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# Non-Gaussian effects and multifractality in the Bragg glass

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**Abstract** – We study, beyond the Gaussian approximation, the decay of the translational order correlation function for a $d$-dimensional scalar periodic elastic system in a disordered environment. We develop a method based on functional determinants, equivalent to summing an infinite set of diagrams. We obtain, in dimension $d = 4-\varepsilon$, the even $n$-th cumulant of relative displacements as $\overline{\langle[u(r)-u(0)]^n\rangle}^c \simeq \mathcal{A}_n \ln r$ with $\mathcal{A}_n = -(\varepsilon/3)^n \Gamma(n-\frac{1}{2})\zeta(2n-3)/\sqrt{\pi}$, as well as the multifractal dimension $x_q$ of the exponential field $e^{qu(r)}$. As a corollary, we obtain an analytic expression for a class of $n$-loop integrals in $d = 4$, which appear in the perturbative determination of Konishi amplitudes, also accessible via AdS/CFT using integrability.



**Introduction.** – Periodic elastic systems in quenched disorder model numerous applications, from charge-density waves in solids [1], vortex lattices in superconductors [2,3] Wigner crystals [4], Josephson junction arrays [5], to liquid crystals [6]. The competition between elastic energy, which favors periodicity, and disorder, which favors distortions, produces a complicated energy landscape with many metastable states. While we know since Larkin [7] that weak disorder destroys perfect translational order, it was realized later that topological order (*i.e.* no dislocations) may survive, leading to the Bragg glass phase (BrG) [3,8] and validating the elastic description. A key observable, measured from the structure factor in diffraction experiments [9], is the translational correlation function $C_K(\mathbf{r}) = \overline{\langle e^{iK[u(\mathbf{r})-u(0)]}\rangle}$, where $u(\mathbf{r})$ is the ($N$-component) displacement of a node from its position in the perfect lattice, and $K$ is chosen as a reciprocal lattice vector (RLV). Overlines stand for disorder averages, and brackets for thermal averages. Thermal fluctuations are subdominant, and we focus on $T = 0$. It was established [8,10] that at large scale $u(\mathbf{r})$ is *a log-correlated field*,

$$\overline{\langle[u(\mathbf{r})-u(0)]^2\rangle} \simeq \mathcal{A}_2 \ln\frac{r}{a}, \tag{1}$$

where $a$ is a microscopic cutoff, and $r := |\mathbf{r}|$. If one further assumes $u(\mathbf{r})$ to be Gaussian, one obtains

$$C_K(\mathbf{r}) \sim r^{-\eta_K}, \tag{2}$$

with $\eta_K = \eta_K^{\mathrm{G}} := \frac{1}{2}\mathcal{A}_2 K^2$, hence quasi–long-range translational order and sharp diffraction peaks, a characteristic of the BrG [8,9]. This holds for space dimension $d_{\mathrm{lc}} < d < d_{\mathrm{uc}}$ (*i.e.*, $\mathbf{r} \in \mathbb{R}^d$) with $d_{\mathrm{lc}} = 2$, $d_{\mathrm{uc}} = 4$ for standard local elasticity. It was obtained by variational methods and confirmed by the Functional renormalization group (FRG) [8,10], a field-theoretic method developed in recent years [11–16], which allows to treat multiple metastable states. The FRG predicts the universal amplitude $\mathcal{A}_2$ in a dimensional expansion in $d = d_{\mathrm{uc}} - \varepsilon$. In this letter we restrict for simplicity to the scalar case $N = 1$, *i.e.* $u(\mathbf{r}) \in \mathbb{R}$, and choose the periodicity of $u$ to be one, hence the RLV to be $K = 2\pi k$ with $k$ integer. Then, within a 2-loop FRG calculation [13], $\mathcal{A}_2 = \frac{\varepsilon}{18} + \frac{\varepsilon^2}{108} + \mathcal{O}(\varepsilon^3)$ in agreement with numerics [17,18] for $d = 3$.

The rationale for the Gaussian approximation is that around $d_{\mathrm{uc}}$ one can decompose $u = \sqrt{\varepsilon}u_1 + \varepsilon u_2 + \dots$ into independent fields $u_i$, where $u_1$ is Gaussian (see appendix G of [16]). Hence non-Gaussian corrections to $\eta_K$ are expected only to $\mathcal{O}(\varepsilon^4)$. However they grow rapidly with $K$ and surely become important for secondary Bragg peaks. This motivates a calculation of the higher cumulants of $u(\mathbf{r})$. We also want to study $C_K(\mathbf{r})$ for arbitrary $K = 2\pi k$ with $k$ not necessary an integer. This is needed, *e.g.*, in the context of the roughening transition [19] to determine whether the BrG is stable to a small periodic perturbation $V_K = \int \mathrm{d}^d\mathbf{r} \cos(Ku(\mathbf{r}))$. Finally, for the algebraic decay (2) to hold for all $K$ all cumulants need to grow as $\ln r$, a property which we demonstrate.

Another motivation to study the higher cumulants of $u(\mathbf{r})$ comes from multifractal statistics, with examples ranging from turbulence [20] to localization of quantum particles [21]. Although $u(\mathbf{r})$ exhibits single-scale fractal statistics, we show here that the *exponential field* $e^{u(\mathbf{r})}$ exhibits multifractal scaling, *i.e.* its moments behave with system size $L$ as

$$\overline{\langle e^{qu(\mathbf{r})}\rangle} \sim \left(\frac{a}{L}\right)^{x_q}, \tag{3}$$

with a scaling dimension $x_q$. This provides an interesting example beyond the well-studied Gaussian case [22,23] of the general correspondence between exponentials of log-correlated fields and statistically self-similar and homogeneous multifractal fields [24].

The aim of this letter is thus to go beyond the Gaussian approximation: We calculate the multifractal exponents $x_q$ and obtain the higher cumulants of the log-correlated displacement field $u$ as

$$\overline{\langle [u(\mathbf{r}) - u(0)]^n\rangle}^{\mathrm{c}} \simeq \mathcal{A}_n \ln(r/a) \tag{4}$$

for $r \gg a$, $n$ even, where each $\mathcal{A}_n$ is calculated to leading order in $\varepsilon = 4 - d$ (odd cumulants vanish by parity $u \to -u$). We use the FRG and develop a method based on the asymptotic evaluation of functional determinants, which allows us to sum up an *infinite subset of diagrams*. Amazingly, it can also be applied to compute integrals appearing in a perturbative calculation on the field-theory side of AdS/CFT, known as Konishi integrals [25].

Let us mention that for the same model in $d = d_{\mathrm{lc}} = 2$ (the Cardy-Ostlund model) such a summation was achieved using conformal perturbation theory [26]. While for $d > 2$ the $\mathcal{A}_n$ are $T$ independent, in $d = 2$ the glass phase is marginal and exists for $T < T_{\mathrm{c}}$. The higher cumulants, as well as $C_K(\mathbf{r})$ for $k \leq 1$, were obtained to leading order in $T_{\mathrm{c}} - T$.

**The model.** – The Hamiltonian of an elastic system in a disordered environment can be written as

$$\mathcal{H}[u] = \int_{\mathbf{x}} \frac{1}{2}[\nabla u(\mathbf{x})]^2 + \frac{m^2}{2}u^2(\mathbf{x}) + V(u(\mathbf{x}), \mathbf{x}), \tag{5}$$

with $\int_{\mathbf{x}} := \int \mathrm{d}^d\mathbf{x}$. The first term is the elastic energy. The second term is a confining potential with curvature $m^2$ which effectively divides the system into independent subsystems of size $L_m = 1/m$, hence provides an infrared (IR) cutoff. The random potential $V(u, \mathbf{x})$ is a Gaussian with zero mean and correlator

$$\overline{V(u, \mathbf{x})V(u', \mathbf{x}')} = R_0(u - u')\delta^d(\mathbf{x} - \mathbf{x}'), \tag{6}$$

where $R_0(u)$ is a function of period unity, reflecting the periodicity of the unperturbed crystal [3]. The partition function in a given disorder realization, at temperature $T$, is $\mathcal{Z} := \int \mathcal{D}[u]\,\mathrm{e}^{-\mathcal{H}[u]/T}$. To average over the disorder, we introduce replicas $u_\alpha(\mathbf{x})$, $\alpha = 1, \ldots, \mathsf{n}$ of the original system. This leads to the bare replicated action

$$\mathcal{S}_{R_0}[u] = \frac{1}{T}\sum_\alpha \int_{\mathbf{x}} \frac{1}{2}[\nabla u_\alpha(\mathbf{x})]^2 + \frac{m^2}{2}u_\alpha^2(\mathbf{x}) - \frac{1}{2T^2}\sum_{\alpha\beta}\int_{\mathbf{x}} R_0\big(u_\alpha(\mathbf{x}) - u_\beta(\mathbf{x})\big). \tag{7}$$

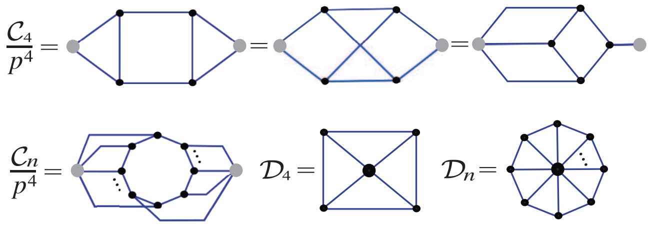



Fig. 1: (Colour on-line) Diagrammatic representation of the integrals contributing to the translational correlation function to leading order. The $C_n$ have two external points (big circles, grey) where the external momentum $p$ enters. They are constructed from a polygon with $n$ vertices each attached to one of the two external points. They are finite in $d = 4$ and $\sim 1/p^4$. $\mathcal{D}_n$ has one external point (big circle, not integrated over) all other points are integrated over. It is log-divergent in $d = 4$.

The observables of the disordered model can be obtained from those of the replicated theory in the limit $\mathsf{n} \to 0$.

**FRG basics.** – The central object of the FRG is the renormalized disorder correlator, the $m$-dependent function $R(u)$. Appropriately defined from the effective action $\Gamma[u]$ associated to $\mathcal{S}_{R_0}[u]$, the function $R(u)$ is an observable [14], which has been measured in numerics [27] and in experiments [28]. It satisfies a FRG flow equation as $m$ is decreased to zero ($R = R_0$ for $m = \infty$). Under rescaling, $R(u) = A_d m^{\varepsilon - 4\zeta}\tilde{R}(m^\zeta u)$, with $A_d = \frac{(4\pi)^{d/2}}{\varepsilon\Gamma(\varepsilon/2)}$, $\tilde{R}(u)$ admits a periodic fixed point (FP) with $\zeta = 0$, and $u \in [0, 1]$,

$$\tilde{R}^*(u) - \tilde{R}^*(0) = \tilde{R}^{*\prime\prime}(0)\frac{1}{2}u^2(1 - u)^2. \tag{8}$$

This form is valid for any $d < 4$, and $-\tilde{R}^{*\prime\prime}(0) = \frac{\varepsilon}{36} + \frac{\varepsilon^2}{54}$ to two-loop accuracy, in agreement with numerics [27]. The salient feature is that the renormalized force correlator $-R''(u)$ acquires a cusp at $u = 0$, which we denote by $\tilde{\sigma} = \tilde{R}^{*\prime\prime\prime}(0^+) = \frac{\varepsilon}{6} + \frac{\varepsilon^2}{9}$. This cusp, seen in experiments [28], is the hallmark of the multiple metastable states and is directly related to the statistics of shocks and avalanches which occur when applying an external force [16].

**Determinant formula.** – The cumulants (4) can be computed from (7) in perturbation theory in $R_0$ at $T = 0$, the leading order being $\mathcal{O}(R_0'''(0^+)^n)$. This perturbation theory involves (complicated) replica combinatorics, see, *e.g.*, [13]. It also requires the evaluation of multi-loop integrals represented in fig. 1, a formidable task. We now show how to shortcut these difficulties. We first reduce the problem to the calculation of a functional determinant using the method developed in [29] to evaluate averages of the form $\mathcal{G}[\lambda] := \overline{\langle \exp(\int_{\mathbf{x}} \lambda(\mathbf{x})u(\mathbf{x}))\rangle} = \lim_{\mathsf{n}\to 0}\langle \exp\left(\int_{\mathbf{x}} \lambda(\mathbf{x})u_1(\mathbf{x})\right)\rangle_{\mathcal{S}}$ where $u_1(\mathbf{x})$ stands for one of the $\mathsf{n}$ replicas. The function $C_K(\mathbf{r})$ can then be computed using the charge density of a dipole, $\lambda_{\mathrm{D}}(\mathbf{x}) := iK[\delta(\mathbf{x} - \mathbf{r}) - \delta(\mathbf{x})]$. For an arbitrary $\lambda(\mathbf{x})$, the average is expressed as $\mathcal{G}[\lambda] = \exp(\int_{\mathbf{x}} \lambda(\mathbf{x})u^\lambda(\mathbf{x}) - \Gamma[u^\lambda])$, where $u^\lambda(\mathbf{x})$ extremizes the exponential, *i.e.* is solution of $\partial_{u_a(\mathbf{x})}\Gamma[u]\big|_{u=u^\lambda} = \lambda(\mathbf{x})\delta_{a1}$. The effective action was calculated in an expansion in $R$ (*i.e.* in $\varepsilon$) to leading order (one loop) as $\Gamma[u] = \mathcal{S}_R[u] + \Gamma_1[u]$ where $\mathcal{S}_R[u]$ is the improved action with the bare correlator $R_0$ replaced by the renormalized one $R$, and $\Gamma_1[u]$ is displayed, *e.g.*, in [29,30]. Performing the extremization at $T = 0$, a slight generalization of sect. IV.A of ref. [29] leads to

$$\overline{\left\langle e^{\int_{\mathbf{x}} \lambda(\mathbf{x})u(\mathbf{x})}\right\rangle} = \mathcal{G}_{\text{Gauss}}[\lambda]e^{-\Gamma_\lambda}. \tag{9}$$

Here $\mathcal{G}_{\text{Gauss}}[\lambda] = e^{\frac{1}{2}\int_{\mathbf{x}\mathbf{x}'} \lambda(\mathbf{x})\lambda(\mathbf{x}')\overline{\langle u(\mathbf{x})u(\mathbf{x}')\rangle}}$ is the Gaussian approximation, $\overline{\langle u(\mathbf{x})u(\mathbf{x}')\rangle}$ the exact 2-point correlation function, and the effective action is

$$-\Gamma_\lambda = \frac{1}{2}\big\{\ln \mathcal{D}_{\text{reg}}[\sigma U(\mathbf{r})] + \ln \mathcal{D}_{\text{reg}}[-\sigma U(\mathbf{r})]\big\}. \tag{10}$$

The effective disorder is $\sigma := R'''(0^+)$, and we define

$$\mathcal{D}[\sigma U(\mathbf{r})] := \frac{\det(-\nabla^2 + \sigma U(\mathbf{r}) + m^2)}{\det(-\nabla^2 + m^2)}. \tag{11}$$

Its logarithm, $\ln(\mathcal{D}[\pm\sigma U]$, has a perturbative expansion in $\sigma$. The first two terms, of order $\sigma$ and $\sigma^2$, which contain ultraviolet divergences in $d = 4$, are included in the Gaussian part. The remaining terms, *i.e.* all $\mathcal{O}(\sigma^p)$ with $p \geq 3$, define the regularized determinant $\ln(\mathcal{D}_{\text{reg}}[\pm\sigma U])$. Thus (10) contains only information about higher cumulants[1]. We have introduced the potential

$$U(\mathbf{r}) := \int_{\mathbf{x}} (-\nabla^2 + m^2)^{-1}_{\mathbf{r},\mathbf{x}}\, \lambda(\mathbf{x}), \tag{12}$$

which in the limit $m \to 0$ satisfies the $d$-dimensional Poisson equation $\nabla^2 U(\mathbf{r}) = -\lambda(\mathbf{r})$. Note that two copies of the determinant appear in the present static problem in eq. (9) as $\sqrt{\mathcal{D}[\sigma U]\mathcal{D}[-\sigma U]}$, which can thus be interpreted as originating from an *effective fermionic* field theory with two flavors of real fermions. A related observation was made in a dynamical calculation of the distribution of pinning forces at the depinning transition [31], where only one copy appears, as $\mathcal{D}[\sigma U]$. Note also, from fig. 1, that to this order we have an effective *cubic* field theory with coupling $\sigma$. The 2-point correlation function in Fourier[2] reads $\overline{\langle u_p u_{-p}\rangle} = c_d p^{-d} f(p/m)$, with $f(z) \sim \tilde{c}_d z^d/c_d$ for small $z$, $f(\infty) = 1$, $\tilde{c}_d = -A_d \tilde{R}^{*\prime\prime}(0)$ and $c_d = \tilde{c}_d(1 - \varepsilon + \ldots)$. Inserting this with the 1-loop FP value into $\mathcal{G}_{\text{Gauss}}[\lambda]$ leads to the above Gaussian result for $\eta_K^{\text{G}}$ with $\mathcal{A}_2 = \frac{2S_d c_d}{(2\pi)^d}$, and $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

**Evaluation of the determinant.** – We now have to evaluate the functional determinant (11). Unfortunately, there is no general method in $d > 1$ for a non-spherically-symmetric potential. However, as we show below, it is sufficient to calculate the determinant for a spherically symmetric potential, and then apply a multifractal scaling analysis [24,32,33]. Thus, we start by computing the scaling dimension $x_q = x_{-q}$, as defined from (3). To this aim we calculate $\mathcal{G}[\lambda]$ for a (regularized) point-like charge $\lambda_{\text{p}}(\mathbf{r}) := q\delta_a(\mathbf{r})$ in a finite-size system. Since the corresponding potential is spherically symmetric, to obtain the determinant ratio (11) we can employ the Gel'fand-Yaglom method [34], generalized to $d$ dimensions [35]. We separate the radial and angular parts of the eigenfunctions as $\Psi(r,\vec{\theta}) = \frac{1}{r^{(d-1)/2}}\psi_l(r)\, Y_l(\vec{\theta})$, where the angular part is given by a hyperspherical harmonic $Y_l(\vec{\theta})$, labeled in part by a non-negative integer $l$. The radial part $\psi_l(r)$ is an eigenfunction of the 1D (radial) Schrödinger-like operator $\mathcal{H}_l + \sigma U(r) + m^2$, where

$$\mathcal{H}_l := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\left(l + \frac{d-3}{2}\right)\left(l + \frac{d-1}{2}\right)}{r^2}. \tag{13}$$

The logarithm of (11) can be written as a sum of the logarithms of the 1D determinant ratios $\mathcal{B}_l$ for partial waves weighted with the degeneracy of angular momentum $l$,

$$\ln\left(\mathcal{D}[\sigma U]\right) = \sum_{l=0}^{\infty} \frac{(2l + d - 2)(l + d - 3)!}{l!(d-2)!} \ln \mathcal{B}_l. \tag{14}$$

The Gel'fand-Yaglom method gives the ratio of the 1D functional determinants for each partial wave $l$ as

$$\mathcal{B}_l := \frac{\det\left[\mathcal{H}_l + \sigma U(r) + m^2\right]}{\det\left[\mathcal{H}_l + m^2\right]} = \frac{\psi_l(L)}{\tilde{\psi}_l(L)}. \tag{15}$$

Here $\psi_l(r)$ is the solution of the initial-value problem for

$$\left[\mathcal{H}_l + \sigma U(r) + m^2\right]\psi_l(r) = 0, \tag{16}$$

satisfying $\psi_l(r) \sim r^{l+(d-1)/2}$ for $r \to 0$. Equation (15) holds for the boundary conditions $u(|\mathbf{r}| = L) = 0$, taking the large-$L$ limit afterwards[3]. The function $\tilde{\psi}_l(r)$ solves (16) with the same small-$r$ behavior, but for $\sigma = 0$.

We can now calculate $\overline{\langle e^{qu(\mathbf{r})}\rangle}$ to leading order in $d = 4 - \varepsilon$. Since $\sigma = \mathcal{O}(\varepsilon)$ we can perform the calculation in $d = 4$. A point-like charge distribution leads to a potential $U(r) \sim 1/r^{d-2}$ which is too singular at the origin in $d = 4$. We introduce an UV cutoff via a uniformly charged ball of radius $a$, $\lambda_{\text{B}}(\mathbf{r}) = \frac{qd}{S_d a^d}\Theta(a - |\mathbf{r}|)$. Since $L$ is finite, we solve Poisson's equation setting $m \to 0$ and obtain

$$U(r) = \begin{cases} \dfrac{qa^{2-d}}{2S_d}\left(\dfrac{d}{d-2} - \dfrac{r^2}{a^2}\right) & \text{for } 0 < r < a, \\[2ex] \dfrac{q}{S_d(d-2)}\dfrac{1}{r^{d-2}} & \text{for } a < r < L. \end{cases} \tag{17}$$

We insert this potential in the Gaussian approximation which reads $\ln \mathcal{G}_{\text{Gauss}} = -\frac{1}{2}R''(0)\int_{\mathbf{r}} U(r)^2$, to lowest order

[1] A simpler version of (10) was considered in appendix G of [16] for a uniform source; it yields the cumulants of $\int_{\mathbf{r}} u(\mathbf{r})$.

[2] It was calculated to $\mathcal{O}(\varepsilon^2)$ in [13], sect. VI A.

[3] To work directly in an infinite system, the electric field must vanish fast enough. One can either use $m = 0$ with a neutral charge configuration (dipole), or $m > 0$ (screening, exponential decay).

$\mathcal{O}(\varepsilon)$. The log-divergence of this integral in $d = 4$ leads to $x_q^{\rm G} = -\tilde{c}_4 q^2/(8S_4) = -\varepsilon q^2/72$. More generally, eq. (1) requires by consistency that $\overline{u(\mathbf{r})^2} \simeq \frac{1}{2}\mathcal{A}_2 \ln(L/a)$ hence $x_q^{\rm G} = -\mathcal{A}_2 q^2/4$, fixing the quadratic part $\mathcal{O}(q^2)$ of $x_q$.

To calculate the leading non-Gaussian corrections to $x_q$ via (11), we find the solution of (16) in $d = 4$ with the potential (17). It reads, for $r < a$

$$\psi_l(r) = \frac{r^{l+\frac{3}{2}}}{e^{\frac{ir^2\sqrt{s}}{2a^2}}}\, {}_1F_1\left(\frac{l+2-i\sqrt{s}}{2}+1; l+2; \frac{ir^2\sqrt{s}}{a^2}\right), \tag{18}$$

and for $a < r < L$,

$$\psi_l(r) = c_1 r^{\frac{1}{2}-\sqrt{(l+1)^2+s}} + c_2 r^{\sqrt{(l+1)^2+s}+\frac{1}{2}}. \tag{19}$$

We introduced $s := \sigma q/(2S_d)$. One can find $c_{1,2}$ by matching at $r = a$. Using eq. (15) we obtain the partial-wave determinant, which is universal at large $L$,

$$\ln \mathcal{B}_l = \left[\sqrt{(l+1)^2+s} - (l+1)\right] \ln(L/a) + \mathcal{O}(L^0). \tag{20}$$

The term $\mathcal{O}(L^0)$ can be calculated from the $c_i$; it is not universal. Note that the massive problem also leads to (20) with $\ln(L)$ replaced by $\ln(1/m)$.

Substituting this result into eq. (14) yields the result for $\ln(\mathcal{D}[\sigma U])$. However, the sum over $l$ diverges, indicating that this functional determinant requires regularization in $d \geq 2$ [35]. However in (10) we only need the regularized determinant $\mathcal{D}_{\rm reg}[\pm\sigma U] \sim (L/a)^{-F_{\rm reg}(\pm s)}$ where the first two orders in $s$ are subtracted,

$$F_{\rm reg}(s) = -\sum_{l=0}^{\infty}(l+1)^2\Bigg(\sqrt{(l+1)^2+s} - (l+1) - \frac{s}{2(l+1)} + \frac{s^2}{8(l+1)^3}\Bigg). \tag{21}$$

Summing over $l$, it can also be written as a series in $s$,

$$F_{\rm reg}(s) = \sum_{n=3}^{\infty} f_n s^n, \quad f_n = (-1)^n \frac{\Gamma(n-\frac{1}{2})\zeta(2n-3)}{2\sqrt{\pi}\,\Gamma(n+1)}. \tag{22}$$

Putting together the two copies we obtain the multi-fractal scaling exponent, an even function of $s$ (and $q$),

$$x_q = -\frac{1}{4}\mathcal{A}_2 q^2 + F(s), \quad s = \frac{\varepsilon}{3}q, \tag{23}$$

$$F(s) := \frac{1}{2}\left[F_{\rm reg}(s) + F_{\rm reg}(-s)\right] = \sum_{n=2}^{\infty} f_{2n} s^{2n}. \tag{24}$$

To leading order we used $\sigma = A_d\tilde{\sigma}$, $\tilde{\sigma} = \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2)$ and $S_4 = 2\pi^2$. The final result is finite, as we avoided divergences by i) using perturbation theory in the renormalized $R$ rather than in the bare $R_0$, ii) by separating the non-Gaussian part $F(s)$ from the Gaussian one. For completeness we also defined the single-copy exponent $F_{\rm reg}(s)$ since it appears in the theory of depinning[4].

[4]At depinning, there is an additional tadpole diagram associated to the non-zero average $\overline{u(\mathbf{r})} = -F_c/m^2$, where $F_c$ is the threshold force. Similarly separating the non-Gaussian parts leads to $F_{\rm reg}(s)$.

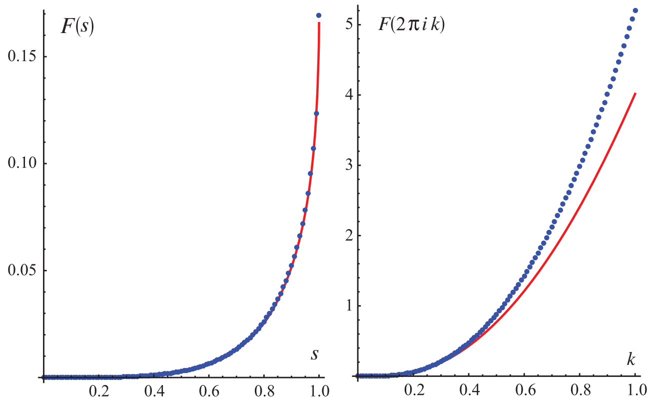



Fig. 2: (Colour on-line) Numerical evaluation (blue dots) of $F(s)$ (left) and $F(2\pi i k)$ (right). The red solid line is the contribution of the mode $l = 0$.

**Analysis of the result.** – Equation (23) is an even series in $s$ with a radius of convergence of $|s| = 1$. At $s = \pm 1$, $F(s)$, plotted in fig. 2, has a square-root singularity given by its $l = 0$ term. On the other hand, the exponent $x_q$ must satisfy[5] $q\frac{\rm d}{{\rm d}q}x_q \leq 0$, and convexity $\frac{{\rm d}^2}{{\rm d}q^2}x_q \leq 0$, both requirements for multifractal field theories [33]. While the Gaussian part $x_q^{\rm G} = -\frac{1}{4}\mathcal{A}_2 q^2$ does, the correction term $F(s)$ does not, since $F''(s) \geq 0$. Since $F''(s) \sim \frac{1}{8(1-|s|)^{3/2}}$ diverges at $s = \pm 1$ ($q = q_p \simeq \frac{3}{\varepsilon}$) one cannot trust the calculation in that region[6]; it surely fails when $F''(\frac{q\varepsilon}{3}) > \frac{1}{4\varepsilon}$.

**Calculation of 2-point correlations.** – To obtain the cumulants (4) and the translational correlation function (2) we would need a dipole source, for which we cannot solve the Schrödinger problem. One way to proceed is to *assume* that the exponential field $e^{u(\mathbf{r})}$ obeys the conventional multifractal scaling formula [24,32,33]:

$$\overline{\langle e^{q_1 u(\mathbf{r}_1)} e^{q_2 u(\mathbf{r}_2)}\rangle} \sim \left(\frac{r_{12}}{a}\right)^{x_{q_1+q_2}-x_{q_1}-x_{q_2}} \left(\frac{L}{a}\right)^{-x_{q_1+q_2}}, \tag{25}$$

with $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$. Since we already calculated $x_q$, this formula, taken for $q_1 = -q_2 = q$ immediately yields

$$\overline{\langle e^{q[u(\mathbf{r})-u(0)]}\rangle} \sim \left(\frac{r}{a}\right)^{-2x_q}, \tag{26}$$

using that $x_q = x_{-q}$ and $x_0 = 0$. Let us define the expansion $x_q = \sum_{n=1}^{\infty}\frac{1}{n!}a_n q^n$. Using the standard formula

$$\ln \overline{\langle e^A\rangle} = \sum_{n=1}^{\infty}\frac{1}{n!}\overline{\langle A^n\rangle}^c, \tag{27}$$

we obtain one of the main results of this letter, eq. (4), with the amplitudes for even $n \geq 4$,

$$\mathcal{A}_n = -2a_n = -\frac{\Gamma(n-\frac{1}{2})\zeta(2n-3)}{\sqrt{\pi}}\left(\frac{\varepsilon}{3}\right)^n. \tag{28}$$

[5]Since $\overline{\langle qu \sinh qu\rangle} \geq 0$ and from Cauchy-Schwarz the inequality $\overline{\langle u^2 e^{qu}\rangle}\;\overline{\langle e^{qu}\rangle} \geq \overline{\langle ue^{qu}\rangle}^2$ must hold.

[6]Our result is a summation of a convergent series in $q\varepsilon$, but there is no guarantee that there are no non-perturbative corrections.

There is actually more information in eq. (25): Using (27) and expanding in powers of $q_1^j q_2^{n-j}$ we obtain

$$\overline{\langle u(\mathbf{r}_1)^j u(\mathbf{r}_2)^{n-j}\rangle}^c \simeq a_n \ln(r_{12}/L), \tag{29}$$
$$\overline{\langle u(\mathbf{r}_1)^n\rangle}^c \simeq -a_n \ln(L/a). \tag{30}$$

While we already know (30) from (3) and (27), eq. (29), valid for any $1 \le j \le n-1$ represents strong constraints.

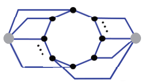
Formula (25) is, at this stage, an *educated guess*, since we do not know the exact solution to the corresponding 2-charge (dipole) Schrödinger problem. We now close this gap via a careful examination of the integrals appearing in the expansion of the determinant in powers of $\sigma$, represented by the diagrams in fig. 1. We show two properties:

i) All terms of the form eq. (29) are equal, and independent of $j$: This *proves* that both eqs. (25) and (26) hold.

ii) The topologically distinct integrals with the same $j$ are also all equal. This remarkable property goes beyond what is needed for eq. (29), and provides simple expressions for such integrals; as announced in the introduction, they are of interest in the AdS/CFT context.

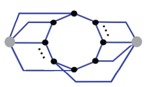
For clarity, let us detail the term $n = 4$ (setting $m = 0$). The calculation of $\overline{\langle u(\mathbf{r}_1)^2 u(\mathbf{r}_2)^2\rangle}$ involves two 3-loop integrals, $I_{\{2,2\}_1}(p)$ and $I_{\{2,2\}_2}(p)$, which are represented by the first two (topologically distinct) diagrams in fig. 1. The first is *equal* to the integral, with entering momentum $p$, $I_{\{2,2\}_1}(p) := \int_{\mathbf{q}} \frac{I(\mathbf{p},\mathbf{q})^2}{q^2(\mathbf{p}-\mathbf{q})^2}$ with $I(\mathbf{p},\mathbf{q}) := \int_{\mathbf{k}} \frac{1}{k^2(\mathbf{k}+\mathbf{p})^2(\mathbf{k}+\mathbf{q})^2}$, $\int_{\mathbf{q}} := \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}$. The third diagram (*i.e.* integral) is the only one entering in the calculation of $\overline{\langle u(\mathbf{r}_1)^3 u(\mathbf{r}_2)\rangle}$. By power counting, these integrals are *both UV and IR finite* in $d = 4$, and scale as $p^{-4}$; we now determine their amplitude.

First we show that, for given $n$, the diagrams with two external points depicted in fig. 1 are *independent of how these points are attached to the polygon vertices*. In a nutshell this is because they all scale as $p^{-4}$, and if we identify the two external points, we obtain *the same* integral $\mathcal{D}_n$ in fig. 1. Explicitly, for $m = 0$ and $d = 4$, any of these diagrams has $n-1$ loops and $2n$ propagators, and reads



$$= \frac{\mathcal{C}_n}{p^4}, \tag{31}$$

where *a priori* $\mathcal{C}_n$ depends on how we attach the $n$ points of the polygon to the two external points. In a massive scheme, and $d = 4-\varepsilon$, by power counting this changes to



$$= \frac{\mathcal{C}_n}{p^{4+(n-1)\varepsilon}}\, g_n\!\left(\frac{p}{\alpha_n m}\right), \tag{32}$$

where $g_n(x) \to 1$ for $x \to \infty$, $g_n(0) = 0$ and $\alpha_n$ parameterizes the crossover point with $g_n(1) = \frac{1}{2}$. Now $\mathcal{D}_n$ is obtained from $\mathcal{C}_n$ by integrating over the external momentum:

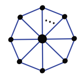
$$\begin{aligned}\mathcal{D}_n &= \int_{\mathbf{p}} \frac{\mathcal{C}_n}{p^{4+(n-1)\varepsilon}}\, g_n\!\left(\frac{p}{\alpha_n m}\right) \simeq \mathcal{C}_n \frac{S_d}{(2\pi)^d} \int_{\alpha_n m}^{\infty} \frac{\mathrm{d}p}{p^{1+n\varepsilon}} \\ &= \frac{\mathcal{C}_n(\alpha_n m)^{-n\varepsilon}}{8\pi^2 n\varepsilon} + \mathcal{O}\left(\varepsilon^0\right) = \frac{\mathcal{C}_n m^{-n\varepsilon}}{8\pi^2 n\varepsilon} + \mathcal{O}\left(\varepsilon^0\right). \end{aligned} \tag{33}$$

The leading pole in $\varepsilon$ does not depend on $\alpha_n$, and is universal. Since all these diagrams lead to the same value of $\mathcal{D}_n$, all integrals of the type (31) are *equal*, and in $d = 4$ equal to $\mathcal{C}_n/p^4$.

We already know the integral $\mathcal{D}_n$ in $d = 4$ from eqs. (21) and (22), by matching powers of $q$ in the expansion of the determinant with a point source, $\ln \mathcal{D}[\sigma U] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathcal{D}_n (q\sigma)^n$ which yields $\mathcal{D}_n \simeq (-1)^n n f_n/(2\pi)^{2n} \ln(\frac{L}{a})$ for any $n \ge 3$. Interestingly, the Gel'fand-Yaglom method allows us to calculate $\mathcal{D}_n$ directly in $d = 4-\varepsilon$. For $d < 4$ we can set $a = 0$ in the potential (17). The corresponding radial Schrödinger problem can be solved *exactly* as

$$\psi_l(r) = r^{l+\frac{d-1}{2}} z_l(r), \quad z_l(r) = {}_0F_1\left(\frac{2(l+1)}{\varepsilon}; \frac{2sr^\varepsilon}{(2-\varepsilon)\varepsilon^2}\right).$$

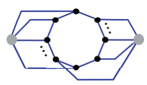
Using the identity $\lim_{\varepsilon\to 0} \varepsilon \ln_0 F_1(\frac{2(l+1)}{\varepsilon}, \frac{\tilde{s}}{\varepsilon^2}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\Gamma(n-\frac{1}{2})\tilde{s}^n}{2n\sqrt{\pi}\Gamma(n+1)(l+1)^{2n-1}}$ we calculate to leading order in $\varepsilon$, $\ln \mathcal{D}[\sigma U] \simeq \sum_{l=0}^{\infty} (l+1)^2 \ln z_l(L)$. This yields the polygon integrals for $n \ge 3$ in the massive scheme,



$$\mathcal{D}_n = \ldots = \frac{m^{-n\varepsilon}}{n\varepsilon} \frac{\Gamma(n-1/2)\zeta(2n-3)}{2\sqrt{\pi}(2\pi)^{2n}\Gamma(n)} + \mathcal{O}(\varepsilon^0). \tag{34}$$

Note that $\frac{L^{n\varepsilon}}{n\varepsilon}$ changed to $\frac{m^{-n\varepsilon}}{n\varepsilon}$. Further substituting this factor by $\ln(L/a)$ reproduces the above estimate for $d = 4$.

Using eqs. (33) and (34) we now obtain $\mathcal{C}_n$ in $d = 4$,



$$\mathcal{C}_n = p^4 \ldots = \frac{\Gamma(n-\frac{1}{2})\zeta(2n-3)}{\sqrt{\pi}\Gamma(n)(2\pi)^{2n-2}}. \tag{35}$$

This allows to expand the determinant in the presence of two charges $q_1$, $q_2$, in terms of 2-point diagrams, and obtain, using (27) and (10) in $d = 4$ with $m = 0$:

$$\begin{aligned}\sum_{n\ge 4} \frac{1}{n!} \overline{\langle [q_1 u(\mathbf{r}) + q_2 u(0)]^n\rangle}^c &= \sum_{n\ \mathrm{even}\ge 4} \frac{(-1)^{n+1}}{n}\sigma^n \\ \times\left[(q_1^n + q_2^n)\mathcal{D}_n + \int_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}} \sum_{j=1}^{n-1} \binom{n}{j} q_1^j q_2^{n-j} \frac{\mathcal{C}_n}{p^4}\right]. \end{aligned} \tag{36}$$

Here we used that all $\mathcal{C}_n$ integrals are the same. Since $\binom{n}{j}$ appears on both sides it implies (29) with $a_n = -\frac{S_4}{(2\pi)^4}\mathcal{C}_n(n-1)!\sigma^n$ in agreement with (28). Choosing $q_2 = -q_1$ rederives our main result for the cumulants (4) and (28) since $\sum_{j=1}^{n-1}\binom{n}{j}(-1)^j = -2$. We thus proved that the multifractal scaling relations (25) and (26) hold.

Performing the analytical continuation $q = iK$ we obtain the decay exponent[7] of the translational correlations,

$$\eta_K = \Big[\frac{\varepsilon}{36} + \frac{\varepsilon^2}{216} + \mathcal{O}(\varepsilon^3)\Big]K^2 + 2F\Big(iK\frac{\varepsilon}{3}\Big). \qquad (37)$$

The wave vector $K$ is arbitrary, not necessarily a RLV[8]. Although non-Gaussian corrections start at $\mathcal{O}(\varepsilon^4)$, setting directly $\varepsilon = 1$ and $K = K_0 = 2\pi$ yields[9] $\eta^{\rm G}_{K_0}|_{\text{1-loop}} = 1.097$, $\eta^{\rm G}_{K_0}|_{\text{2-loop}} = 1.279$ while $\eta_{K_0} - \eta^{\rm G}_{K_0} = 0.569$. Even if these corrections may be an overestimate, and higher-loop corrections are needed, non-Gaussian effects[10] appear to be non-negligible for $d = 3$ [18]. Comparison with the elastic term [19] then shows that a small periodic perturbation $V_K$ becomes relevant for $K < K_c$ with $2 - \eta_{K_c} = 0$.

**Conclusion.** – Using functional determinants we obtained the scaling exponents of the (real and imaginary) exponential correlations of the displacement field in a disordered elastic system. We leave the calculation of the spectrum of fractal dimensions[11], and the extension to a more general elastic kernels for the future. As a surprising corollary, our method yields, in an elegant way and for arbitrary $n$, exact expressions for the integrals $\mathcal{C}_n$ (we numerically checked formula (35) for $n = 3, 4, 5$). Similar integrals appear in $N = 4$ SYM, on the field-theory side of two theories related via AdS/CFT: *E.g.*, $\mathcal{C}_5$ contributes to the Konishi anomalous dimension in $N = 4$ SYM at five-loop order, and an elaborate formalism was put in place to calculate it [25]. We hope that our method, and possible generalizations, will also allow for a further-reaching check of the AdS/CFT duality[12].

* * *

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[7]Note that $e^{iKu(r)}$ obeys ordinary field-theory scaling, while $e^{qu(r)}$ obeys multifractal scaling [33].

[8]In $d = 2$, $C_K(r)$ was argued [36] to exhibit cusps for integer $K/(2\pi)$ due to screening of the 2-point function by the interaction.

[9]We used eq. (21) which can be considered as the analytic continuation of eq. (22), whose radius of convergence is $K = 3$.

[10]In $d = 4$ the second cumulant grows as $\ln(\ln(r))$, while higher ones reach a (non-universal) finite limit.

[11]The Gibbs measure of a particle diffusing on top of the elastic object with potential energy $\sim u(\mathbf{r})$ provides a normalized multifractal measure $\mu(\mathbf{r}) = \frac{e^{\gamma u(\mathbf{r})}}{\int_{\mathbf{x}} e^{\gamma u(\mathbf{x})}}$ from which one can calculate a spectrum of dimensions.

[12]Reciprocally, the results in [37] yield the full 4-point function for the Bragg glass.