Advanced quantum mechanics

Chapter 3 – Coherence and correlations in many-body systems

Sylvain Nascimbene

M2 ICFP – École Normale Supérieure

Outline

1. Coherence and correlations in many-body systems
   - First-order correlation function
   - Pair distribution and particle fluctuations
   - Example 1: ideal Bose and Fermi gases
   - Example 2: impenetrable bosons in one dimension
   - Example 3: superfluid helium

Chapter 3 – Coherence and correlations

Correlation functions describe many-body systems in terms of simple observables. They

- identify phases of matter — long-range phase order, broken translation symmetry;
- are intimately linked to the spatial structure of entanglement.

We will study three examples:

1. ideal Bose and Fermi gases,
2. impenetrable bosons in one dimension (Tonks–Girardeau),
3. superfluid helium.

The g_1 function

$$g_1(\mathbf{r}, \mathbf{r}') = \langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}') \rangle.$$

This is the expectation value of the one-body operator $\hat{b} = |\mathbf{r}\rangle\langle\mathbf{r}'|$, so that

$$g_1(\mathbf{r}, \mathbf{r}') = \mathrm{Tr}\,(|\mathbf{r}\rangle\langle\mathbf{r}'|\hat{\rho}_1) = \langle\mathbf{r}'|\hat{\rho}_1|\mathbf{r}\rangle$$

— a matrix element of the single-particle density matrix, *with the arguments in reverse order*. For a pure state,

$$g_1(\mathbf{r}, \mathbf{r}') = \int \psi^*(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, \psi(\mathbf{r}', \mathbf{r}_2, \ldots, \mathbf{r}_N)\, \mathrm{d}^3 r_2 \cdots \mathrm{d}^3 r_N.$$

Any one-body observable follows:

$$\langle \hat{B} \rangle = \int \langle \mathbf{r}|\hat{b}|\mathbf{r}'\rangle\, g_1(\mathbf{r}, \mathbf{r}')\, \mathrm{d}^3 r\, \mathrm{d}^3 r'.$$

Properties, and the coherence length

1. **Diagonal = density.** $g_1(\mathbf{r},\mathbf{r}) = \rho(\mathbf{r})$, with $\int g_1(\mathbf{r},\mathbf{r})\,\mathrm{d}^3 r = N$.
2. **Hermiticity.** $g_1(\mathbf{r}',\mathbf{r}) = g_1(\mathbf{r},\mathbf{r}')^*$.
3. **Positivity.** $\hat{\rho}_1 \geq 0$, so Cauchy–Schwarz bounds the off-diagonal elements,

$$|g_1(\mathbf{r},\mathbf{r}')| \leq \sqrt{\rho(\mathbf{r})\,\rho(\mathbf{r}')}.$$

It is therefore natural to normalize:

$$g^{(1)}(\mathbf{r},\mathbf{r}') = \frac{g_1(\mathbf{r},\mathbf{r}')}{\sqrt{\rho(\mathbf{r})\,\rho(\mathbf{r}')}}, \qquad \left|g^{(1)}\right| \leq 1.$$

$g^{(1)}$ measures the degree of coherence between two points, independently of the local density. It equals 1 in modulus for a perfectly coherent field, and the distance over which it decays defines the coherence length.

Link with the momentum distribution

For a translationally invariant system, $g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{0})\hat{\psi}(\mathbf{r})\rangle$. With $\hat{a}_\mathbf{k} = L^{-d/2}\int \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\hat{\psi}(\mathbf{r})\,\mathrm{d}^d r$,

$$g_1(\mathbf{r}) = \frac{1}{L^d}\sum_{\mathbf{k},\mathbf{k}'} \mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}\langle \hat{a}^\dagger_\mathbf{k}\hat{a}_{\mathbf{k}'}\rangle.$$

Under a translation by $\mathbf{a}$ the mode operators acquire a phase, $\hat{a}_\mathbf{k} \to \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}}\hat{a}_\mathbf{k}$, so $\langle \hat{a}^\dagger_\mathbf{k}\hat{a}_{\mathbf{k}'}\rangle$ is multiplied by $\mathrm{e}^{\mathrm{i}(\mathbf{k}'-\mathbf{k})\cdot\mathbf{a}}$. Being unchanged for *every* $\mathbf{a}$, it must vanish unless $\mathbf{k} = \mathbf{k}'$:

$$\langle \hat{a}^\dagger_\mathbf{k}\hat{a}_{\mathbf{k}'}\rangle = n_\mathbf{k}\,\delta_{\mathbf{k},\mathbf{k}'}.$$

The double sum collapses, and

$$g_1(\mathbf{r}) = \frac{1}{(2\pi)^d}\int \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, n_\mathbf{k}\,\mathrm{d}^d k.$$

Consequences. Coherence length and momentum width are conjugate: a sharply peaked $n_\mathbf{k}$ (a condensate) means long-range coherence. And since $n_\mathbf{k}$ is measured directly by time-of-flight, this is the standard experimental route to g_1.

The g_2 function

$$g_2(\mathbf{r},\mathbf{r}') = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})\rangle = \langle \mathbf{r},\mathbf{r}'|\hat{\rho}_2|\mathbf{r},\mathbf{r}'\rangle,$$

the expectation value of the two-body observable $\hat{c} = |\mathbf{r},\mathbf{r}'\rangle\langle\mathbf{r},\mathbf{r}'|$. For a pure state,

$$g_2(\mathbf{r},\mathbf{r}') = N(N-1)\int |\psi(\mathbf{r},\mathbf{r}',\mathbf{r}_3,\ldots,\mathbf{r}_N)|^2\, \mathrm{d}^3 r_3 \cdots \mathrm{d}^3 r_N.$$

Normal ordering is essential

g_2 is the joint probability density of detecting one particle at $\mathbf{r}$ *and another one* at $\mathbf{r}'$. It differs from $\langle\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')\rangle$ by the self-detection term $\delta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})$.

Properties: symmetric, positive, and $\int g_2\, \mathrm{d}^3 r\, \mathrm{d}^3 r' = \langle \hat{N}(\hat{N}-1)\rangle$. For fermions $\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) = 0$ gives the Pauli hole

$$g_2(\mathbf{r},\mathbf{r}) = 0.$$

Normalization and particle fluctuations

$$g^{(2)}(\mathbf{r},\mathbf{r}') = \frac{g_2(\mathbf{r},\mathbf{r}')}{\rho(\mathbf{r})\,\rho(\mathbf{r}')} \longrightarrow 1 \text{ at large separation (no long-range order).}$$

$$g^{(2)} > 1: \text{ bunching} \qquad\qquad g^{(2)} < 1: \text{ antibunching}$$

For the number of particles $\hat{N}_\Omega = \int_\Omega \hat{\psi}^\dagger\hat{\psi}\,\mathrm{d}^3 r$ in a subregion Ω, moving $\hat{\psi}(\mathbf{r})$ to the right with $\hat{\psi}(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}') = \hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')$ gives

$$\Delta N_\Omega^2 = \langle \hat{N}_\Omega \rangle + \int_\Omega [g_2(\mathbf{r},\mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}')]\;\mathrm{d}^3 r\,\mathrm{d}^3 r'.$$

- The first term is the self-detection contribution: the Poissonian result $\Delta N^2 = \langle \hat{N} \rangle$ for uncorrelated particles.
- The deviation from Poisson is entirely governed by $g^{(2)} - 1$: bunching $\Rightarrow$ super-Poissonian, antibunching $\Rightarrow$ sub-Poissonian.

Ideal gases: momentum distributions

Grand-canonical ensemble, $\hat{\rho} = \Xi^{-1} \exp[-\beta(\hat{H} - \mu\hat{N})]$:

$$n_{\mathbf{k}} = \frac{1}{\mathrm{e}^{\beta(\varepsilon_k - \mu)} - \varepsilon}, \qquad \varepsilon = +1 \text{ (Bose)}, \ -1 \text{ (Fermi)}.$$

Classical limit $\beta(\varepsilon_k - \mu) \gg 1$:

$$n_{\mathbf{k}} = \rho \, \lambda_{\mathrm{dB}}^3 \, \exp\left[-\frac{(k\lambda_{\mathrm{dB}})^2}{4\pi}\right], \qquad \lambda_{\mathrm{dB}} = \frac{h}{\sqrt{2\pi m k_{\mathrm{B}} T}},$$

the prefactor being the phase-space density $\rho\lambda_{\mathrm{dB}}^3 \ll 1$. Fourier transforming this Gaussian,

$$g_1(\mathbf{r}) = \rho \, \exp\left[-\pi \left(\frac{r}{\lambda_{\mathrm{dB}}}\right)^2\right].$$

One checks $g_1(\mathbf{0}) = \rho$. In a classical gas, coherence does not extend beyond the thermal wavelength.

Condensed Bose gas and degenerate Fermi gas

Bose–Einstein condensate ($T < T_c$, i.e. $\rho\lambda_{\mathrm{dB}}^3 > 2.612$; then $\mu = 0$):

$$n_{\mathbf{k}} = n_0\,\delta_{\mathbf{k},0} + \frac{1}{\mathrm{e}^{\beta\varepsilon_k} - 1}, \qquad n_0 = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right],$$

$$g_1(\mathbf{r}) = \rho_0 + \frac{1}{(2\pi)^3}\int \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} n_{\mathbf{k}}\,\mathrm{d}^3k.$$

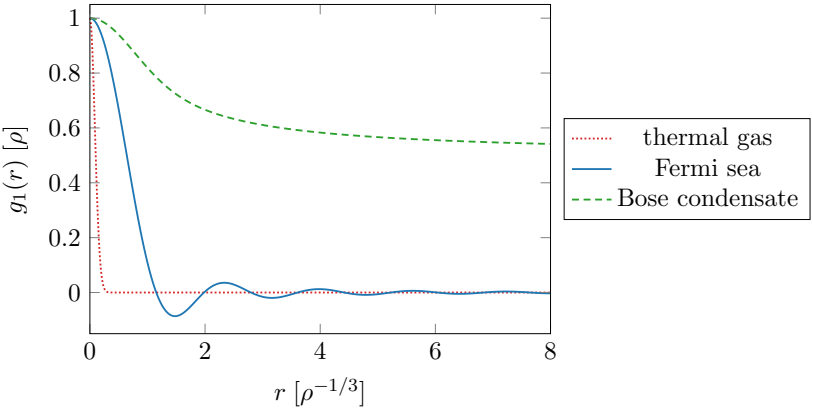
The second term decays over λ_{dB}; the first survives at all distances. The condensate exhibits long-range phase coherence, $g_1(\infty) = \rho_0$.

Fermi sea at $T = 0$ ($n_{\mathbf{k}} = 1$ for $k < k_{\mathrm{F}}$, else 0):

$$g_1(x) = \rho\ \mathrm{sinc}(k_{\mathrm{F}}x) \ \text{ in 1D}, \qquad g_1(\mathbf{r}) = \frac{\sin(k_{\mathrm{F}}r) - (k_{\mathrm{F}}r)\cos(k_{\mathrm{F}}r)}{2\pi^2 r^3} \ \text{ in 3D}.$$

Oscillating, and decaying only algebraically (*Friedel oscillations*) — a consequence of the discontinuity of $n_{\mathbf{k}}$ at the Fermi surface.

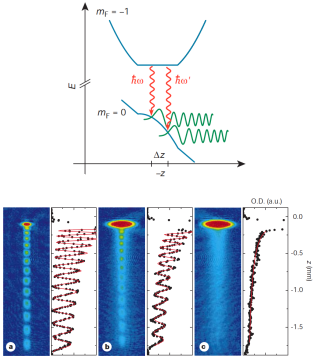
First-order coherence of ideal gases

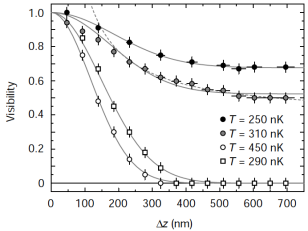


$g_1(r)$ for a thermal gas (red dotted, $\rho\lambda_{\mathrm{dB}}^3 = 0.01$, no discernible effect of quantum statistics), a $T = 0$ Fermi gas, and a Bose gas with 50% condensed fraction, in three dimensions.

Note the three qualitatively different behaviours: Gaussian decay over λ_{dB}, algebraic Friedel oscillations, and a non-zero plateau set by the condensed density.

Measuring g_1 in an atomic gas





Measurement of $g_1(r)$ in a weakly interacting Bose gas of rubidium [1]. A radiofrequency field driven with two tones outcouples atoms from two positions separated by r; the two emitted beams interfere, and the visibility of the interference pattern is proportional to $g_1(r)$. The measurements agree with the prediction for a non-interacting Bose gas.

Bunching and antibunching from Wick's theorem

Ideal gases at thermal equilibrium are Gaussian states, so

$$\begin{aligned} g_2(\mathbf{r},\mathbf{r}') &= \langle\hat\psi^\dagger(\mathbf{r})\hat\psi(\mathbf{r})\rangle\langle\hat\psi^\dagger(\mathbf{r}')\hat\psi(\mathbf{r}')\rangle + \varepsilon\,\langle\hat\psi^\dagger(\mathbf{r})\hat\psi(\mathbf{r}')\rangle\langle\hat\psi^\dagger(\mathbf{r}')\hat\psi(\mathbf{r})\rangle \\ &= \rho(\mathbf{r})\rho(\mathbf{r}') + \varepsilon\,|g_1(\mathbf{r},\mathbf{r}')|^2\,, \end{aligned}$$

only two of the three pairings contributing, the anomalous one vanishing at fixed particle number. Equivalently

$$g^{(2)}(\mathbf{r},\mathbf{r}') = 1 + \varepsilon\left|g^{(1)}(\mathbf{r},\mathbf{r}')\right|^2.$$

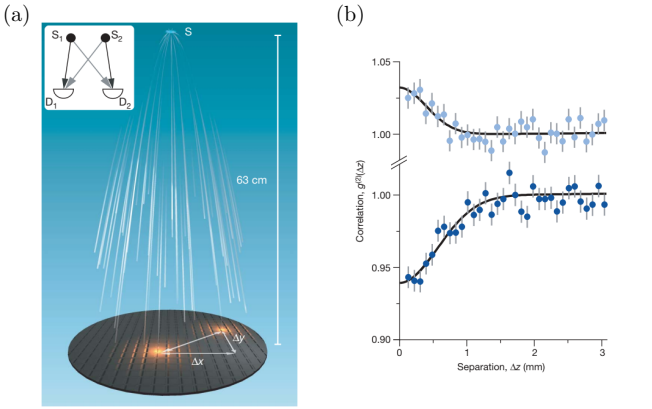
For an ideal gas, pair correlations are entirely determined by the first-order coherence.

At short distance, where $|g^{(1)}| \to 1$:

$$g_2(\mathbf{r},\mathbf{r}) = 2\,\rho(\mathbf{r})^2 \quad \text{(bosons)}, \qquad g_2(\mathbf{r},\mathbf{r}) = 0 \quad \text{(fermions)}.$$

The correlations extend over the coherence length — λ_{dB} for a thermal gas, $1/k_{\mathrm{F}}$ for a degenerate Fermi gas. These are purely quantum-statistical effects: no interaction is involved.

Bunching and antibunching in the same experiment



(a) An ultracold gas of ^{3}He (fermion) or ^{4}He (boson) is released and detected on a multichannel plate; x, y from the detection point, z from the arrival time. (b) Fermions (bottom) show antibunching, bosons (top) bunching [2]. The deviation from perfect (anti)bunching is due to the finite detector resolution.

Particle fluctuations in a 1D Fermi gas

At $T = 0$ the gas is Gaussian, so $g_2(x) = \rho^2 \left[1 - \mathrm{sinc}^2(k_F x)\right]$ and

$$\Delta N^2 = \langle N \rangle - \rho^2 \int_{-L}^{L} (L - |X|)\, \mathrm{sinc}^2(k_F X)\, \mathrm{d}X,$$

the triangular weight being the overlap of the segment with its translate by X. Already this shows $\Delta N < \sqrt{\langle \hat{N} \rangle}$: sub-Poissonian. With $\int_{-A}^{A} (A - |u|)\, \mathrm{sinc}^2\, u\, \mathrm{d}u = \pi A - [\log(2A) + \gamma + 1] + O(A^{-2})$,

$$\Delta N^2 = \frac{\log(2\pi\rho L) + \gamma + 1}{\pi^2} + O(1/L^2)$$

— only *logarithmic* in the system size, far below the Poisson law. In 3D, $\Delta N^2 \propto (k_F L)^2 \log(k_F L)$ [3].

Area law and entanglement

Truly short-range correlations would give $\Delta N^2 \propto L^{d-1}$ (*area law*), as for a band insulator. For the entanglement entropy $S(\hat{\rho}_\Omega) = -\,\mathrm{Tr}(\hat{\rho}_\Omega \log \hat{\rho}_\Omega)$ of free fermions [4]:

$$S \sim L^{d-1} \log L \text{ (gapless: free fermions, metals)}, \quad S \sim L^{d-1} \text{ (gapped: insulators)}.$$

Impenetrable bosons: the Tonks–Girardeau limit

One-dimensional bosons with contact interactions — the Lieb–Liniger model [5]:

$$\hat{H} = \int \left[-\frac{\hbar^2}{2m} \hat{\psi}^\dagger \partial_x^2 \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right] \mathrm{d}x.$$

We take the limit $g \to \infty$ of impenetrable bosons [6, 7]. In first quantization: wave-functions vanish whenever $x_i = x_j$.

Discretizing space (spacing $\mathrm{d}x$), the hardcore constraint amounts to projecting with $\hat{P}$ onto states with at most one boson per site. The projected operators $\hat{P}\hat{\psi}_x\hat{P}$ are

- not bosonic: $(\hat{P}\hat{\psi}_x^\dagger\hat{P})^2 = 0$;
- not fermionic either: they still *commute* at different points.

Interacting — yet exactly solvable, thanks to a mapping to free fermions [7].

Fermionization: the Jordan–Wigner transformation

Attach to each operator the parity of the number of particles to its left [8]:

$$\hat{\psi}_x^F = \hat{\Pi}_{0,x}\,\hat{P}\hat{\psi}_x\hat{P}, \qquad \hat{\Pi}_{x,x'} = \exp\left[\mathrm{i}\pi \sum_{x\le x''<x'} \hat{\psi}_{x''}^\dagger \hat{\psi}_{x''}\right].$$

The string operator $\hat{\Pi}$ is non-local. It commutes with $\hat{\psi}_{x''}$ for x'' outside the interval, and anticommutes for x'' inside.

One checks that the $\hat{\psi}_x^F$ obey fermionic anticommutation relations: at equal points $\{\hat{\psi}_x^F, \hat{\psi}_x^{F\dagger}\} = (1-\hat{n}_x) + \hat{n}_x = 1$, and at different points the strings regroup to give

$$\{\hat{\psi}_x^F, \hat{\psi}_{x'}^{F\dagger}\} = \hat{\Pi}_{x,x'}\left(-\hat{P}\hat{\psi}_x\hat{P}\hat{\psi}_{x'}^\dagger\hat{P} + \hat{P}\hat{\psi}_{x'}^\dagger\hat{P}\hat{\psi}_x\hat{P}\right) = 0.$$

The Hamiltonian becomes

$$\hat{H} = -\frac{\hbar^2}{2m\mathrm{d}x^2}\sum_x \hat{\psi}_x^{F\dagger}\left(\hat{\psi}_{x-\mathrm{d}x}^F - 2\hat{\psi}_x^F + \hat{\psi}_{x+\mathrm{d}x}^F\right)$$

— free fermions.

The ground state, and what is shared with free fermions

At the level of wave-functions, multiply by the "unit antisymmetric function"

$$\psi_F(x_1, \ldots, x_N) = A(x_1, \ldots, x_N)\, \psi(x_1, \ldots, x_N), \qquad A = \prod_{i<j} \mathrm{sgn}(x_i - x_j),$$

well defined because two particles never coincide. The ground state is thus

$$\psi = |\psi_F|, \qquad \psi_F = \text{Slater determinant of } \mathrm{e}^{\mathrm{i}k_j x_l},\ |k_j| < k_\mathrm{F} = \pi\rho.$$

Two distinct reasons for coincidence

- The energies coincide, because the two Hamiltonians are mapped onto one another.
- Any observable diagonal in the positions takes the same value, because $|\psi| = |\psi_F|$ (since $|A| = 1$). This covers the density and all density correlations.

In particular, as g_2 involves only $|\psi|^2$,

$$g_2(x, x') = g_2^F(x, x') = \rho^2 \left[1 - \mathrm{sinc}^2\left(k_\mathrm{F}(x - x')\right)\right], \qquad g_2(0) = 0.$$

The bunching of ideal bosons is lost, as expected for impenetrable particles.

Why g_2 is shared but g_1 is not

Inverting the transformation, $\hat{\psi}_x = \hat{\Pi}_{0,x}\hat{\psi}^F_x$ and $\hat{\psi}^\dagger_x = \hat{\psi}^{F\dagger}_x\hat{\Pi}_{0,x}$. For the pair correlation,

$$g_2(x,x') = \langle \hat{\psi}^{F\dagger}_x\, \hat{\Pi}_{0,x}\, \hat{\psi}^{F\dagger}_{x'} \hat{\psi}^F_{x'}\, \hat{\Pi}_{0,x}\, \hat{\psi}^F_x \rangle.$$

Move the first string to the right through the *pair* $\hat{\psi}^{F\dagger}_{x'}\hat{\psi}^F_{x'}$:

- $x' \geq x$: x' lies outside $[0,x[$ and the string commutes with both operators;
- $x' < x$: it anticommutes with each, and the two signs compensate.

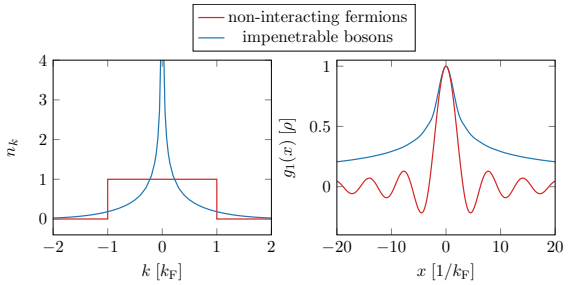
The two strings meet and cancel, $\hat{\Pi}^2_{0,x} = 1$, giving $g_2 = g_2^F$. The essential point: the two field operators sit at the *same* point, in a pair.

For the first-order correlation function the operators sit at *different* points:

$$g_1(x,x') = \langle \hat{\psi}^\dagger_x \hat{\psi}_{x'} \rangle = \langle \hat{\psi}^{F\dagger}_x\, \hat{\Pi}_{x,x'}\, \hat{\psi}^F_{x'} \rangle.$$

The strings no longer cancel: they combine into $\hat{\Pi}_{x,x'}$, the parity of the particles *between* the two points. g_1, and hence the momentum distribution, differ from the fermionic ones — consistently with the fact that g_1 probes the *phase* of the wave-function, which is exactly what A changes.

Momentum distribution and first-order coherence



Asymptotic behaviours [9, 10]:

$$n_k \sim A\,\rho\sqrt{\frac{2\pi}{k_\mathrm{F} k}}, \qquad k \ll k_\mathrm{F}$$

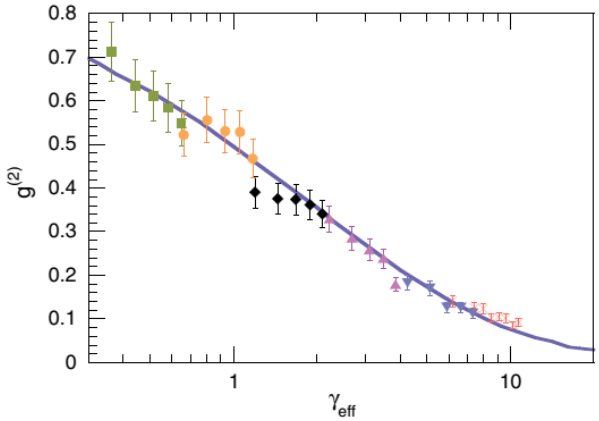
$$g_1(x) \sim \frac{A\,\rho}{\sqrt{k_\mathrm{F}|x|}}, \qquad k_\mathrm{F} x \gg 1$$

with $A = 0.92418\ldots$

Consistent with each other: the Fourier transform of $|x|^{-1/2}$ is $\sqrt{2\pi/|k|}$.

n_k diverges as $k \to 0$, but the $k = 0$ mode is not macroscopically populated: there is no Bose–Einstein condensation [11], as confirmed by the decay of g_1 at large distance.

Observing the Tonks–Girardeau regime



Reached with ultracold rubidium atoms tightly confined to one dimension [12, 13].

The local pair correlation $g^{(2)}(0)$ is measured through the rate of photoassociation of pairs of atoms, as a function of the effective Lieb–Liniger parameter

$$\gamma_{\mathrm{eff}} = \frac{m^* g_{1\mathrm{D}}}{\hbar^2 \rho},$$

large for strong interactions or low density.

The antibunching of the Tonks–Girardeau regime appears in the strongly interacting limit $\gamma_{\mathrm{eff}} \to \infty$ [14].

Superfluid helium

Superfluidity was discovered in 1937 by Kapitza, Allen and Misener, through the viscosity of liquid helium at cryogenic temperatures [15, 16], and soon attributed to the formation of a Bose–Einstein condensate.

Neutron scattering measures the atomic momentum distribution [17]. The central peak — the $n_0 \delta_{\mathbf{k},\mathbf{0}}$ term broadened by the experimental resolution — gives a condensed fraction

$$f_0 = \rho_0/\rho = 7\%$$

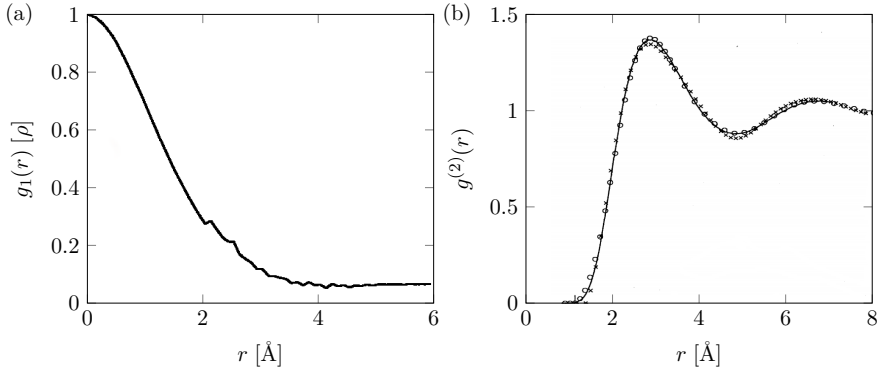
at zero temperature and saturating vapour pressure.

Partial condensation, complete superfluidity

The system is *not* fully condensed at $T = 0$, because of the strong interactions between atoms. This does not prevent it from being fully superfluid: it has no viscosity and does not respond to moderate rotations of its container.

Bogoliubov theory, developed later in the course, provides the framework for partial condensation at $T = 0$.

Correlation functions of superfluid helium



(a) g_1 from quantum Monte-Carlo [18, 19]; the asymptotic value $g_1(\infty)/\rho \simeq 7\%$ is the condensed fraction, consistent with the neutron measurement. (b) Normalized pair distribution $g^{(2)}(r)$: Monte-Carlo (solid line) and neutron scattering (symbols).

At short distance $g^{(2)} \to 0$, because of the short-range repulsion between helium atoms. It exceeds one near $r \simeq 3.5$ Å, but this bunching is not a signature of bosonic statistics: the same peak is found in Monte-Carlo simulations of an equivalent system *without* exchange statistics [19].

Summary – Chapter 3

- g_1 is the Fourier transform of the momentum distribution; its decay length is the coherence length. A condensate shows long-range order, $g_1(\infty) = \rho_0$.
- g_2 measures pair correlations, with normal ordering essential. The deviation of $g^{(2)}$ from 1 controls the deviation of the number fluctuations from the Poisson law.
- **Ideal gases:** $g^{(2)} = 1 + \varepsilon |g^{(1)}|^2$ — bunching for bosons, the Pauli hole for fermions, with no interaction involved. Fermionic fluctuations are sub-Poissonian and only logarithmic in L.
- **Impenetrable bosons** map onto free fermions: identical energies and density correlations, but different g_1 and momentum distribution, because of the Jordan–Wigner string. No condensation in 1D.
- **Superfluid helium:** only 7% condensed at $T = 0$ because of strong interactions, yet fully superfluid.

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