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Optimization problems and replica symmetry breaking in finite connectivity spin glasses

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Abstract. A formalism capable of handling the first step of hierarchical replica symmetry breaking (RSB) in finite-connectivity models is introduced. The emerging order parameter is claimed to be a probability distribution over the space of field distributions (or, equivalently magnetization distributions) inside the cluster of states. The approach is shown to coincide with previous works in the replica-symmetric case and in the two limiting cases $m = 0$ and 1 where m is Parisi's break point. As an application to the study of optimization problems, the GS properties of the random 3-satisfiability problem are investigated and we present a first RSB solution improving replica-symmetric results.

1. Introduction

It is now commonly thought that spin glasses may exhibit highly interesting and nontrivial features already at the mean-field level [1, 2]. This statement stems from the extensive studies performed during the last 20 years on models with infinite connectivity (IC), especially on the celebrated Sherrington–Kirkpatrick (SK) model [1]. Such models strongly differ from realistic finite-dimensional systems on two points, that is their large connectivity and the absence of any geometrical underlying structure. As was realized some years ago, only the latter aspect is intrinsic to mean-field theory while the unrealistic nature of the connectivity may be cured [3, 5–10]. Mean-field spin glasses with finite connectivity (FC) are of importance for at least two reasons. First, they are expected to share common properties with finite-dimensional physical systems that IC models cannot exhibit [4]. Secondly, it is now well known that optimization or decision problems can rigorously be mapped onto the ground state (GS) of spin-glass models with FC [11]. The understanding of complex optimization problems [12], which would be of practical use for algorithmic design, therefore requires the introduction of sophisticated techniques that were invented in the IC models context, e.g. replica symmetry breaking (RSB) [2]. As regards to their potential interest, the investigation of the spin-glass phase of FC models has attracted little attention in the past years [7–10]. This situation is probably due mainly to the technical difficulties arising in the analytical calculations.

In this paper, we show that the complexity of FC models with respect to IC ones manifests itself through the emergence of new and richer order parameters (section 2). We expose how to compute the free-energy of FC models within the one-step RSB scheme

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in section 3. Section 4 is devoted to the derivation of the saddle-point equation for the functional order parameter. Throughout the paper, we check that this formalism gives back the known results for the replica symmetric (RS) theory of FC models and the RSB theory of IC models. Our approach can be applied to any Ising FC spin glass and is therefore of particular relevance to optimization problems. As an illustration, we concentrate in section 5 upon the so-called random 3-satisfiability (3-SAT) problem, of central importance in complexity theory [12, 13]. Recent theoretical interpretations [14, 15] of the numerical data accumulated so far on 3-SAT [16] speak indeed for the existence of a spin-glass phase, making 3-SAT a valuable testing ground for RSB calculations. A first RSB solution is found and shown to improve the RS result.

2. Order parameter for finite-connectivity models

2.1. Occupation densities and the multilevel gas picture

Let $\mathcal{H}$ be a Hamiltonian depending on N Ising spins S_i, $i = 1, \ldots, N$ and on some quenched degrees of freedom. To compute the equilibrium properties, we resort to the replica method [2]. Once the system has been replicated n times and the disorder averaged out, we obtain an effective model of 2^n-states spins $\boldsymbol{S}_i = (S_i^1, S_i^2, \ldots, S_i^n)$. In the absence of any underlying geometry (e.g. lattice), the effective Hamitonian $\mathcal{H}_{\rm eff}$ is invariant under any relabelling of the sites. As a consequence, $\mathcal{H}_{\rm eff}$ depends upon the spins *only* through the 2^n occupation densities $c(\boldsymbol{\sigma})$ defined as the normalized fractions of sites i such that $\boldsymbol{S}_i = \boldsymbol{\sigma}$ [15, 17]. This is the very meaning of a mean-field theory: the effective Hamiltonian depends on the spins only through a finite set of global fields rather than an extensive set of local ones. For instance, the effective Hamiltonian of the SK model at inverse temperature β [1, 17] reads

$$\begin{aligned} N\mathcal{H}_{\rm eff}^{\rm SK} &= -\frac{\beta}{2N}\sum_{i<j}\left(\sum_{a=1}^{n} S_i^a S_j^a\right)^2 \\ &\simeq -\frac{\beta N}{4}\sum_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2} c(\boldsymbol{\sigma}_1)c(\boldsymbol{\sigma}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)^2 \end{aligned} \tag{1}$$

up to O(1) irrelevant terms.

In this picture, we may interpret any mean-field IC or FC replicated system as a 'gas' of N particles living on 2^n interacting levels. Each level is labelled by a vector $\boldsymbol{\sigma}$ comprised of n binary components $\sigma^a = \pm 1$ and is filled in with $Nc(\boldsymbol{\sigma})$ particles. Taking into account the entropic contribution coming from the combinatorial choices of the sites, we obtain the general expression for the nth moment of the partition function Z,

$$\overline{Z^n} = \int_0^1 \prod_{\boldsymbol{\sigma}} \mathrm{d}c(\boldsymbol{\sigma})\,\delta\left(\sum_{\boldsymbol{\sigma}} c(\boldsymbol{\sigma}) - 1\right)\mathrm{e}^{-N\beta\mathcal{F}(\{c\})} \tag{2}$$

where the bar denotes the average over the disorder and the free-energy functional reads, to the largest order in N,

$$\mathcal{F}(\{c\}) = \mathcal{H}_{\rm eff}(\{c\}) + \frac{1}{\beta}\sum_{\boldsymbol{\sigma}} c(\boldsymbol{\sigma})\ln c(\boldsymbol{\sigma}). \tag{3}$$

In the thermodynamical limit, the occupation densities $c(\boldsymbol{\sigma})$ are determined through the 2^n saddle-point equations corresponding to the optimization of the free-energy functional $\mathcal{F}$,

$$c(\boldsymbol{\sigma}) = \lambda(n)\exp\left(-\beta\frac{\partial\mathcal{H}_{\rm eff}}{\partial c(\boldsymbol{\sigma})}\right) \tag{4}$$

where $\lambda(n)$ is to be chosen to ensure the normalization of the c's. All equilibrium properties can then be computed.

Let us briefly comment on the pros and the cons of the above formalism with respect to the usual formulation which involves overlaps between spins belonging to different replicas. Expressions (2) and (3) always require us to compute 2^n order parameters c. In the case of IC models, most of them contain redundant information since only $\frac{1}{2}n(n-1)$ distinct overlaps are required. As we shall see in the following, the structure of the order parameters of IC models is indeed very simple and does not account for the whole spectrum of mean-field disordered models. For FC systems, the present formalism proves to be much more tractable than the usual formulation from the analytical standpoint. It enables us to encode all overlaps in a concise way through the generating function $c(\boldsymbol{\sigma})$†. Another advantage of this approach appears when focusing on Hamiltonians with p-spins interactions ($p \geqslant 3$) [18]. The use of the occupation densities c's avoids the physically unclear introduction of Lagrange parameters, that are necessary to define overlaps even in the IC case.

2.2. Replica-symmetric ansatz

With a view to undertake RSB calculations, a short discussion of the RS ansatz is illuminating. The RS theory of FC models was worked out ten years ago [5, 6] and may be reformulated within the Thouless–Anderson–Palmer (TAP) framework [19]. Inside the single RS state, the spins S_i fluctuate around their Gibbs averages $\langle S_i \rangle$. All relevant information rests in the histogram $p_{\rm rs}(h)$ of the effective fields $h_i = \frac{1}{\beta}\tanh^{-1}(\langle S_i \rangle)$ [5]. The order parameter thus proves to be a function $p_{\rm rs}$, belonging to the space V of the probability distributions over real numbers. How are these results recovered in the present formalism? RS corresponds to the invariance of the saddle-point $c(\boldsymbol{\sigma})$ under any permutation of its components. In other words, $c(\boldsymbol{\sigma})$ is a function of the magnetization $s = \sum_{a=1}^{n} \sigma^a$ only. Inserting the above ansatz into the extremization conditions of $\mathcal{F}(\{c\})$, one easily finds all thermodynamical quantities through the one-to-one correspondence

$$c(\boldsymbol{\sigma}) = \int_R \mathrm{d}h\; p_{\rm rs}(h) \prod_{a=1}^{n} \left(\frac{\mathrm{e}^{\beta h \sigma^a}}{\mathrm{e}^{-\beta h} + \mathrm{e}^{\beta h}} \right) \tag{5}$$

or, equivalently,

$$c(s) = \int_R \mathrm{d}h\; p_{\rm rs}(h) \frac{\mathrm{e}^{\beta h s}}{(2\cosh \beta h)^n} \tag{6}$$

where the symbol $\int_R$ stands for an integral over the whole real axis. Note that the integral over h in (6) converges if the real part of the magnetization s is smaller than n (in absolute value). Therefore, when the number of replicas n tends to zero, the order parameter $c(s)$ may be analytically continued on the imaginary axis. This is precisely what one needs to go back to the field distribution

$$p_{\rm rs}(h) = \int_R \frac{\mathrm{d}s}{2\pi} c(\mathrm{i}s/\beta) \mathrm{e}^{-\mathrm{i}sh} \qquad (n \to 0). \tag{7}$$

Let us underline that a drastic simplification takes place in IC models. The field distribution $p_{\rm rs}$ (and c) becomes Gaussian; it is fully described by a variance, the RS overlap q [2], and the functional nature of the order parameter is hidden.

† Another generating function of all overlaps was already introduced by Mottishaw and De Dominicis for technical reasons. Their order parameter, however, includes some disorder dependence (see [7, equation (3)]) and its interpretation is therefore less clear than $c(\boldsymbol{\sigma})$s one. See also [10].

2.3. Replica-symmetry broken ansatz

We now consider the first step of Parisi's hierarchical RSB scheme [2], regardless of the possible existence of any new RSB pattern in FC spin glasses. Each replica a is labelled by a couple of integers $(b, \tilde{b})$ according to the number b of the block it belongs to ($1 \leqslant b \leqslant n/m$) and its position $\tilde{b}$ inside this block ($1 \leqslant \tilde{b} \leqslant m$). Following Mottishaw and De Dominicis [8], we see that $c(\boldsymbol{\sigma})$ is left unchanged under any permutation inside the blocks and depends only on the block magnetizations $s_b = \sum_{\tilde{b}=1}^{m} \sigma^{(b,\tilde{b})}$. To guess the structure of the order parameter, we again resort to the TAP approach. Consider spin S_i. Due to the presence of numerous states in the single cluster, the thermal average $\langle S_i \rangle$ fluctuates from state to state. Consequently, the effective field h_i is distributed according to a function $\rho_i(h) \in V$. We must keep in mind that ρ_i is in turn a random variable depending on the particular spin under consideration. The variety of ρ's may be taken into account by introducing their histogram, that is a functional $\mathcal{P}[\rho]$. This is a normalized distribution over V: $\int_V \mathcal{D}\rho\ \mathcal{P}[\rho] = 1$. We thus look for $\mathcal{P}$ such that

$$c(\boldsymbol{\sigma}) = \int_V \mathcal{D}\rho\ \mathcal{P}[\rho] \prod_{b=1}^{n/m} \int_R \mathrm{d}h\ \rho(h) \frac{\mathrm{e}^{\beta h s_b}}{(2\cosh \beta h)^m} \tag{8}$$

satisfies the saddle-point equations for $\mathcal{F}$. The generic order parameter $\mathcal{P}$ for FC models appears to be much more complex than in the IC case.

As a simple check of the above formalism, let us see how to find back the RS theory. In the latter case, there exists a unique state. The effective fields can therefore not fluctuate from 'state' to 'state' and ρ_i is simply a Dirac distribution in h_i. The fields h_i fluctuate according to their distribution p_{rs}. Defining

$$\mathcal{P}_{\mathrm{rs}}[\rho] = \int_R \mathrm{d}\tilde{h}\ p_{\mathrm{rs}}(\tilde{h}) \delta[\rho(h) - \delta(h - \tilde{h})] \tag{9}$$

we indeed find that the order parameter (8) simplifies to the RS expressions (5) and (6). In equation (9), the symbol δ denotes the Dirac functional, i.e. the product over all values of h is omitted for simplicity. It was first remarked by Wong and Sherrington (in a different formalism) [9] that a generalization of the above equation to the RSB case may be obtained by replacing the inner Dirac distribution in equation (9) with a function to be optimized over. This ansatz may be correct if the distributions ρ with a nonzero weight can be labelled by (i.e. are not more 'numerous' than) real numbers. In the generic case, a full functional $\mathcal{P}$ is needed *a priori*.

How can expression (8) be analytically continued to real n, m? We first define $\nu(y)$ as the number of blocks b of magnetizations $s_b = y$, with $y = -m, -m+2, \ldots, m-2, m$. It is easy to check on (8) that $c(\boldsymbol{\sigma})$ depends only on the set of $\nu(y)$'s, as expected from the invariance of the order parameter under permutations between blocks [7]. The discrete nature of y is merely due to the integer value of m and can be omitted to define an analytical continuation of the order parameters. Consequently, $\nu(y)$ may be any function in the range $-m \leqslant y \leqslant m$ satisfying the constraint

$$\int_{-m}^{m} \mathrm{d}y\ \nu(y) = \frac{n}{m} \to 0 \tag{10}$$

in the small n limit. Finally, the order parameter c becomes a functional over the set of all possible functions ν and reads, from (8),

$$c[\nu] = \int_V \mathcal{D}\rho \mathcal{P}[\rho] \exp\left(\int_{-m}^{m} \mathrm{d}y\ \nu(y) \ln\left[\int_R \mathrm{d}h\ \rho(h) \frac{\mathrm{e}^{\beta h y}}{(2\cosh \beta h)^m} \right] \right). \tag{11}$$

Note that the RS case is recovered by injecting the order parameter (9) into (11); one recovers (6) with $s = \int_{-m}^{m} \mathrm{d}y\, \nu(y)\, y$. In the generic RSB case, $c[\nu]$ depends on the whole ν function and not only on its first moment†.

3. Expression of the one-step free-energy functional

3.1. Methods to compute the free energy

Two procedures can be followed to access the thermodynamical properties, depending on the starting point of the calculation.

• *From the saddle-point equation*: we inject the expression of the order parameter (11) into the saddle-point equation (4). The resulting equation for $P[\rho]$ has to be solved. Then, one can compute the free-energy (3), possibly using the saddle-point equation to simplify the calculation (especially the entropic term whose calculation is not easy).

• *From the free-energy functional*: since we know that the saddle-point equation is closed within the one-step algebra, we may first compute the free-energy functional $\mathcal{F}$ given in (3) restricted to one-step order parameters (8). Once the analytical continuation of $\mathcal{F}$ to real $m, n(\to 0)$ has been carried out, we obtain the saddle-point equation for $P[\rho]$ by differentiating $\mathcal{F}$ with respect to the latter.

While both methods lead to the same result, the second one has an important practical advantage. The (total) derivative of $\mathcal{F}$ with respect to any parameter is equal to the partial derivative, while this is not necessarily true in the first procedure. This is of little interest for most of the control parameters which appear in the effective Hamiltonian only (it is usually easy to compute the partial derivative of $\mathcal{H}_{\mathrm{eff}}$ in (3) and to perform the analytical continuation—see next paragraph). However, considerable simplifications arise when computing the derivative of $\mathcal{F}$ with respect to m. In the following, we shall therefore adopt the second procedure.

3.2. Energetic contribution

According to the interpretation given in section 2, the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$ describes the interactions between the different levels $\boldsymbol{\sigma}$. If the levels interact K by K, the corresponding effective Hamiltonian typically reads

$$\mathcal{H}_{\mathrm{eff}} \sim \sum_{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_K} c(\boldsymbol{\sigma}_1) c(\boldsymbol{\sigma}_2) \ldots c(\boldsymbol{\sigma}_K)\; I(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_K) \tag{12}$$

where the interaction function I is invariant under global permutations of both replica and level labels. For all usual IC or FC models, the computation of $\mathcal{H}_{\mathrm{eff}}$ within the ansatz (8) as well as taking the limit $n \to 0$ do not present any difficulty.

As an illustration, we consider three examples:

• *the SK model*: inserting $c(\boldsymbol{\sigma})$ (8) in $\mathcal{H}_{\mathrm{eff}}$ (1), the trace over $\boldsymbol{\sigma}$ is straightforward to carry out by writing

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\bigg|_{z=0} \exp\left(z \sum_{a=1}^{n} \sigma_1^a \sigma_2^a \right). \tag{13}$$

We obtain:

$$\frac{1}{n} \mathcal{H}_{\mathrm{eff}}^{\mathrm{SK}} = -\frac{\beta}{4n} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\bigg|_{z=0} \int_V \mathcal{D}\rho_1 \mathcal{D}\rho_2 \mathcal{P}[\rho_1] \mathcal{P}[\rho_2]$$

† For IC models, the only non trivial moment appearing in $c[\nu]$ is the variance $\int_{-m}^{m} \mathrm{d}y\, \nu(y) y^2$. This oversimplification reflects the Gaussian nature of the distribution in (42).

$$\times\left[\int_R \mathrm{d}h_1\,\mathrm{d}h_2\;\rho_1(h_1)\rho_2(h_2)(\cosh z+\sinh z\tanh\beta h_1\tanh\beta h_2)^m\right]^{n/m}$$
$$=-\frac{\beta}{4}(1-q_1^2+m(q_1^2-q_0^2))\qquad(n\to 0)\tag{14}$$

where

$$q_0=\int_V \mathcal{D}\rho\mathcal{P}[\rho]\left(\int_R \mathrm{d}h\;\rho(h)\tanh\beta h\right)^2$$
$$q_1=\int_V \mathcal{D}\rho\mathcal{P}[\rho]\int_R \mathrm{d}h\;\rho(h)(\tanh\beta h)^2.\tag{15}$$

As expected for IC models, the functional nature of the order parameter is indeed drastically washed out since only the two moments q_0, q_1 of $\mathcal{P}[\rho]$ are relevant.

• *The Viana–Bray model*: this model was introduced as a FC version of the SK model [3]. The Viana–Bray Hamiltonian reads

$$\mathcal{H}_{\mathrm{eff}}^{\mathrm{VB}}=\frac{\alpha}{\beta}\left[1-\sum_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2} c(\boldsymbol{\sigma}_1)\;c(\boldsymbol{\sigma}_2)\cosh(\beta\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right]\tag{16}$$

where α denotes the mean connectivity per spin. We obtain

$$\frac{1}{n}\mathcal{H}_{\mathrm{eff}}^{\mathrm{VB}}=-\frac{\alpha}{\beta m}\int_V \mathcal{D}\rho_1\mathcal{D}\rho_2\mathcal{P}[\rho_1]\mathcal{P}[\rho_2]$$
$$\times\ln\left[\int_R \mathrm{d}h_1\,\mathrm{d}h_2\;\rho_1(h_1)\rho_2(h_2)(\cosh\beta+\sinh\beta\tanh\beta h_1\tanh\beta h_2)^m\right]\tag{17}$$

which explicitly depends upon on the whole distribution $\mathcal{P}[\rho]$.

• *the satisfiability problem*: this FC model will be described in section 5. At this stage, we only recall its Hamiltonian based on K-spins interactions:

$$\mathcal{H}_{\mathrm{eff}}^{\mathrm{SAT}}=-\frac{\alpha}{\beta}\ln\left[\sum_{\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_K} c(\boldsymbol{\sigma}_1)\ldots c(\boldsymbol{\sigma}_K)\exp\left(-\beta\sum_{a=1}^{n}\prod_{\ell=1}^{K}\delta[\sigma_\ell^a;1]\right)\right]\tag{18}$$

where $\delta[.;.]$ denotes the Kronecker function and α is a positive real parameter [14]. We obtain

$$\frac{1}{n}\mathcal{H}_{\mathrm{eff}}^{\mathrm{SAT}}=-\frac{\alpha}{\beta}\int_V \mathcal{D}\rho_1\ldots\mathcal{D}\rho_K\mathcal{P}[\rho_1]\ldots\mathcal{P}[\rho_K]\frac{1}{m}\ln\left[\int_R \mathrm{d}h_1\ldots\mathrm{d}h_K\;\rho_1(h_1)\ldots\rho_K(h_K)\right.$$
$$\left.\times\left(1+(\mathrm{e}^{-\beta}-1)\frac{\mathrm{e}^{\beta\sum_{j=1}^{K}h_j}}{\prod_{j=1}^{K}2\cosh\beta h_j}\right)^m\right].\tag{19}$$

Note the additional complexity of (19) with respect to (17) due to the presence of multispins interactions.

3.3. Entropic contribution

The calculation of the model-independent entropic contribution

$$S=-\sum_{\boldsymbol{\sigma}} c(\boldsymbol{\sigma})\ln c(\boldsymbol{\sigma})\tag{20}$$

to the free energy is far more complicated. Indeed, in contrast to $\mathcal{H}_{\mathrm{eff}}$, S is not a function of a finite number of the integer moments

$$C_\ell=\sum_{\boldsymbol{\sigma}}[c(\boldsymbol{\sigma})]^\ell\tag{21}$$

of the order parameter. The computation of the entropy S is exposed in appendix A. The final expression, in the small n limit

$$\frac{1}{n}S = -\int \mathcal{D}\hat{\nu}\mathcal{D}\nu \exp\left(-\mathrm{i}\int_{-m}^{m} \mathrm{d}y\, \hat{\nu}(y)\nu(y)\right) c[\mathrm{i}\nu] \ln c[\mathrm{i}\nu] \times \frac{1}{m} \ln\left[\int_R \frac{\mathrm{d}x}{2\pi} \int_{-m}^{m} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}xy}(2\cos x)^m \exp \hat{\nu}(y)\right] \tag{22}$$

involves a double functional integrals over $\hat{\nu}$ and ν functions, see appendix A. The order parameter P enters expression (22) through the c functional (11) as expected.

3.4. Back to RS: the $m = 0$ and $m = 1$ cases

In order to be self-consistent, the above formalism has to ensure that the RS free energy is found when $m = 0$ or 1. We shall now see that this is the case.

We first look at the energetic part of $\mathcal{F}$. Within the RS ansatz, the order parameter $P[\rho]$ simplifies to (9). It is a simple exercise to obtain the resulting expressions for the energy, see e.g. (14), (17) and (19). We observe that the same expressions are found if we identify the RS field distribution $p_{\mathrm{rs}}(h)$ with the following.

- Case $m = 0$:

$$p_0(h) = \int_V \mathcal{D}\rho P[\rho]\rho(h). \tag{23}$$

More generally, the expansion of the one-step free-energy functional in powers of m coincides with an expansion in terms of the cumulants of $P[\rho]$. For instance, the first nontrivial term (of order m) includes the two functions correlation

$$\Gamma(h, h') = \int_V \mathcal{D}\rho P[\rho]\rho(h)\rho(h') \tag{24}$$

which is diagonal in the RS assumption and shows off-diagonal contributions otherwise [17].

- Case $m = 1$:

$$p_1(h) = \int_V \mathcal{D}\rho P[\rho]\delta(h - H[\rho]). \tag{25}$$

where

$$H[\rho] = \frac{1}{\beta} \tanh^{-1}\left(\int_R \mathrm{d}h\, \rho(h) \tanh \beta h\right). \tag{26}$$

When $m = 1$, an exponential number of states with exponentially small weights contribute to the partition function. As a result, these states may be considered as effective micro-configurations and their cluster as a single effective 'state'. Equations (25) and (26) express that the mean-spin magnetizations in the latter are simply the averages of the spins magnetizations over all physical states (with vanishingly small weights).

The validity of the above relationship between RS theory and the $m = 0$ and 1 cases also holds for the entropic part of $\mathcal{F}$, see appendix B. In addition, we show in appendix B how to compute the path integrals in (22) in the RS scheme to obtain the RS entropy

$$\frac{1}{n}S_{\mathrm{rs}} = \int_R \frac{\mathrm{d}\hat{\nu}\,\mathrm{d}\nu}{2\pi} \mathrm{e}^{-\mathrm{i}\hat{\nu}\nu} \ln(2\cosh \beta\hat{\nu}) c_{\mathrm{rs}}(\mathrm{i}\nu)(1 - \ln c_{\mathrm{rs}}(\mathrm{i}\nu)). \tag{27}$$

with

$$c_{\mathrm{rs}}(\mathrm{i}\nu) = \int_R \mathrm{d}h\, p_{\mathrm{rs}}(h)\mathrm{e}^{\mathrm{i}\nu h}. \tag{28}$$

The field distribution $p_{\rm rs}$ is obtained through the optimization of both entropic and energetic contributions to the RS free energy. This approach, that gives the same result as previous works in the RS framework, we shall now extend to the first step of RSB.

4. Saddle-point equation for the functional order parameter

We shall now obtain the saddle-point equation fulfilled by the order parameter $P[\rho]$. To do so, we differentiate both entropic and energetic parts of the free-energy functional.

4.1. Differentiation of the entropic part

The entropy (22) depends on the order parameter through c (11) only. It is convenient to introduce the operator

$$\mathcal{K}[\rho,\nu]=\int\mathcal{D}\hat{\nu}\exp\left[-\mathrm{i}\int_{-m}^{m}\mathrm{d}y\,\nu(y)\left(\hat{\nu}(y)-\ln\left(\int_R\mathrm{d}h\,\rho(h)\frac{\mathrm{e}^{\beta hy}}{(2\cosh\beta h)^m}\right)\right)\right]$$
$$\times\frac{1}{m}\ln\left[\int_R\frac{\mathrm{d}x}{2\pi}\int_{-m}^{m}\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}xy}(2\cos x)^m\exp\hat{\nu}(y)\right] \tag{29}$$

which depends on both functions $\rho(h)$ and $\nu(y)$. Using the above definition, we rewrite the derivative of the entropic contribution to the free energy as

$$\frac{1}{n}\frac{\partial S}{\partial P[\rho]}=-\int\mathcal{D}\nu\mathcal{K}[\rho,\nu]\ln c[\mathrm{i}\nu] \tag{30}$$

which depends on the field distribution ρ through (29).

4.2. Differentiation of the energetic part

As emphasized in section 3.2, the calculation of the (model-dependent) energetic part of the free-energy, and consequently of its derivative does not present any difficulty. It turns out that the latter may always be written in the convenient form

$$\frac{1}{n}\frac{\partial\mathcal{H}^{\rm eff}}{\partial P[\rho]}=\int\mathcal{D}\nu\mathcal{K}[\rho,\nu]\Omega[\mathrm{i}\nu]. \tag{31}$$

We now give the expressions of Ω for the different models of interest.

- *The SK model*:

$$\Omega^{\rm SK}[\nu]=-\frac{\beta}{2}(m(1-q_1)\nu_0+q_0(\nu_1)^2+(1-q_1)\nu_2) \tag{32}$$

where q_0, q_1 have been defined in (15) and

$$\nu_j=\int_{-m}^{m}\mathrm{d}y\,\nu(y)y^j. \tag{33}$$

- *The Viana–Bray model*:

$$\Omega^{\rm VB}[\nu]=-\frac{2\alpha}{\beta}\int_V\mathcal{D}\rho_1\mathcal{P}[\rho_1]\frac{1}{2}\sum_{\sigma=\pm1}\exp\left(\int_{-m}^{m}\mathrm{d}y\,\nu(y)\right.$$
$$\left.\times\ln\left[\int_R\mathrm{d}h_1\,\rho_1(h_1)\exp\left(\frac{1}{2}(m(A_+ + A_-)+\sigma y(A_+ - A_-))\right)\right]\right) \tag{34}$$
$$A_\epsilon\equiv\ln\left[\frac{\cosh\beta(h_1+\epsilon)}{\cosh\beta h_1}\right]\qquad(\epsilon=\pm1).$$

- *The satisfiability problem*:

$$\Omega^{\mathrm{SAT}}[\nu] = -\frac{\alpha K}{\beta}\int_V \mathcal{D}\rho_1 \ldots \mathcal{D}\rho_{K-1}\mathcal{P}[\rho_1]\ldots\mathcal{P}[\rho_{K-1}]\frac{1}{2}\sum_{\sigma=\pm 1}\exp\left(\int_{-m}^{m} \mathrm{d}y\, \nu(y) \times \ln\left[\int_R \prod_{j=1}^{K-1}\mathrm{d}h_j\, \rho_j(h_j)\exp\left(\frac{1}{2}(m+\sigma y)A_{K-1}\right)\right]\right) \tag{35}$$

$$A_{K-1} \equiv \ln\left[1 + (\mathrm{e}^{-\beta}-1)\frac{\mathrm{e}^{\beta\sum_{j=1}^{K-1}h_j}}{\prod_{j=1}^{K-1}2\cosh\beta h_j}\right].$$

4.3. Self-consistency equation for $P[\rho]$

Gathering derivatives (30) and (31) together, we obtain the following saddle-point equation

$$\int \mathcal{D}\nu\mathcal{K}[\rho,\nu](-\ln c[\mathrm{i}\nu] - \beta\Omega[\mathrm{i}\nu]) = \lambda_0 \tag{36}$$

which has to be satisfied for any $\rho(h)$. The Lagrange multiplier λ_0 in (36) is determined through the normalization of $P[\rho]$. It does not depend upon $\rho(h)$ and may be computed for e.g. $\rho(h) = \delta(h)$. Therefore, the saddle-point equation (36) simply means that $\Omega[\mathrm{i}\nu] - \ln c[\mathrm{i}\nu]$ is a zero mode of the operator

$$\mathcal{Q}[\rho,\nu] = \mathcal{K}[\rho,\nu] - \mathcal{K}[\delta(h),\nu]. \tag{37}$$

The kernel of $\mathcal{Q}$ contains all constant functionals λ_1 due to the normalization of $\rho(h)$. Therefore, we end up with the following saddle-point equation

$$\ln c[\nu] + \beta\Omega[\nu] = \lambda_1 \tag{38}$$

where λ_1 can be determined for e.g. $\nu(y) = 0$. From (11) and (38), we obtain $\lambda_1 = \beta\Omega[0]$, or equivalently

$$c[\nu] = \exp(-\beta\Omega[\nu] + \beta\Omega[0]). \tag{39}$$

We now have to find the ν functions for which equation (39) has to be satisfied. P is a distribution of normalized probabilities $\rho(h)$ merely by convention. Both $P[\rho(h)]$ and $P[a(\rho)\rho(h)]$ must lead to the same physics independently of the irrelevant a factors. This is ensured if

$$\int_{-m}^{m}\mathrm{d}y\,\nu(y) = 0 \tag{40}$$

that is if condition (10) is satisfied. Consequently, $P[\rho]$ has to be such that equation (39) is correct for any function $\nu(y)$ over the range $-m \leqslant y \leqslant m$ with zero integral ν_0 (33).

4.4. A simple application: infinite-connectivity models

For the SK model, the self consistency equation (39) is obtained from (11) and (32) and reads

$$\int_V \mathcal{D}\rho\mathcal{P}[\rho]\exp\left(\int_{-m}^{m}\mathrm{d}y\,\nu(y)\ln\left[\int_R \mathrm{d}h\,\rho(h)\frac{\mathrm{e}^{\beta h y}}{(2\cosh\beta h)^m}\right]\right) = \exp\left[\frac{\beta^2}{2}\left(q_0(\nu_1)^2 + (q_1-q_0)\nu_2\right)\right] \tag{41}$$

for any ν such that $\nu_0 = 0$. Solving the above equation, one finds the SK order parameter

$$\mathcal{P}^{\mathrm{SK}}[\rho] = \int_R \mathrm{d}\tilde{h}\, w_{q_0}(\tilde{h})\delta\left[\rho(h) - \frac{w_{q_1-q_0}(h-\tilde{h})(2\cosh\beta h)^m}{\int_R \mathrm{d}h'\, w_{q_1-q_0}(h'-\tilde{h})(2\cosh\beta h')^m}\right] \tag{42}$$

where $w_a(z) = \mathrm{e}^{-z^2/2a}/\sqrt{2\pi a}$. In equation (42), the symbol δ denotes the Dirac functional, i.e. the product over all values of h is omitted for simplicity. As expected, equations (15) are identical to the usual self-consistent equations for the RSB overlaps q_0 and q_1, see [2, equation (III.41)]. Note that the expression of ρ in (42) is in full agreement with the cavity derivation of the effective field distribution, see [2, equation (V.29)].

5. Application to the 3-satisfiability problem

For generic FC models, the exact resolution of the saddle-point equation (39) appears to be extremely difficult. At zero temperature, the limit of interest for optimization problems, some analytical simplifications take place. We shall now see on 3-SAT how quantitative results may be obtained this way.

5.1. Presentation of the K-satisfiability problem

The satisfiability (SAT) problem is the paradigm of the class of hard (NP–complete) computational problems arising in complexity theory [12]. A pedagogical introduction to the K-SAT problem, a version of SAT and some of the current open issues in theoretical computer science may be found in [13].

K-SAT is defined as follows. Let us consider N Boolean variables $\{x_i = 0, 1\}_{i=1,\ldots,N}$. Choose randomly K among the N possible indices i and then, for each of them, a literal that is the corresponding x_i or its negation $\bar{x}_i$ with equal probabilities of one half. A clause C is the logical OR of the K previously chosen literals, that is C will be true (or satisfied) if and only if at least one literal is true. Next, repeat this process to obtain M independently chosen clauses $\{C_\ell\}_{\ell=1,\ldots,M}$ and ask for all of them to be true at the same time (i.e. we take the logical AND of the M clauses). A logical assignment of the $\{x_i\}$'s satisfying all clauses, if any, is called a solution of the K-satisfiability problem.

For large instances ($M, N \to \infty$), K-SAT exhibits a striking threshold phenomenon as a function of the ratio $\alpha = M/N$ of the number of clauses per variable. Numerical simulations indicate that the probability of finding a solution falls abruptly from one down to zero when α crosses a critical value $\alpha_{\mathrm{c}}(K)$. Above $\alpha_{\mathrm{c}}(K)$, all clauses cannot be satisfied any longer. This scenario is rigorously established in the $K = 2$ case, where $\alpha_{\mathrm{c}} = 1$[20]. For $K \geqslant 3$, much less is known; $K(\geqslant 3)$-SAT belongs to the class of hard computational problems, roughly meaning that running times of search algorithms are thought to scale exponentially in N. Some upper and lower bounds on $\alpha_{\mathrm{c}}(K)$ have been derived[21] and numerical simulations have recently allowed us to find precise estimates of α_{c}, e.g. $\alpha_{\mathrm{c}}(3) \simeq 4.2$ [16].

In order to study the K-SAT problem, we map it onto a random diluted system by introducing some spin variables $S_i = \pm 1$ (a simple shift of the Boolean variables) and a quenched (unbiased) matrix $C_{\ell,i} = 1$ (resp. -1) if x_i (resp. $\bar{x}_i$) belongs to the clause C_ℓ, 0 otherwise. Then the energy-cost function

$$E[C, S] = \sum_{\ell=1}^{M} \delta\left[\sum_{i=1}^{N} C_{\ell,i} S_i; -K\right] \tag{43}$$

equals the number of violated clauses and therefore its GS properties describe the transition from the SAT phase ($E_{\rm GS} = 0$) to the UNSAT phase ($E_{\rm GS} > 0$). Note that a similar cost function was first introduced for neural networks by Gardner and Derrida [22].

The effective Hamiltonian corresponding to the cost function (43) was given in (18). Previous studies have shown that the RS theory was able to find back the critical threshold of the 2-SAT problem but became wrong for $K(\geqslant 3)$-SAT instances [14, 15]. In the following, we therefore concentrate upon the most interesting $K = 3$ case. We briefly recall the RS solution and then present a first RSB solution.

5.2. Saddle-point equation and RS solution

Following the above-mentioned procedure, the saddle-point equation for $\mathcal{P}$ reads

$$\ln c[\nu] = 3\alpha \int_V \mathcal{D}\rho_1 \mathcal{D}\rho_2 \mathcal{P}[\rho_1]\mathcal{P}[\rho_2](\tfrac{1}{2}\mathrm{e}^{\Phi_-} + \tfrac{1}{2}\mathrm{e}^{\Phi_+} - 1) \tag{44}$$

where (for $\epsilon = \pm 1$)

$$\Phi_\epsilon = \int_{-m}^{m} \mathrm{d}y\, \nu(y) \ln \left\{ \int_R \mathrm{d}h_1\, \mathrm{d}h_2\, \rho_1(h_1)\rho_2(h_2) \times \exp\left[\frac{m+\epsilon y}{2} \ln\left(1 + \frac{\mathrm{e}^{-\beta} - 1}{(1+\mathrm{e}^{-2\beta h_1})(1+\mathrm{e}^{-2\beta h_2})} \right) \right] \right\}.$$

Equation (44) has to be satisfied for all ν of integral zero. The GS properties are obtained by sending $\beta \to \infty$.

Let us briefly recall the RS result [14]. Inserting ansatz (9) into (44), we find that the simplest RS solution includes half-integer fields only [5, 14, 17]†

$$p_{\rm rs}(h) = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-\gamma} I_\ell(\gamma) \delta\left(h - \frac{\ell}{2}\right) \tag{45}$$

where I_ℓ is the ℓth modified Bessel function and γ is self-consistently determined through

$$\gamma = \frac{3\alpha}{4}(1 - \mathrm{e}^{-\gamma} I_0(\gamma))^2. \tag{46}$$

In addition to the SAT phase solution $\gamma = E_{\rm GS} = 0$, there appears a metastable solution $\gamma > 0$, $E_{\rm GS} \neq 0$ above $\alpha = 4.67$, see figure 1. This solution becomes thermodynamically stable at $\alpha = 5.18$, well above the 'experimental' threshold $\alpha \simeq 4.2$. The latter is indeed thought to coincide with a first-order spin-glass transition [14, 15]. Moreover, the presence of 3-spins interactions in (18) suggests that the one-step RSB solution could be exact for 3-SAT and that $m = 1$ at the threshold [18] .

5.3. Simplest RSB solution

We now turn to the RSB solution. In view of (45), we first restrict to distributions ρ's on half-integer fields h. Secondly, the cavity theory teaches us that ρ is biased in favour of large fields (see [2, section V.2] for a clear explanation of this point). For IC models, large hs indeed benefit from a Boltzmann factor $\mathrm{e}^{\beta m|h|}$ as can be seen in (42). We thus propose the following form for the distributions ρ with a nonzero weight $\mathcal{P}[\rho]$ (for $m = 1$),

$$\rho(h) = \omega(\beta) \sum_{\ell=\ell^{(-)}}^{\ell^{(+)}} \rho_\ell \exp\left(\beta \frac{|\ell|}{2}\right) \delta\left(h - \frac{\ell}{2}\right) \tag{47}$$

† The field h equals half the difference between the numbers of violated clauses when flipping a Boolean variable (see [14, section IX]). It may thus take half-integer values [17].

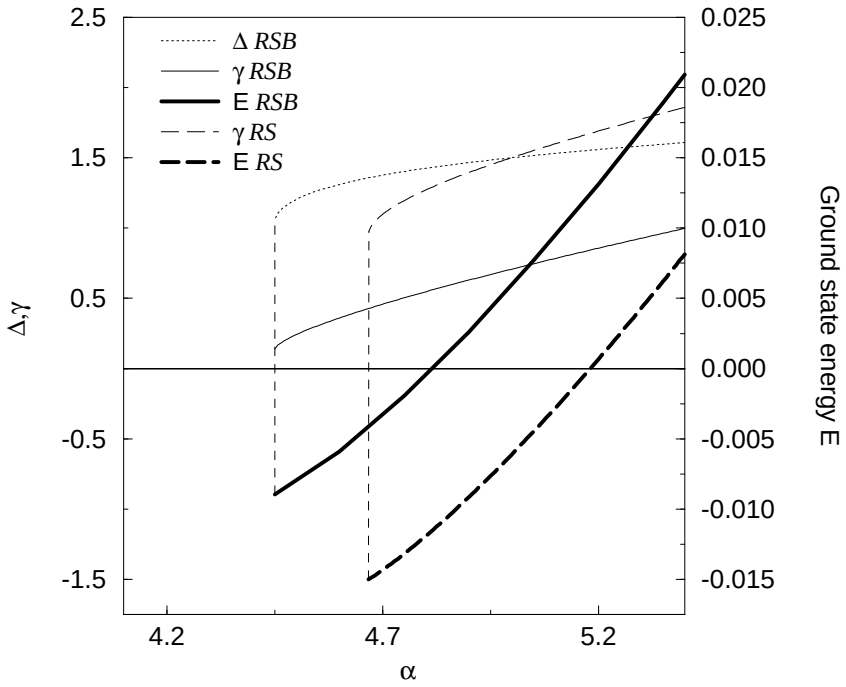

Figure 1. Decay parameters γ (light curves—left side scale) and GS energies E (heavy curves—right side scale) for the RSB (full curves) and the RS (long-broken curves) solutions as functions of the number of clauses per variables α. The RSB width parameter Δ is shown on the upper dotted curve. The vertical broken lines indicate the spinodal points $\alpha_{\rm RSB} = 4.45$ and $\alpha_{\rm RS} = 4.67$.

where $\omega(\beta)$ is a normalization factor and the ρ_ℓ's do not contain exponential terms in β. Identity (47) is merely the simplest hypothesis compatible with the saddle-point equation; we shall come back to this point at the end of the letter. Sending the temperature to zero, we find that $\mathcal{P}[\rho]$ is a function of only two variables which are computed from ρ, namely the ends of its support $\ell^{(-)}$ and $\ell^{(+)}$. This results from the zero-temperature expression of the order parameter

$$\lim_{\beta\to\infty} c[\nu/\beta] = \sum_{\ell^{(-)}\leqslant\ell^{(+)}} \mathcal{P}_{\ell^{(-)},\ell^{(+)}} \mathrm{e}^{\nu_- \ell^{(-)} + \nu_+ \ell^{(+)}} \tag{48}$$

where $\mathcal{P}_{\ell^{(-)},\ell^{(+)}}$ equals the sum of the weights of the ρ functions having support $[\ell^{(-)}, \ell^{(+)}]$. Note that (48) depends on ν through

$$\begin{aligned} \nu_- &= \int_{-1}^{0} \mathrm{d}y\, \nu(y) y \\ \nu_+ &= \int_{0}^{1} \mathrm{d}y\, \nu(y) y \end{aligned} \tag{49}$$

in agreement with the statement that (47) is the simplest non-RS solution to (44). After some algebra, $\Omega^{\rm SAT}[\nu]$ (35) can also be shown to depend on ν_-, ν_+ only. The self-consistent equation for $\mathcal{P}[\rho]$ simplifies into an implicit equation for the matrix $\mathcal{P}_{\ell^{(-)},\ell^{(+)}}$ which can be

easily solved,

$$\mathcal{P}_{\ell^{(-)},\ell^{(+)}} = \mathrm{e}^{-\Delta}\left(\frac{\Delta}{2}\right)^{\ell^{(+)}-\ell^{(-)}} \sum_{\ell=\ell^{(-)}}^{\ell^{(+)}} \frac{\mathrm{e}^{-\gamma} I_\ell(\gamma)}{(\ell^{(+)}-\ell)!(\ell-\ell^{(-)})!}. \tag{50}$$

As in the RS solution (45), γ controls the decay of the probability weights of large fields. The new parameter Δ sets the width of the supports of the ρ's, that is the magnitude of the fluctuations of the effective fields from state to state. When $\Delta = 0$, the supports of the ρ's shrink to single points and the RS solution (45) is recovered, in agreement with (9). The two self-consistent equations on γ and Δ as well as the expression of the corresponding GS energy E_{GS} are given in appendix C. The results are displayed figure 1. In addition to the trivial solution $\gamma = \Delta = E_{\mathrm{GS}} = 0$, there appears another solution for $\alpha > 4.45$ with $0 < \gamma < \Delta$ and a metastable, i.e. negative, GS energy. This solution becomes thermodynamically stable at $\alpha = 4.82$.

6. Conclusion

The formalism we have presented in this article has permitted us to find an explicit RSB solution improving the RS saddle-point. Yet, the predicted threshold $\alpha_{\mathrm{c}} = 4.82$ exceeds the 'experimental' value. This discrepancy may stem from assumption (47). In contrast to the IC case, the pay-off for large fields appearing in ρ might be a nonlinear function of $|h|$. If so, an inspection of the saddle-point equation (44) shows that h is not constrained to take half-integer values anymore. This refinement scheme is reminiscent of the iterative procedure used in the RS theory to improve the simplest solution (45). By increasing the resolution on the fields, the RS thresholds decreased from 5.18 down to 4.60 [15].

However, the meaning of nonhalf-integer fields is far from being clear. From a physical point of view, the effective fields should be half-integer valued at zero temperature. Previous studies performed for the Viana–Bray model have nevertheless shown that the half-integer field distribution is unstable with respect to longitudinal (within the RS sector) fluctuations [8]. Then, we face the following dilemna. The RS saddle-point equation admits a physically sensible but unstable solution and many other ones, whose significances are dubious. In this context, the RSB solution presented in the last Section has a remarkable property. While the supports of the ρs with nonzero weights contain physical (i.e. half-integer) values only, the resulting effective distribution $P(h)$ (25) includes Dirac peaks on the integer multiples of a quarter! More precisely, the RSB GS energy is equal to the GS energy of the RS solution with fields which are integer multiples of a quarter [15]. Work is in progress to reach a better understanding of this puzzling coincidence[23].

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Appendix A. Calculation of the RSB entropy

Clearly, the analytical continuation of the C_ℓ's to real ℓ in the vicinity of the unity will allow us to compute the entropy (20),

$$S = -\frac{\mathrm{d}}{\mathrm{d}\ell}\bigg|_{\ell=1} C_\ell. \tag{A1}$$

For integer ℓ, the moments C_ℓ are easily expressed from (8) as $C_\ell \simeq 1 + n\Gamma_\ell$ in the small n limit with

$$\Gamma_\ell = \int_V \mathcal{D}\rho_1 \ldots \mathcal{D}\rho_\ell \mathcal{P}[\rho_1] \ldots \mathcal{P}[\rho_\ell] \times \frac{1}{m} \ln \left[\int_R \mathrm{d}h_1 \ldots \mathrm{d}h_\ell \, \rho_1(h_1) \ldots \rho_\ell(h_\ell) \left(\frac{2\cosh(\beta \sum_{j=1}^{\ell} h_j)}{\prod_{j=1}^{\ell} 2\cosh \beta h_j} \right)^m \right]. \tag{A2}$$

In order to achieve an analytical continuation of Γ_ℓ, we proceed in two steps. First, consider the argument A of the logarithm in (A2). Defining

$$\hat{\nu}(y; \rho_1, \ldots, \rho_\ell) = \sum_{j=1}^{\ell} \ln \left[\int_R \mathrm{d}h \, \rho_j(h) \frac{\mathrm{e}^{\beta h y}}{(2\cosh \beta h)^m} \right] \tag{A3}$$

the latter may be rewritten as

$$A[\hat{\nu}(y; \rho_1, \ldots, \rho_\ell)] = \int_R \frac{\mathrm{d}x \, \mathrm{d}y}{2\pi} \mathrm{e}^{-\mathrm{i}xy} (2\cosh x)^m \exp \hat{\nu}(\mathrm{i}y; \rho_1, \ldots, \rho_\ell) \tag{A4}$$

$$= \int_R \frac{\mathrm{d}x}{2\pi} \int_{-m}^{m} \mathrm{d}y \, \mathrm{e}^{-\mathrm{i}xy} (2\cos x)^m \exp \hat{\nu}(y; \rho_1, \ldots, \rho_\ell) \tag{A5}$$

where the last expression has been obtained through the rotation $(x, y) \to (\mathrm{i}x, -\mathrm{i}y)$. Note that the range of integration over y in (A5) makes the integral over h in (A3) convergent. We now introduce the probability distribution $\mathcal{Q}_\ell[g]$ of the functions $g(y; \rho_1, \ldots, \rho_\ell)$,

$$\mathcal{Q}_\ell[\hat{\nu}] = \int_V \mathcal{D}\rho_1 \ldots \mathcal{D}\rho_\ell \mathcal{P}[\rho_1] \ldots \mathcal{P}[\rho_\ell] \delta[\hat{\nu}(y) - \hat{\nu}(y; \rho_1, \ldots, \rho_\ell)] \tag{A6}$$

and rewrite Γ_ℓ as a functional integral over all possible g functions (with support $[-m, m]$) weighted with measure $\mathcal{Q}_\ell$,

$$\Gamma_\ell = \int \mathcal{D}\hat{\nu} \mathcal{Q}_\ell[\hat{\nu}] \frac{1}{m} \ln A[\hat{\nu}(y)]. \tag{A7}$$

The second step of the calculation lies in the analytical continuation of the measure $\mathcal{Q}_\ell$ to real ℓ. This may be achieved through an exponential representation of the functional Dirac in (A6). We thus obtain

$$\left. \frac{\mathrm{d}}{\mathrm{d}\ell} \right|_{\ell=1} \mathcal{Q}_\ell[\hat{\nu}] = \int \mathcal{D}\nu \exp \left(-\mathrm{i} \int_{-m}^{m} \mathrm{d}y \hat{\nu}(y) \nu(y) \right) c[\nu] \ln c[\nu] \tag{A8}$$

where the functional $c[\nu]$ is given in (11). In equation (A8), the measure $\mathcal{D}\hat{\nu}$ ensures the correct normalization of the Dirac functional, that is factors $1/2\pi$ have to be included when discretising the path integral over $\hat{\nu}$. The resulting expression of the entropy is given in (22).

Appendix B. RS entropy and the $m = 0, 1$ cases

In this appendix, we first derive the RS expression of the entropy (22). We then show how the latter is recovered when $m = 0$ or $m = 1$.

B.1. Calculation of the RS entropy

Consider the RS expression (9) for $P[\rho]$. The functional $c[\nu]$ (11) simplifies to

$$c(\nu_0, \nu_1) = \int_R \mathrm{d}h\; p_{\mathrm{rs}}(h) \exp(-\mathrm{i}\nu_0 m \ln(2\cosh\beta h) + \mathrm{i}\nu_1 \beta h). \tag{B1}$$

It depends on ν through the two moments ν_0, ν_1 only, see (33). We now introduce the series expansion of $\hat{\nu}(y)$ around $y = 0$,

$$\hat{\nu}(y) = \sum_{j=0}^{\infty} \hat{\nu}_j y^j. \tag{B2}$$

Combining the coefficients definitions (33) and (B2), we rewrite the coupling term between $\hat{\nu}$ and ν in (22) as follows

$$\exp\left(-\mathrm{i}\int_{-m}^{m} \mathrm{d}y\; \hat{\nu}(y)\nu(y)\right) = \exp\left(-\mathrm{i}\sum_{j=0}^{\infty} \hat{\nu}_j \nu_j\right). \tag{B3}$$

The integration over all coefficients ν_j with $j \geqslant 2$ gives $\hat{\nu}_j = 0$, $\forall j \geqslant 2$. Therefore, only the $\hat{\nu}$'s that are linear functions of their argument y survive. We obtain

$$\begin{aligned}\frac{1}{n}S = -\int_R \frac{\mathrm{d}\hat{\nu}_0\,\mathrm{d}\nu_0}{2\pi}\frac{\mathrm{d}\hat{\nu}_1\,\mathrm{d}\nu_1}{2\pi} c(\nu_0, \nu_1)\ln c(\nu_0, \nu_1)\mathrm{e}^{-\mathrm{i}(\hat{\nu}_0\nu_0+\hat{\nu}_1\nu_1)} \\ \times\frac{1}{m}\ln\left[\int_R \frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}xy}(2\cosh x)^m \exp(\hat{\nu}_0 + \hat{\nu}_1 \mathrm{i}y)\right].\end{aligned} \tag{B4}$$

The argument of the logarithm in (B4) simply reads $\mathrm{e}^{\hat{\nu}_0}(2\cosh\hat{\nu}_1)^m$. The remaining integrals on $\hat{\nu}_0$, ν_0, $\hat{\nu}_1$, ν_1 may then be carried out and the final result is given in (27).

B.2. Case $m = 0$

We expand the logarithm in (22) and find

$$\begin{aligned}\frac{1}{m}\ln\left[\int_R \frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}xy}(2\cosh x)^m \exp\hat{\nu}(\mathrm{i}y)\right] \\ = \frac{\hat{\nu}(0)}{m} + \int_R \frac{\mathrm{d}y\,\mathrm{d}\hat{y}}{2\pi}\mathrm{e}^{-\mathrm{i}y\hat{y}}\ln(2\cosh y)\mathrm{e}^{\hat{\nu}(\mathrm{i}\hat{y})-\hat{\nu}(0)} + \mathrm{O}(m)\end{aligned} \tag{B5}$$

in the small m limit. We now consider the first term on the r.h.s. of (B5) and integrate out all modes $\hat{\nu}(y \neq 0)$ in (22). The resulting $c[\nu]$ reads, for small m,

$$c[\nu] = 1 + \mathrm{i}m\nu(0)\int_R \mathrm{d}h\; p_0(h)\ln(2\cosh\beta h) + \mathrm{O}(m^2) \tag{B6}$$

where p_0 has been defined in (23). Integrating $g(0)$, $\nu(0)$ out, we obtain a first contribution to the entropy (22),

$$\frac{1}{n}S_1 = \int_R \mathrm{d}h\; p_0(h)\ln(2\cosh\beta h). \tag{B7}$$

We now focus on the second part of the r.h.s. of (B5). By integrating the $\hat{\nu}$s out, we obtain $\nu(y) = 0$ except $\nu(\mathrm{i}\hat{y}) = -\nu(0) = \mathrm{i}$. The corresponding functional (11) reads

$$c[\nu] \equiv c(\hat{y}) = \int_R \mathrm{d}h\; p_0(h)\mathrm{e}^{\mathrm{i}\beta h\hat{y}} \tag{B8}$$

giving rise to the following entropy

$$\frac{1}{n}S_2 = -\int_R \frac{\mathrm{d}y\,\mathrm{d}\hat{y}}{2\pi}\mathrm{e}^{-\mathrm{i}y\hat{y}}\ln(2\cosh y)c(\hat{y})\ln c(\hat{y}). \tag{B9}$$

Summing up S_1 and S_2, the RS entropy (27) is recovered with $p_{\rm rs} = p_0$.

B.3. Case $m = 1$

For $m = 1$, the argument of the logarithm in (22) reads $\mathrm{e}^{\hat{\nu}_+} + \mathrm{e}^{\hat{\nu}_-}$, with $\hat{\nu}_\epsilon = \hat{\nu}(\mathrm{i}\epsilon)$, $\epsilon = \pm 1$. For all $y \neq \pm 1$, the integration over $\hat{\nu}(y)$ gives $\nu(y) = 0$. The entropy then reads

$$\frac{1}{n}S = -\int_R \frac{\mathrm{d}\hat{\nu}_+\,\mathrm{d}\nu_+}{2\pi}\frac{\mathrm{d}\hat{\nu}_-\,\mathrm{d}\nu_-}{2\pi}c(\nu_+,\nu_-)\ln c(\nu_+,\nu_-)\mathrm{e}^{\mathrm{i}(\hat{\nu}_+\nu_+ + \hat{\nu}_-\nu_-)}\ln[\mathrm{e}^{\hat{\nu}_+} + \mathrm{e}^{\hat{\nu}_-}] \tag{B10}$$

where, using definition (25),

$$c(\nu_+,\nu_-) = \int_R \mathrm{d}h\; p_1(h)\exp(\mathrm{i}(\nu_+ + \nu_-)\ln(2\cosh\beta h) - \mathrm{i}(\nu_+ - \nu_-)\beta h). \tag{B11}$$

We make the following unitary change of variables

$$\begin{aligned} \hat{\nu}_0 &= \tfrac{1}{2}(\hat{\nu}_+ + \hat{\nu}_-) \\ \nu_0 &= \nu_+ + \nu_- \\ \hat{\nu}_1 &= \tfrac{1}{2}(\hat{\nu}_+ - \hat{\nu}_-) \\ \nu_1 &= \nu_+ - \nu_- \end{aligned} \tag{B12}$$

and find the RS expression (B4). Therefore, when $m = 1$, the RS entropy is recovered with $p_{\rm rs} = p_1$.

Appendix C. Self-consistency equations for γ, Δ and ground-state energy E

The self-consistent equations for γ and Δ read

$$\gamma = 3\alpha\left(\frac{1}{2} - \frac{r_0}{2} - r_1\right)^2 \tag{C1}$$

$$\Delta = 3\alpha\left(\frac{1}{2} - \frac{r_0}{2}\right)^2 - \gamma \tag{C2}$$

where

$$\begin{aligned} r_0 &= \sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\gamma-\Delta}I_\ell(\gamma)I_{2\ell}(\Delta) \\ r_1 &= \tfrac{1}{2}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\gamma-\Delta}I_\ell(\gamma)[I_{2\ell-1}(\Delta) + I_{2\ell+1}(\Delta)]. \end{aligned} \tag{C3}$$

To compute the GS energy E, we insert the saddle-point solution for $c(\boldsymbol{\sigma})$ into the free-energy (3) and keep the linear term in n only. We then send $\beta \to \infty$ and find

$$\begin{aligned} E = -\alpha\left[\left(\frac{1-r_0}{2}\right)^3 + \left(\frac{1-r_0}{2} - r_1\right)^3\right] + \frac{3\alpha}{4}\left[\left(\frac{1-r_0}{2}\right)^2 + \left(\frac{1-r_0}{2} - r_1\right)^2\right] \\ -\frac{\Delta}{4}(r_0 + r_1) - \frac{\gamma}{4}(r_0 + r_2 + 2r_1) \end{aligned} \tag{C4}$$

$$r_2 = \tfrac{1}{2}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\gamma-\Delta}I_\ell(\gamma)[I_{2\ell-2}(\Delta) + I_{2\ell+2}(\Delta)].$$

Note that $\Delta = 0$ is always a solution of the self-consistent equation (C2); equation (C1) then gives back the RS saddle-point constraint on γ (46). In this case, the GS energy E (C4) simplifies to

$$E_{\mathrm{RS}} = \frac{\gamma}{6}(1 - \mathrm{e}^{-\gamma} I_0(\gamma) - 3\mathrm{e}^{-\gamma} I_1(\gamma)) \tag{C5}$$

in agreement with the findings of [14, 15].

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