Advanced quantum mechanics
TD 2 : Spontaneous emission and reabsorption : the Wigner-Weisskopf model

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Consider a scalar model for the electromagnetic radiation in a one-dimensional box of size $[0, L]$ with periodic boundary conditions. We restrict to positive wave vectors $k > 0$; physically this corresponds to a situation where the radiation propagates only in one direction. The dispersion relation for the free field is given by $\omega = ck$. A two-level atom is placed at position $x = 0$ and is coupled to the radiation. The Hamiltonian of the system can be written in the form $\hat{H} = \hat{H}_0 + \hat{V}$ with

$$\hat{H}_0 = \sum_n \hbar\omega_n \hat{a}_n^\dagger \hat{a}_n + \hbar\omega_0 \left|e\right\rangle \left\langle e\right| \qquad \text{and} \qquad \hat{V} = v \sum_n \left(\left|e\right\rangle \left\langle g\right| \hat{a}_n + \hat{a}_n^\dagger \left|g\right\rangle \left\langle e\right| \right) . \tag{1}$$

The operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ annihilate and create a photon in the mode n, $\hbar\omega_n = \hbar c k_n = \hbar c (2\pi n/L)$. We write the following relations :

$$\hbar\omega_n = n\delta \qquad \text{with} \qquad \delta = \frac{2\pi\hbar c}{L} . \tag{2}$$

We use the notation $E_0 = \hbar\omega_0 \equiv n_0\delta$. The two-level atom transition is at resonance with a photon of angular frequency ω_0 and wave vector k_0, and we will use a cutoff on the coupling v at an energy $E = 2E_0$ such that the photonic states with $k > 2k_0$ are not coupled to the atom. During the whole exercise we will restrict to states with zero or one photon.

1. Calculate $\hat{P}\hat{G}(z)\hat{P}$, where $\hat{G}(z)$ is the resolvent of the Hamiltonian $\hat{H}$ and $\hat{P}$ projects on the state $|\phi\rangle = |e; 0\rangle$ (i.e. atom in excited state and zero photons in the cavity), in the limit where $E_0 \to \infty$ with $z - E_0$ fixed. See the appendix for some reminders. We will use

   $$\tilde{z} = z - E_0 \; ; \quad \tilde{n} = n - n_0 \; ; \quad \tilde{E} = E - E_0 \; , \tag{3}$$

   and the identity

   $$\sum_{l=-\infty}^{+\infty} \frac{1}{z - l} = \pi \, \text{cotan}(\pi z) \; , \tag{4}$$

   with z being a non-integer complex number.
2. How can one determine the spectrum of the effective Hamiltonian from this expression for $\hat{P}\hat{G}(z)\hat{P}$? Graphically solve the equation for the eigenvalues $\tilde{E}_m$ and show that there is only one eigenvalue in each of the intervals $]m\delta, (m+1)\delta[$ for integer m.
3. Determine the probability amplitude at time $t > 0$ for the system to remain in the initial state $|\psi(0)\rangle = |e; 0\rangle$ in the form of a sum over m (which labels the eigenvalues $\tilde{E}_m$).
4. In Fig. 1, we show a numerical evaluation of the expression found in the previous question. Explain qualitatively the presence of revivals in the probability of finding the system in the initial state by using an approximate property of the spectrum of $\hat{H}$. Use the non-perturbed spectrum of $\hat{H}$ to estimate the time when the first revival takes place.
5. Take the continuum limit $\delta, v \to 0$ at constant $\tilde{v} \equiv v^2/\delta$, replace the sum with an integral, and interpret the result. Use the notation : $\Gamma = \frac{2\pi}{\hbar}\tilde{v}$.

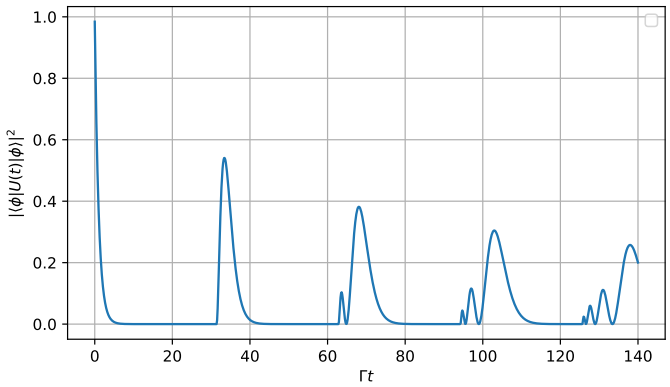


FIGURE 1 – The probability $|\langle\phi|\hat{U}(t)|\phi\rangle|^2$ for the system to remain in the initial state $|\phi\rangle$ at function of the time t. We took $\delta = 0.2\hbar\Gamma$ with $\hbar\Gamma = 2\pi v^2/\delta$.

6. Show that one can obtain the same result by taking the limit of a continuous spectrum directly in $\hat{P}\hat{G}(z)\hat{P}$. Calculate $\hat{P}\hat{G}(z)\hat{P}$ in the continuous limit for $\mathrm{Im}(z) > 0$ and perform an analytical continuation to the lower half-plane.
7. We finally consider the model with continuous spectrum : the discrete excited state $|e\rangle$ of bare energy E_0 is coupled to a continuum of field modes $|E\rangle$ with energy $0 \leq E \leq \Lambda$, and $\langle E|E'\rangle = \delta(E - E')$. Like before, the model has a sharp high-energy cutoff $\Lambda = 2E_0$. The Hamiltonian is

$$\hat{H} = E_0\,|e\rangle\langle e| + \int_0^\Lambda dE\,E\,|E\rangle\langle E| + \int_0^\Lambda dE\,\big(V(E)\,|E\rangle\langle e| + V^*(E)\,|e\rangle\langle E|\big), \tag{5}$$

where $V(E) = \langle E|\hat{V}|e\rangle$ is the matrix element coupling $|e\rangle$ to the continuum mode at energy E. We take the constant coupling matrix element

$$V(E) = V_0 \quad \text{for all } E \geq 0.$$

(a) Determine $\langle e|\hat{H}_{\text{eff}}(z)|e\rangle$. Then $\langle e|\hat{G}(z)|e\rangle$ and its analytic structure.
(b) Determine again the survival probability amplitude $\langle e|\hat{U}(t)|e\rangle$. Evalulate the integral by performing analytic continuation of $\langle e|\,\hat{G}(z)\,|e\rangle$ to the lower-half plane.

Appendix : some reminders from the lecture

The resolvant of the Hamiltonian $\hat{H}$ is defined as the operator

$$\hat{G}(z) = \frac{1}{z - \hat{H}}\,. \tag{6}$$

In the theory of effective Hamiltonians, one generally introduces a subspace-projection operator $\hat{P}$ that commutes with the non-perturbed Hamiltonian $\hat{H}_0$. The complementary projection operator is $\hat{Q} = 1 - \hat{P}$. The central result is

$$\hat{P}\hat{G}(z)\hat{P} = \frac{1}{z\hat{P} - \hat{H}_{eff}(z)} \tag{7}$$

Note that we written $z\hat{P}$ instead of z in the denominator, just for clarity. The operator $\hat{H}_{eff}(z)$ is given by

$$\hat{H}_{eff}(z) = \hat{P}\hat{H}\hat{P} + \hat{P}\hat{V}\hat{Q}\frac{\hat{Q}}{z\hat{Q} - \hat{Q}\hat{H}\hat{Q}}\hat{Q}\hat{V}\hat{P} \, , \tag{8}$$

with $\hat{V}$ the perturbation, such that $\hat{H} = \hat{H}_0 + \hat{V}$.
Let us recall the link between the evolution operator and the resolvant operator. The so-called retarded Green's function $\hat{U}_+(t) \equiv \hat{U}(t)\Theta(t)$, with $\Theta(t)$ the Heaviside function, is solution of

$$i\hbar\frac{d}{dt}\hat{U}_+(t) = \hat{H}\hat{U}_+(t) + i\hbar\delta(t) \, , \tag{9}$$

which corresponds to the usual equation for $\hat{U}(t)$ with an extra inhomogeneous term $i\hbar\delta(t)$. The Fourier transform of $\hat{U}_+(t)$ gives

$$\begin{aligned} FT[\hat{U}_+](E) &= \int_{-\infty}^{+\infty} dt e^{iEt/\hbar}\hat{U}_+(t)e^{-\eta t/\hbar} & (10)\\ &= \int_{0}^{+\infty} dt e^{i(E+i\eta-\hat{H})t/\hbar} & (11)\\ &= \frac{i\hbar}{E + i\eta - \hat{H}} & (12)\\ &\equiv i\hbar\, \hat{G}(E + i\eta) & (13) \end{aligned}$$

where a convergence factor $e^{-\eta t/\hbar}$ has been included. We let η go to zero at the end of the calculation. Taking the inverse Fourier transform, $\hat{U}_+(t)$ is expressed in terms of the resolvant :

$$\begin{aligned} \hat{U}_+(t) &= \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} e^{-iEt/\hbar}\, i\hbar\, \hat{G}(E + i\eta) & (14)\\ &= \int_{C_+} \frac{dz}{2\pi i} e^{-izt/\hbar}\hat{G}(z) \, , & (15) \end{aligned}$$

with C_+ the path shown in Fig. 2 Similarly, the advanced Green's function is $\hat{U}_-(t) \equiv -\hat{U}(t)\Theta(-t)$. Its Fourier transform

$$FT[\hat{U}_-](E) = i\hbar\hat{G}(E - i\eta) \, . \tag{16}$$

Again one can take the inverse Fourier transform. Subtracting $\hat{U}_+(t) - \hat{U}_-(t) \equiv \hat{U}(t)$ gives the general result :

$$\hat{U}(t) = \int_{C_+ \cup\, C_-} \frac{dz}{2\pi i} e^{-izt/\hbar}\hat{G}(z) \, , \tag{17}$$

with the paths shown in Fig. 3. Note : if we are only interested in times $t > 0$, we could just use (15), as was done in the lecture.

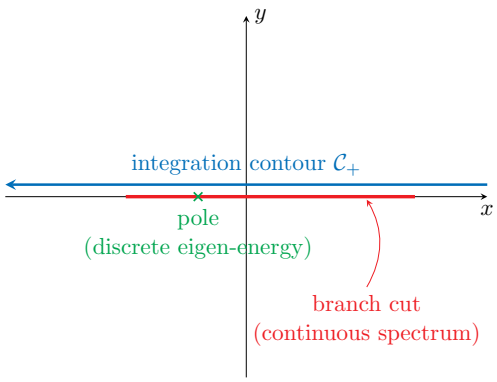


FIGURE 2 – Scheme of the complex plane defining the argument of the resolvent operator $\hat{G}(z)$. The Hermitian nature of the Hamiltonian ensures analyticity of the resolvent away from the x-axis. One expects poles on the x-axis corresponding to discrete eigenstates (green cross), and a branch cut corresponding to the continuous eigen-spectrum (red segment). The contour C_+ is shown in blue. The model discussed in this TD is discrete, so there is no branch cut in our case.

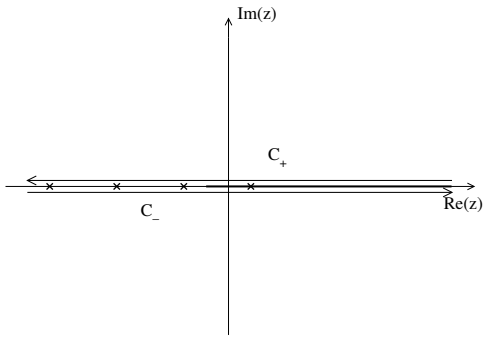


FIGURE 3 – The contours C_+ and C_-.