled by Monte Carlo methods. For details, including the extension to open boundary conditions, see [12, 20]

In Fig. 2 we compare, for different models, the exact scaling function $\Phi(z)$ to $\Phi_{\text{gauss}}(z)$, using $G(n)=C/n^{d+2\zeta}$. The roughness exponent was previously obtained using both field theory [7] and numerical methods [17]. The agreement between Φ and Φ_{gauss} is clearly spectacular. The scatter of the data, visible in Fig. 2, is mostly due to the finite width of histogram bins.

Tiny—yet significant—differences between Φ and Φ_{gauss} are best resolved in the integrated probability distributions, which need no discretization. The difference between the integrated distributions is

$$\Delta H(z)=\int_0^z \mathrm{d}t(\Phi_{\text{gauss}}(t)-\Phi(t)), \tag{15}$$

where $H(z)=\int_0^z \mathrm{d}t\Phi(t)$ is the fraction of samples with a renormalized width below z. In Fig. 3, we show $\Delta H(z)$ obtained from $N=2\,10^5$ independent samples. Statistical fluctuations in this quantity are of order $1/\sqrt{N}$ and the signal-to-noise ratio would be smaller than 1 if N was an order of magnitude smaller. The absence of systematic finite-size effects shows that the asymptotic regime of large interfaces has been reached and thus to conclude that the exact distribution for large systems is not Gaussian.

We now discuss the field theoretical calculation. To lowest order in perturbation theory, we show that the generalized Gaussian approximation appears naturally. This is instructive since it identifies the diagrams which are obtained by assuming the theory to be Gaussian, albeit non-trivial, since it involves a non-trivial roughness exponent ζ. Using dynamical field theoretic methods [7], one starts again from the Laplace transform $W(\lambda)$ and expands in powers of the correlator of the pinning force $\Delta(u)$. To lowest order one finds [15] that $\log W(\lambda)$ is the sum of all connected 1-loop diagrams. The loop with N disorder vertices and N insertions of w^2 is

$$\frac{1}{2N}\sum_q\left(\frac{-2\tilde{\lambda}\Delta(0)}{(q^2)^2}\right)^N\ , \tag{16}$$

where the sums over q thus run over a d-dimensional lattice with spacing $\frac{2\pi}{L}$, and the 0-mode is excluded, as appropriate for periodic BC. Resumming (16) over q would give Eq. (14) with $G(q)\sim 1/q^4$, i.e. the dimensional reduction (Larkin) result. In fact, the FRG tells us that the calculation should be performed with the running disorder $\Delta(0)\to\Delta_l(0)=\mathrm{e}^{(\epsilon-2\zeta)l}\tilde{\Delta}^*(0)$ where $\tilde{\Delta}^*(0)$ is the (non-universal) value of the fixed point [7]. For the present case of periodic boundary conditions and momentum infrared cutoff, one can replace $l\to\log(1/q)$, and finally obtains Eq. (14) with $G(q)=C/q^{d+2\zeta}$. This calculation is valid to dominant order in $\epsilon=4-d$, i.e. near $d=4$. If the same class of diagrams are resummed in any d it leads to the GA, as we now illustrate considering e.g. the second connected cumulant of the WD. This cumulant is *not connected* w.r.t. h, and thus there is an exact relation:

$$\overline{(w^2)^2}^c = \overline{(w^2)^2} - \overline{(w^2)}^2 = 2\,(1+R)\int_{x,y} G_{xy}^2. \qquad (17)$$

The first term results from Wick's theorem and would be the full result if the measure were Gaussian. Analogous formulae exist for higher cumulants, and if the measure of h is purely Gaussian can be resummed into Eq. (14). Even though the GA is not exact, the deviations, given by the last term in (17) are expected to be small; indeed they are of order ϵ^4. Thus the GA is already exact to the *two lowest leading orders* ϵ^2 and ϵ^3, which explains why it is so accurate even in low dimension.

The calculation of the deviations using the field theory is delicate [15]. The kurtosis R in Eq. (4) which characterizes the importance of non-Gaussian effects is found to be $R = -0.13\epsilon^2$ to lowest order for small $\epsilon = 4-d$. It is easy to see that this strongly overestimates R in low dimensions. Another method is to work in fixed dimension and to truncate to one loop, yielding $R = -0.036$ ($d = 3$), $R = -0.048$ ($d = 2$), $R = -0.01$ ($d = 1$), which in view of the numerical results below seems to underestimate R. The small values obtained in low dimensions arise from kinematic constraints in the diagrams, presumably a genuine effect indicating large corrections from higher orders in ϵ to the above $O(\epsilon^2)$ even in $d = 3$. Note that the *sign* of the result indicates a distribution more peaked than a Gaussian and is in agreement with the only other known exact result [21] for the (random field) statics in $d = 0$, $R = -0.080865$...

We have computed from Eq. (4) the generalized kurtosis function, in a model-independent way. We have checked on the $1-d$ harmonic model that, as $\Phi(z)$, R is not affected by finite size effects and – using 10^6 samples – we find: $R = -0.054 \pm 0.002$ ($1-d$ harmonic $L = 256$); $R = -0.067 \pm 0.002$ ($1-d$ anharmonic $L = 64$); $R = -0.053 \pm 0.002$ ($2-d$ harmonic $L = 32$). As proven by FRG calculations, R is small but definitely different from zero. Direct information on the non-Gaussian effects can also be obtained from the Fourier transforms of the interfaces $u(x)$ in Eq. (5). In Fourier space, for $d = 1$, the expression of R is:

$$R = \frac{\sum_{n_1,n_2} \overline{(a_{n_1}^2 + b_{n_1}^2)(a_{n_2}^2 + b_{n_2}^2)}^c}{2\sum_n \overline{(a_n^2 + b_n^2)}^2}. \qquad (18)$$

We remark that R detects correlations in the disorder-averaged fourth moments $\overline{|u(q_1)|^2|u(q_2)|^2}$, which cannot be expressed simply through the second moments $\overline{|u(q)|^2}$.

To summarize, we have computed both numerically and within field theory the width distribution of critical configurations at depinning, with consistent results. The shapes of the distributions are strongly dominated by the value of ζ. On the other hand, it will be difficult to distinguish different universality classes from the forms of $\Phi(z)$, if their roughness exponents are similar. Other universal quantities, such as R defined here, directly involve the non-Gaussian part of the distribution. Their precise determination however requires more work, both numerically and within field theory. Also, since the WD is so tied up to ζ, finite size effects in both quantities are connected. Finite-size effects will need to be well understood in order to resolve open issues [6, 19, 22] concerning $d_{\rm uc}$ for the anisotropic depinning class. It would be interesting to carry similar calculations on other pure, and disordered models, in particular for the equivalent static system.

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