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Avalanches in mean-field models and the Barkhausen noise in spin-glasses

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Abstract – We obtain a general formula for the distribution of sizes of "static avalanches", or shocks, in generic mean-field glasses with replica-symmetry-breaking (RSB) saddle points. For the Sherrington-Kirkpatrick (SK) spin-glass it yields the density $\rho(\Delta M)$ of the sizes of magnetization jumps ΔM along the equilibrium magnetization curve at zero temperature. Continuous RSB allows for a power-law behavior $\rho(\Delta M) \sim 1/(\Delta M)^\tau$ with exponent $\tau = 1$ for SK, related to the criticality (marginal stability) of the spin-glass phase. All scales of the ultrametric phase space are implicated in jump events. Similar results are obtained for the sizes S of static jumps of pinned elastic systems, or of shocks in Burgers turbulence in large dimension. In all cases with a 1-step solution, $\rho(S) \sim Se^{-AS^2}$. A simple interpretation relating droplets to shocks, and a scaling theory for the equilibrium analog of Barkhausen noise in finite-dimensional spin-glasses are discussed.

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Many disordered systems crackle when driven slowly, reacting with abrupt responses over a broad range of scales [1]. These avalanche phenomena occur in granular materials [2], earthquakes [3], fracture [4], liquid fronts [5], vortex lattices [6], and other pinned elastic objects such as domain walls in disordered ferromagnets [7], where jumps in magnetization are known as Barkhausen noise [8]. Also in electronic glasses, where striking memory effects are observed upon gating [9], one expects crackling phenomena. The size S of these events is power law distributed, *i.e.* scale-free, $\rho(S) \sim S^{-\tau}$. This property, often termed self-organized criticality, emerges naturally in sandpile models, where analytical results were obtained [10]. But even there, $\rho(S)$ is difficult to compute. Scale-free response also occurs in pinned elastic systems, where quenched disorder leads to glassiness and metastability at all scales. The distribution of avalanche sizes for a single elastic interface was obtained from the functional renormalization group (FRG) [11], and compared with numerics [12,13] and with wetting experiments at the depinning transition [5]. The random-field Ising model with short-range interactions, much studied in this context, exhibits a transition between non-critical and infinite avalanches as disorder is varied [14,15], with scale-free avalanche distributions only at a special point in the phase diagram. While domain-wall motion plays an important role in soft magnets, a description without nucleation, long-range dipolar interactions and the ensuing frustration between domains would be incomplete [8].

The situation is less explored in strongly frustrated spin-glasses, whose complex energy landscape shares many features with that of pinned elastic systems. In particular, the spin-glass phase exhibits criticality with power-law spin correlations, as predicted in mean-field theory [16] and in the droplet picture [17]. This property is difficult to access by standard experimental protocols. However, the statistics of magnetization bursts in a hysteresis experiment (the Barkhausen noise) should be sensitive to the criticality of the glass state, and thus serve as a probe of spin-glasses, both experimental and numerical.

The aim of this letter is to compute the statistics of *equilibrium* (*i.e.* static) magnetization jumps in the Sherrington-Kirkpatrick (SK) mean-field spin-glass. We obtain a formula, (7) below, which applies more generally to any mean-field model described by a replica-symmetry-breaking (RSB) saddle point. The strategy is similar to ref. [11] for elastic interfaces: a static avalanche, or shock, occurs when the system jumps discontinuously between two degenerate global minima as the energy landscape is tilted with an external force. This phenomenon shows

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up as non-analytic cusps in all moments of the effective (*i.e.* renormalized) force. Calculation of these cusps allowed [11] to obtain the jump-size distribution in an ϵ expansion around internal dimension $d=4$. To lowest order, it was found to be identical for dynamical avalanches (*i.e.* under a driving force) and for static avalanches, or shocks, with $\tau = 2 - \frac{2}{d+\zeta} + O(\epsilon^2)$, ζ being the roughness exponent. Here, we extend previous mean-field studies of the second moment of the equilibrium effective force [18–20] to all moments. The resulting formula (7) allows for a simple interpretation. We suggest an extension as a scaling theory for finite-dimensional spin-glasses, based on a relation between shock and droplet distributions, which extends an identity for a particle [21]. Similar shocks occur in many systems: We also discuss elastic manifolds [22] and decaying Burgers turbulence [23], embedded in large dimension N, the mean-field limit. We find that 1-step RSB always results in $\rho(S) \sim S e^{-AS^2}$, similar to the shock-size distribution obtained by Kida in Burgers turbulence [24,25]. In contrast, we show that continuous RSB results in novel and interesting scale-free avalanche distributions.

Mean-field spin-glasses and the SK model. – Out-of-equilibrium avalanches in the SK model at $T=0$ were studied numerically [26], and found to exhibit criticality, *i.e.* power-law size distributions. The system self-organizes to remain at the brink of stability. The distribution of the local field $h_i = \sum_{j\neq i} J_{ij}\sigma_j + \overline{h}$, *i.e.*, the energy cost to flip only spin i, displays a linear pseudo-gap [27], marginally satisfying the minimal requirement for metastability. Upon increasing $\overline{h}$ by $\sim 1/\sqrt{N}$ (N being the number of spins) a first spin flips, and with finite probability entails $O(N)$ spin flips, with a change in magnetization of $O(\sqrt{N})$. The thermodynamic criticality of the spin-glass phase suggests similar avalanche phenomena at equilibrium. Indeed, it has long been known [28] that the equilibrium magnetization $M(\overline{h})$ of the SK model undergoes a sequence of small jumps as $\overline{h}$ is increased. These jumps of size $N^{1/2}$ lead to non–self-averaging spikes in the susceptibility. However, the analytical understanding of avalanches in spin-glasses, and their relation to thermodynamics, has remained scarce.

In ref. [29], the equilibrium magnetization of mean-field systems with p-spin interactions ($p>2$) was analyzed and compared to a toy model of a large set of states with random energies E and magnetizations M [30]. When the free energies $E_{1,2} - hM_{1,2}$ of the two lowest states cross as h is increased, a jump in magnetization occurs. This basic picture, with metastable states replacing the random states, remains a good qualitative guide, both in the p-spin model and in the SK model of spin-glasses. In [29], the presence of large equilibrium jumps was demonstrated by a non-analyticity in the second moment $\overline{[M(h_2)-M(h_1)]^2}$ for $h_2 - h_1 \sim N^{-1/2}$, in close analogy to the force correlator of elastic systems [20]. Such shocks are sharply defined only at $T=0$. In order for the magnetization jumps not to be washed out by thermal smearing very low $T \ll 1$ must be considered in mean-field spin models, including SK. Consider now specifically the SK model (the case $p=2$):

$$H = -\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - \overline{h}\sum_{i=1}^{N}\sigma_i, \tag{1}$$

where the J_{ij} are i.i.d. centered Gaussian random variables of variance J^2/N, that couple all N Ising spins, and $\overline{h}$ is the external field. This problem is more involved than the p-spin model with $p>2$, since its glass phase involves infinite-step RSB with marginally stable states, unlike $p>2$, which has a 1-step solution. This is reflected in crucial differences in the avalanche statistics. The equilibrium solution of (1) at $N\to\infty$ is given by Parisi's full RSB ansatz for the overlap matrix $Q_{ab} = \langle\sigma_a\sigma_b\rangle$, which is parametrized by a monotonous order parameter function $q(\mathsf{x})$ on the interval $0<\mathsf{x}<1$ [31]. $q(\mathsf{x})$ exhibits a plateau at large and small x, $q(\mathsf{x}>\mathsf{x}_c) = q_c$, $q(\mathsf{x}<\mathsf{x}_m) = q_m$.

Jumps in the equilibrium configuration as a function of $\overline{h}$ are closely related to chaos in a field. Equilibrium configurations in different fields have minimum overlap as soon as the difference in fields significantly exceeds $1/\sqrt{N}$ [32], which sets a typical scale for large shocks. Interestingly, this is the same scale as suggested by the *dynamical* stability considerations discussed above [26]. In both cases, the average size of an avalanche (total magnetization change) should scale as $\Delta M \sim N\overline{\chi}\delta\overline{h} \sim \sqrt{N}$, with $\overline{\chi}$ the average susceptibility. This is confirmed below.

While we outline the main steps of the calculation on the SK model, the technique extends to any mean-field system described by RSB saddle points. Details will be presented in [33]. The probability for a shock in the interval $[\overline{h}, \overline{h}+\delta\tilde{h}/\sqrt{N}]$ is proportional to $\delta\tilde{h}$ if $\delta\tilde{h} \ll 1$; its fingerprint are non-analyticities in the moments of magnetization differences, $\overline{[M(h)-M(h+\delta\tilde{h}/\sqrt{N})]^k} \sim N^{k/2}|\delta\tilde{h}|$. Calculating the prefactor of $|\delta\tilde{h}|$ for all k allows us to infer the avalanche-size density per unit field for $\Delta m > 0$:

$$\rho_{\overline{h}}(\Delta m) = \lim_{\delta\tilde{h}\downarrow 0} \frac{1}{\delta\tilde{h}} \overline{\delta\left(\Delta m - \frac{M(\overline{h}+\frac{\delta\tilde{h}}{\sqrt{N}}) - M(\overline{h})}{\sqrt{N}}\right)}, \tag{2}$$

where we have introduced the rescaled magnetization $m = M/N^{1/2}$, which jumps by $\Delta m = O(1)$ in typical shocks. To calculate correlators of magnetization in different fields,

$$\overline{M(h_1)\dots M(h_k)} = (-1)^k \partial_{h_1}\dots\partial_{h_k} \overline{F(h_1)\dots F(h_k)}, \tag{3}$$

we consider the generating function of $a=1,\dots,n$ replica

$$\begin{aligned} \exp\Big[W[\{h_a\}]\Big] &:= \overline{\exp\Big[-\beta\sum_{a=1}^{n}F(h_a)\Big]}^{J} \\ &= \exp\Big[\sum_{k=1}^{\infty}\frac{(-\beta)^k}{k!}\sum_{a_1,\dots,a_k=1}^{n}\overline{F(h_{a_1})\cdots F(h_{a_k})}^{J,c}\Big] \end{aligned}$$

$$= \int \prod_{a\neq b} \mathrm{d}Q_{ab} e^{\frac{N}{2}\beta^2 J^2\left(n-\sum_{a\neq b} Q_{ab}^2\right)+NA(Q,\{h_a\})},$$

$$e^{A(Q,\{h_a\})} := \sum_{\sigma_a=\pm 1} \exp\Big(\beta^2 J^2 \sum_{a\neq b} Q_{ab}\sigma_a\sigma_b + \sum_a \beta h_a \sigma_a\Big). \quad (4)$$

Organizing the n replica into k groups subject to the same field $h_{i=1,\dots,k} = \bar{h} + \tilde{h}_i/\sqrt{N}$, with $\sum_a \tilde{h}_a = 0$, and analyzing the cumulant expansion of the potential $W[\{h_i\}]$, the k-point correlator (3) can be extracted in the limit $n\to 0$. Expanding $A(Q,\{h_a\})$ to second order in $\tilde{h}_i$, the potential is evaluated at the saddle point where Q_{ab} assumes Parisi's equilibrium solution $q_{\bar{h}}(\mathsf{x})$. However, due to the explicit breaking of replica symmetry by the external fields h_a, a sum over inequivalent saddle points differing by replica permutations of Q_{ab} has to be performed. Generalizing techniques introduced in [34], we find a compact integral representation for the k'th cumulant

$$\overline{m(h_1)\dots m(h_k)}^{J,c} = \frac{-k}{(-\beta)^k} \times \int \mathrm{d}^k y\, \delta\left(\sum_{i=1}^k y_i\right) \partial_{\tilde{h}_1}\dots\partial_{\tilde{h}_k}\phi(0,y),$$

where $\phi(x,y)$ solves the differential equation

$$\frac{\partial\phi}{\partial \mathsf{x}} = -\frac{\beta^2}{2}\sum_{i,j=1}^k \tilde{h}^i\tilde{h}^j \frac{\mathrm{d}q_{\bar{h}}(\mathsf{x})}{\mathrm{d}\mathsf{x}}\left(\frac{\partial^2\phi}{\partial y_i\partial y_j} + \mathsf{x}\frac{\partial\phi}{\partial y_i}\frac{\partial\phi}{\partial y_j}\right),$$

$$\phi(\mathsf{x}=1;\{y_i\}) = \log\left(\sum_{i=1}^k \exp(y_i)\right). \quad (5)$$

In order to unambiguously identify shocks we need to take the limit $N^{-1/2} \gg T \to 0$. It is known [18,20] that the non-analyticities $\propto |\tilde{h}_i|$ in the cumulants are obtained by an expansion of the diffusion-type equation (5) to first order in the non-linear term. For $k \geqslant 2$, the result encapsulates the full statistical information about jumps [33],

$$\overline{(m_{h_1} - m_{h_2})^k} = \tilde{h}_{12}\int_0^\infty \rho_{\bar{h}}(\Delta m)(\Delta m)^k\, \mathrm{d}\Delta m + \mathcal{O}(\tilde{h}_{12}^2), \quad (6)$$

where $h_{1,2} = \bar{h} + \tilde{h}_{1,2}/\sqrt{N}$, and $\tilde{h}_{12} = \tilde{h}_1 - \tilde{h}_2 > 0$. The density per unit $\delta\bar{h}$ of jump sizes $\Delta m > 0$, cf. eq. (2) is[1]:

$$\rho_{\bar{h}}(\Delta m) = \Delta m \int_{q_m^-}^{q_c} \mathrm{d}q\, \nu_{\bar{h}}(q) \frac{\exp[-\frac{(\Delta m)^2}{4(q_c - q)}]}{\sqrt{4\pi(q_c-q)}}\theta(\Delta m). \quad (7)$$

The weight $\nu_{\bar{h}}(q) \equiv \lim_{T\to 0}[T\, \mathrm{d}q_{\bar{h}}/\mathrm{d}\mathsf{x}]^{-1}$ can be interpreted as the probability density, per unit energy, of finding a metastable state at overlap within $[q, q+\mathrm{d}q]$ with energy close to the ground state [31]. The density of shocks receives contributions from the largest ($q \lesssim q_c(T=0) = 1$) to the smallest overlaps $q_m(\bar{h}) \approx \bar{h}^{2/3}$. Jumps in overlap of order $O(1)$ are indeed expected due to field chaos [32]. A useful check of eq. (7) is provided by the average magnetization jump. It turns out to equal the thermodynamic (field cooled) susceptibility, $\int \rho_{\bar{h}}(\Delta m)\Delta m\, \mathrm{d}\Delta m = \lim_{T\to 0}\frac{1}{T}\int_0^1 \mathrm{d}\mathsf{x}(q_c - q(\mathsf{x})) = \chi_{\mathrm{FC}}^{(T=0)}$. This is expected since the intra-state (zero-field cooled) susceptibility vanishes as $T\to 0$, the response being entirely due to interstate transitions.

The formula (7) has a very natural interpretation. If we take $\tilde{h}_{12} \ll 1$ in (6) we only need to consider the possibility that the ground state and the lowest-lying metastable state cross as we tune $\bar{h}$ from $\tilde{h}_1$ to $\tilde{h}_2$, corrections being of order $\mathcal{O}(\tilde{h}_{12}^2)$. The disorder-averaged density of states of this two-level system is described by $\nu_{\bar{h}}(q)\mathrm{d}q\, \mathrm{d}E$. The two states differ in $N_{\mathrm{fl}} = N(1-q)/2$ flipped spins. In the SK model the magnetization is uncorrelated with the energy, and one thus expects the magnetization difference between the states to be a Gaussian variable of zero mean and variance $\langle\Delta m^2\rangle_q = 4N_{\mathrm{fl}}/N = 2(1-q)$. If $\Delta m > 0$, a jump at equilibrium occurs once $\tilde{h}_{12} = E/\Delta m$. For the shock probability per unit $\tilde{h}$ one thus expects

$$\int_{q_m^-}^{q_c} \mathrm{d}q \int_0^\infty \mathrm{d}E\, \nu_{\bar{h}}(q) \frac{\exp\left[-\frac{(\Delta m)^2}{2\langle\Delta m^2\rangle_q}\right]}{\sqrt{2\pi\langle\Delta m^2\rangle_q}}\,\delta\left(\tilde{h}_{12} - \frac{E}{\Delta m}\right), \quad (8)$$

reproducing eq. (7). The result (7) is generally valid for models described by RSB. It thus applies to p-spin models with only one step of RSB, $q_{\bar{h}=0}(\mathsf{x}) = q_0 + (q_1 - q_0)\theta(\mathsf{x}-\mathsf{x}_1)$. With $\int \mathrm{d}q\, \nu(q) \to \hat{\mathsf{x}}_1 \int \mathrm{d}q\delta(q-q_0)$, $\hat{\mathsf{x}}_1 = \mathsf{x}_1/T$, the avalanche distribution simplifies into the form

$$\rho_{\bar{h}}^{(p>2)}(\Delta m) = \hat{\mathsf{x}}_1 \Delta m \frac{\exp\left[-\frac{(\Delta m)^2}{4(q_1-q_0)}\right]}{\sqrt{4\pi(q_1-q_0)}}\theta(\Delta m). \quad (9)$$

One verifies that its second moment agrees with ref. [29]. The distribution (9) is non-critical, peaking around a typical size $\Delta m \sim 2\sqrt{q_1 - q_0}$, with $\rho(\Delta m) \sim \Delta m$ at small Δm (similar to one of the lower curves in fig. 1). The case of SK with full replica-symmetry breaking is much richer, as there is a $T=0$ limit function $q(\hat{\mathsf{x}})$. The weight with which events at overlap distance $1-q$ contribute is a power law [35],

$$\nu_{\bar{h}}(q|1 \gg 1-q \gg T^2) = C(1-q)^{-3/2}, \quad (10)$$

with $C = 0.32047$ [36]. This holds independently of the external field $\bar{h}$, and of additional random-field disorder [37]. From (10) and (7) it leads to a robust scale-invariant jump density:

$$\rho(\Delta m) \approx \frac{2C}{\sqrt{\pi}}\frac{1}{(\Delta m)^\tau}, \quad \Delta m \ll 1 \quad (11)$$

with $\tau = 1$. The universal exponent $\tau = 1$ for jump sizes $N^{-1/2} \ll \Delta m \ll 1$ results from superimposed contributions from all overlaps, *i.e.* all scales, illustrated in fig. 1.

[1] It contains a piece $\delta(q-q_m)\mathsf{x}_m/T$ when $q(\mathsf{x})$ exhibits a plateau at $\mathsf{x} \leqslant \mathsf{x}_m$ (if $\bar{h} \neq 0$), hence the notation q_m^- in the integral. The integral measure can also be written as $\int_0^{\mathsf{x}_c/T}\mathrm{d}(\mathsf{x}/T)$.

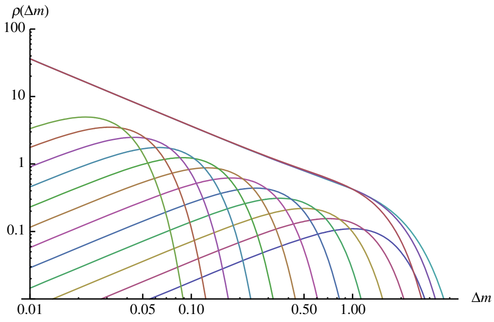

Fig. 1: (Color online) Power-law distribution of jumps for the SK model. The curves in the lower part show the 1-step–like contributions due to overlaps $(1-q)=2^{-k}$, $k=1,\dots,12$. The three nearly coinciding lines on the top show $\rho(\Delta m)$ from Eq. (7), for $\overline{h}=0, 0.25$ and 0.5, respectively. We use the approximation $\hat{\mathsf{x}}(q)=(aq+bq^2)/\sqrt{1-q}$ with $a=1.28$ and $b=-0.64$, and a sharp lower cutoff at $q_{\min}(\overline{h})=1.0\overline{h}^{2/3}$ [35,38,39]. The increase of $\overline{h}$ decreases the cutoff at large Δm, while the avalanche distribution for $\Delta m \ll 1$ is a universal power law.

The cutoff function for larger jumps $\Delta m \gtrsim 1$ depends on the applied field. In zero field, $q(\hat{\mathsf{x}})$ is linear at $\hat{x} \ll 1$. The resulting density $\nu(0)=1.34523$ at $q=0$ leads to the asymptotics

$$\rho(\Delta m) \approx \frac{2\nu(0)}{\sqrt{\pi}} \frac{e^{-(\Delta m)^2/4}}{(\Delta m)^{\tau'}}, \quad \Delta m \gg 1 \tag{12}$$

with $\tau'=1$. Plots at intermediate $\Delta m = O(1)$ are shown in fig. 1 using approximations to $q(\hat{\mathsf{x}})$. A small field produces a plateau at $q_{\min}(\overline{h})=1.0\times\overline{h}^{2/3}$ and while (11) remains unchanged, the asymptotics (12) for $\Delta m \gg \Delta m_h \sim \overline{h}^{-1/3}$ now decays with $\tau'=-1$, as for the 1-step RSB case, replacing in (9) $\hat{\mathsf{x}}_1 \to \hat{\mathsf{x}}_h \approx \nu(0) q_{\min}(\overline{h})$ and $q_1 \to q_{\min}(\overline{h})$.

Accepting eq. (8) to represent the joint distribution of q and Δm, we can infer the probability distribution to flip $N_{\rm fl}$ spins when increasing the magnetic field by δh:

$$\mathcal{D}\left(N_{\rm fl}=\frac{(1-q)N}{2}\right)=\frac{2\sqrt{q_c-q}}{N\sqrt{\pi}}\nu_{\overline{h}}(q) \xrightarrow{\rm SK} \frac{C}{\sqrt{\pi}}\frac{1}{N_{\rm fl}^{\rho}} \tag{13}$$

with $\rho=1$. A very similar density of avalanches with the same exponents $\tau=\rho=1$ was observed in the $T=0$ hysteresis curve of [26]. In both cases, the number of spin flips scales as $N_{\rm fl}\sim N^{\sigma}$, with $\sigma=1$, whereas the magnetization changes as $\Delta m \sim N^{\beta}$, with $\beta=1/2$. This coincidence between equilibrium and driven dynamics is presumably related to the marginality of the spin-glass. It may also be due to the system being in a mean-field limit, which, in the case of elastic manifolds at $N=1$, indeed gives the same exponents [11]. Similar coincidences were reported in other models [40].

Finite-dimensional spin-glasses. – We argue that also in finite dimensions, independently of whether replica-symmetry breaking [31] or the droplet picture [17] describes the glass state, the distribution of equilibrium avalanches is expected to be a power law (in low fields). Indeed, let us assume that the dominant low-energy excitations are droplet-like spin clusters that flip simultaneously. These droplets are clusters that cannot be decomposed into a set of independent smaller excitations with lower energies. For droplets of typical linear size L we assume a typical energy cost L^{θ} and a non-vanishing density of states (per unit volume) down to $E=0$: $\nu_L(E=0)\,{\rm d}E=\nu_0 L^{-d_{\rm f}} L^{-\theta}{\rm d}E$ with a constant ν_0 independent of L. Empirically θ is very small, and $d_{\rm f}$ is the possibly fractal dimension of the droplets. We assume the total magnetization of droplets of size L to be uncorrelated with the energy, and distributed as[2] $P_L(\Delta M)=L^{-d_{\rm m}}\psi_M(\Delta M/L^{d_{\rm m}})$. In a vanishing field, low-energy droplets are believed to exist at all length scales, while recent numerical results [42] suggest that beyond a finite scale $L_h\sim 1/\overline{h}^{\gamma}$ droplets are suppressed. We make the standard assumption that droplets at scale L are uncorrelated with droplets at scales $\geqslant 2L$. With a reasoning analogous to the one leading to eq. (8), we expect a power-law density of avalanche sizes ΔM (per unit volume and unit field, with $\delta h\to 0$):

$$\begin{aligned}\rho_{\overline{h}}(\Delta M) &\approx \int_1^{L_h}\frac{{\rm d}L}{L}\int_0^{\infty}\frac{\nu_0{\rm d}E}{L^{d_{\rm f}+\theta}}\,\delta\left(\delta h-\frac{E}{\Delta M}\right)P_L(\Delta M)\\ &=\frac{1}{(\Delta M)^{\tau}}\frac{\nu_0}{d_{\rm m}}\int_{\Delta M L_h^{-d_{\rm m}}}^{\Delta M}{\rm d}z\,\psi_M(z)z^{\tau},\end{aligned} \tag{14}$$

with exponent $\tau=\frac{d_{\rm f}+\theta}{d_{\rm m}}$ and a cut-off $\Delta M\sim L_h^{d_{\rm m}}$ (see footnote [3]). Numerical investigation of avalanches at small fields could yield insight into the various exponents entering (14). Furthermore, experimental measurements of power-law Barkhausen noise in spin-glasses (*e.g.*, by monitoring magnetization bursts [8,43]) could provide complementary insight to earlier investigations of equilibrium noise [44].

Elastic manifolds. – The above calculation directly applies to the N-component elastic manifold with coordinate $u(x)$ of internal dimension d in a random potential, in presence of a harmonic well of curvature m^2, which forces $\overline{\langle u\rangle}=v$. It has energy

$$\mathcal{H}=\int {\rm d}^d x\frac{1}{2}(\nabla u)^2+V(u(x),x)+\frac{\mathsf{m}^2}{2}(u(x)-v)^2. \tag{15}$$

As v is increased at $T=0$ *along a straight line*, *i.e.* $v_i=v\delta_{i1}$, the minimum-energy configuration jumps and static avalanches occur, of size $S=\int_x \delta u_1(x)$. We study models with Gaussian bare disorder $\overline{V(u,x)V(0,0)}=\delta^d(x)R_0(u)$ with correlator $R_0(u)=NB(u^2/N)$ and $B'(z)=-(1+\frac{z}{\gamma})^{-\gamma}$, $\gamma>0$, in the large-N limit. Notations are as in [20], except here the Parisi variable is denoted $\mathsf{x}=T\hat{\mathsf{x}}$ ($\mathsf{u}=T\hat{\mathsf{u}}$ there). In [20] the second moment of the renormalized disorder correlator, $R(v)$ was computed.

[2] We note that the numerical study [41] found that most likely neither of the assumptions $d_{\rm m}=d_{\rm f}/2=d/2$, made by Fisher and Huse [39], holds.

[3] The mean-field case $\tau=1$ is formally recovered replacing $L^d\to(1-q)$, $d_{\rm m}/d\to 1/2$ and $(d_{\rm f}+\theta)/d\to 1/2$. The latter reflects the typical gap at distance $1-q$, $\Delta_q=(1-q)^{1/2}$ [39].

Here we are interested in the shock-size density $\rho(S)=\rho_0 P(S)$, the total shock density ρ_0 and the normalized size distribution ($\int \mathrm{d}S P(S)=1$). We define its moments as $\langle S^p\rangle=\int \mathrm{d}S S^p P(S)$. The dictionary is: $\tilde{h}\to v_1$, $\Delta m\to \mathsf{m}^2 S$, $\underline{q(\hat{\mathsf{x}})}\to \mathsf{m}^4 L^d G(\hat{\mathsf{x}})$ with $G(\hat{\mathsf{x}})=G(k=0,\mathsf{x}=T\hat{\mathsf{x}})$, where $\overline{\langle u^a_{-k}u^b_k\rangle}=G_{ab}(k)$. This gives the shock density

$$\rho(S)=\mathsf{m}^2 L^{-d/2} S\int_0^{\hat{\mathsf{x}}_c}\mathrm{d}\hat{\mathsf{x}}\,\frac{\exp\Big(-\frac{L^{-d}S^2}{4[G(\hat{\mathsf{x}}_c^+)-G(\hat{\mathsf{x}})]}\Big)}{\sqrt{4\pi[G(\hat{\mathsf{x}}_c^+)-G(\hat{\mathsf{x}})]}}. \tag{16}$$

Two exact relations hold in all cases:

$$\int \mathrm{d}S S\rho(S)\equiv\rho_0\langle S\rangle=L^d\Big(1-\frac{\mathsf{m}^2}{\mathsf{m}_c^2}\Big), \tag{17}$$

$$\Big(1-\frac{\mathsf{m}^2}{\mathsf{m}_c^2}\Big)\frac{\langle S^2\rangle}{2\langle S\rangle}=\frac{\partial_1^3 R(v)|_{v_1=0^+}}{\mathsf{m}^4}. \tag{18}$$

The first one is the total susceptibility $\partial_{v_1}\int_x\overline{u_1(x)}=L^d$ minus the intra-state susceptibility. The factor $(1-\frac{\mathsf{m}^2}{\mathsf{m}_c^2})$ thus gives the fraction of motion which occurs in jumps, which vanishes at $\mathsf{m}>\mathsf{m}_c^0\equiv\mathsf{m}_c(\mathsf{m}_c^0)$. Here $\mathsf{m}_c=\mathsf{m}_c(\mathsf{m})$ is the running Larkin mass, defined as $m_c^2=m^2+[\sigma](u_c^+)$ in the notations of [20]. Equation (18) extends the relation obtained in [11] between size moments and the cusp of the force correlator to the case of a finite fraction of motion in shocks. The size of the cusp is the same as in [20]. We define the large-size cutoff via $S_\mathsf{m}^2=L^d[G(\hat{\mathsf{x}}_c^+)-G(0)]$. For $d<4$ and $\mathsf{m}<\mathsf{m}_c^0=(\frac{4A_d}{\epsilon})^{1/\epsilon}$, where $\epsilon=4-d$ and $A_d=\frac{2\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}}$, the $T=0^+$ saddle point equations admit a RSB solution [20,22]. We now discuss various cases, depending on the energy exponent $\theta=\frac{2+\gamma(d-2)}{1+\gamma}$:

i) 1-step RSB: it occurs for $\theta\leqslant 0$, *i.e.* $d\leqslant 2$ and $\gamma\geqslant\frac{2}{2-d}$. The shock-size distribution depends on the single scale S_m:

$$P(S)=\frac{1}{S_\mathsf{m}}p\Big(\frac{S}{S_\mathsf{m}}\Big),\quad p(s)=\frac{1}{2}s\,e^{-s^2/4}. \tag{19}$$

Hence $\langle S\rangle=\sqrt{\pi}S_\mathsf{m}$ which yields ρ_0 from (17). Here $S_\mathsf{m}^2L^{-d}=\frac{1}{\hat{\mathsf{x}}_c}(\mathsf{m}^{-2}-\mathsf{m}_c^{-2})$ depends on the details of the 1-step solution. In the critical limit $\mathsf{m}\ll\mathsf{m}_c$, for $d>0$ ($d=0$ is treated below), one finds $S_\mathsf{m}=\mathsf{m}_c^{-1}(\mathsf{m}L)^{d/2}\mathsf{m}^{-d-\zeta}$ with $\mathsf{m}_c=\big[\frac{8A_d(\gamma-1)}{\epsilon d\gamma}\big]^{1/\epsilon}$, and a roughness exponent $\zeta=(2-d)/2$ (defined by $u\sim x^\zeta$).

ii) continuous RSB: it occurs for $\theta>0$, with $\mathsf{m}_c(\mathsf{m})=\mathsf{m}_c^0$,

$$AG(\hat{\mathsf{x}})=\frac{8}{(4-\theta^2)}\frac{1}{\mathsf{m}^{2+\theta}}-\frac{2}{2+\theta}\Big(\frac{A}{\hat{\mathsf{x}}}\Big)^{1+2/\theta},\quad \hat{\mathsf{x}}_\mathsf{m}\leqslant\hat{\mathsf{x}}\leqslant\hat{\mathsf{x}}_c \tag{20}$$

and $G(\hat{\mathsf{x}})=G(\hat{\mathsf{x}}_\mathsf{m})$ for $\hat{\mathsf{x}}\leqslant\hat{\mathsf{x}}_\mathsf{m}$ with $\hat{\mathsf{x}}_c=A\mathsf{m}_c^\theta$, $\hat{\mathsf{x}}_\mathsf{m}=A\mathsf{m}^\theta$, $A=\frac{1+\gamma}{\gamma\epsilon}(\frac{4A_d}{\epsilon})^{\frac{\gamma}{1+\gamma}}$ [20]. The total shock density is

$$\rho_0=\frac{\mathsf{m}^2L^{d/2}}{\sqrt{\pi}}\sqrt{\frac{2A}{2+\theta}}\mathsf{m}_c^{\frac{\theta}{2}-1}f\Big(\frac{\mathsf{m}}{\mathsf{m}_c}\Big), \tag{21}$$

with $f(x)=x^\theta(x^{-2-\theta}-1)^{1/2}+\int_{x^\theta}^1 dy(y^{-1-\frac{2}{\theta}}-1)^{1/2}$. For $\mathsf{m}\ll\mathsf{m}_c$, the size distribution becomes $P(S)\approx\frac{1}{S_\mathsf{m}}p(\frac{S}{S_\mathsf{m}})$ with typical size $S_\mathsf{m}\approx\sqrt{\frac{2}{A(2+\theta)}}\frac{(\mathsf{m}L)^{d/2}}{\mathsf{m}^{(d+\zeta)}}$, roughness exponent $\zeta=\frac{4-d}{2(1+\gamma)}$, avalanche-size exponent $\tau=\frac{2\theta}{2+\theta}$, and

$$p(s)=\frac{1-\tau}{2}\left[se^{-\frac{s^2}{4}}+\tau\Big(\frac{2}{s}\Big)^\tau\Gamma\Big(\frac{1+\tau}{2},\frac{s^2}{4}\Big)\right]\sim\frac{1}{s^\tau},\quad s\ll 1 \tag{22}$$

where $\Gamma(a,z)=\int_z^\infty \mathrm{d}t\,t^{a-1}e^{-t}$. One has $\langle S\rangle=\sqrt{\pi}\frac{2-\theta}{2}S_\mathsf{m}$ and $\rho_0=\frac{L^d}{\langle S\rangle}$ from (18), consistent with (21) at small m. Interestingly, the droplet argument (14) can be adapted to the interface, *i.e.* $N=1$. The correspondence $\Delta M\to S$ and standard interface scaling implies $d_\mathrm{m}\to d+\zeta$ and $d_\mathrm{f}\to d$. Together with $\theta=d-2+2\zeta$ it yields $\tau=\tau_\zeta=2-2/(d+\zeta)$ and provides, for static avalanches, a basis for the conjecture made previously at depinning, *i.e.* out of equilibrium [45]. By contrast, the above large-N limit gives $\tau=2-2/(\frac{d}{2}+\zeta)$. In $d=4$, this gives $\tau=1$, which is different from the usual mean-field exponent $\tau=3/2$ at $N=1$ [11], and expected to hold at finite N. There are indications that this is due to a non-commutativity of the limits $N,L\to\infty$, also reflected by the unusual L-dependence of the maximal avalanche size S_m.

Decaying Burgers turbulence. – We now consider the decaying Burgers velocity field $\mathsf{u}(r,t)$ in dimension N, satisfying

$$\partial_t\mathsf{u}+\frac{1}{2}\partial_r\mathsf{u}^2=\nu\nabla^2\mathsf{u} \tag{23}$$

with Gaussian, power-law–correlated, initial condition

$$\overline{\mathsf{u}_i(r,t=0)\mathsf{u}_j(r',t=0)}=-\partial_i\partial_j R_0(r-r')\sim|r-r'|^{-2\gamma}, \tag{24}$$

$R_0(r)=NB(r^2/N)$. From the Cole-Hopf transformation, the velocity at time $t=1/\mathsf{m}^2$ is obtained from the $d=0$ version of the model (15) with $v\equiv r$ and $T\equiv 2\nu$ as $t\mathsf{u}_i(r,t)=r_i-\langle u_i\rangle_\mathcal{H}$. In the large-dimension limit $N\to\infty$ and for LR correlations $0<\gamma<1$, *i.e.* $0<\theta=2\frac{1-\gamma}{1+\gamma}<2$, the above results for the manifold immediately apply, setting $d=0$. In the inviscid limit, $\nu\to 0$, the velocity field develops discontinuities along codimension-one manifolds, *i.e.* shocks, for $t>t_c=(\mathsf{m}_c^0)^{-2}$. Consider a line, say $r_i=r_1\delta_{i1}$, the velocity field along this line, *i.e.* $\mathsf{u}_1(r)$, and its shocks $\Delta\mathsf{u}=\mathsf{u}_1(r_1^+)-\mathsf{u}_1(r_1^-)$. From the identification $\Delta\mathsf{u}\equiv\mathsf{m}^2S$, the shock-size density along this line is given by eqs. (16) and (20) as $\rho(\Delta\mathsf{u})\equiv\mathsf{m}^{-2}\rho(S)$ with $\mathsf{m}^2=1/t$. At large time $t\gg t_c$, the size probability takes the form $P(\Delta\mathsf{u})=(\Delta\mathsf{u}_t)^{-1}p(\Delta\mathsf{u}/\Delta\mathsf{u}_t)$ where $p(s)$ is given by (22) and the shock-size exponent is $\tau=1-\gamma$. The typical shock size is $\Delta\mathsf{u}_t\equiv\mathsf{m}^2S_\mathsf{m}\sim t^{-1+\frac{\zeta}{2}}$, with $\zeta=2/(1+\gamma)$, consistent with the asymptotic scaling of the decaying velocity field: $\mathsf{u}(r,t)\xrightarrow{\text{in law}}t^{-1+\frac{\zeta}{2}}\tilde{\mathsf{u}}(\tilde{r}=rt^{-\zeta/2})$. The total shock density ρ_0 given by (21) vanishes for $t<t_c$, exhibits a maximum near t_c, then decays as $\sim t^{-\zeta/2}$, as the shock separation grows as $t^{\zeta/2}$ from usual scaling. For shorter-ranged initial correlations, $\gamma\geqslant 1$, the solution is 1-step RSB. The reduced size distribution is $p(s)=\frac{1}{2}se^{-s^2/4}$ with $\rho_0=\frac{1}{\sqrt{\pi}t\Delta\mathsf{u}_t}(1-\frac{t_c}{t})$ and i) $\Delta\mathsf{u}_t=t^{-1/2}(1-\frac{1}{2t})^{1/2}$ for the $\gamma=1$ LR class; ii) $\Delta\mathsf{u}_t\approx t^{-1/2}(\frac{\gamma-1}{\gamma}\ln t)^{-1/4}$ at large t, for the SR class, similar to Kida's result [24] for $N=1$.

Conclusion. – Systems whose thermodynamics is described by full RSB exhibit a power-law distribution of equilibrium-avalanche sizes, which can be traced back to their marginal stability. Even though dynamic avalanches are different from our static analysis, the exponents turn out to be the same $\tau = \rho = 1$ in the SK model, and in both cases the scale-free response is a consequence of criticality and marginal stability [26]. We expect a similar critical response upon slow changes of system parameters in many other systems with full RSB. This is of interest for optimization problems on dilute graphs such as minimal vertex cover [46], coloring or Potts glass [47], k-satisfiability [48] around the satisfiability threshold, and in the UNSAT region at large k. Likewise, in models of complex economic systems one expects a power-law–distributed market response to changes in prices and stocks [49]. Avalanches are also expected in electron glasses with unscreened $1/r$ interactions. A stability argument shows that the number of rearrangements upon adding a new electron at $T = 0$ diverges with system size at least as L^{d-2}, presumably with a wide distribution of dynamic responses. Since the mean field yields a full RSB phase [37], we speculate that static avalanches are power law distributed as well.

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