Advanced quantum mechanics
TD 4 : Bose-Einstein Statistics and Beam Splitter

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Consider an ideal beam splitter (BS) as shown in Fig. 1.

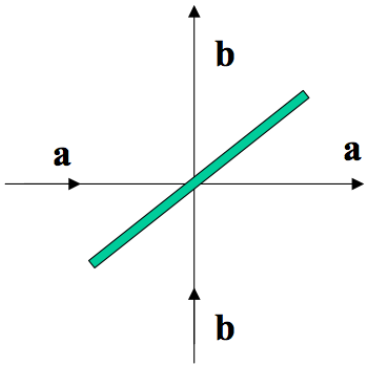


FIGURE 1 – An ideal beam splitter (BS).

A single-particle wave packet arriving on the BS in the mode a generates two single-particle wave packets

— a wave packet which is "transmitted" in the same mode a with a probability amplitude t, t being real

— a wave packet which is "reflected" in the mode b with a probability amplitude r, also real.

Similarly, a single-particle wave packet arriving on the BS in the mode b is transmitted in the same mode with a probability amplitude t, and reflected in the mode a with a probability amplitude $-r$. This corresponds to an evolution operator $\hat{U}^{(1)}$ which in the $|a\rangle, |b\rangle$ basis has the following representation :

$$U^{(1)} = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \tag{1}$$

1. Explain the origin of the $-$ sign and why the condition $t^2 + r^2 = 1$ must apply.
2. We send two identical bosonic particles on the BS, one arriving from the channel a and the other arriving from the channel b. Assuming that the particles do not interact, generalize the operator $U^{(1)}$ to the case of two particles [matrix $U^{(1,2)}$]. Calculate the probability amplitude to have at the exit of the BS respectively :

   — the two bosons in the channel a

   — the two bosons in the channel b

   — one boson in each channel a and b.

   Verify that for a perfect BS (transmission coefficient equal to reflection coefficient, $t = r$), the two bosons exit always from the same channel.

   What would the result be for polarized fermionic particles ?

3. We would like to generalize the matrix U to the case of any number of incident bosons, using second quantization. We denote by $\hat{a}$ and $\hat{b}$ the operators annihilating one particle in the state $|a\rangle$ and $|b\rangle$ respectively.
   We recall the effect of the creation operator of a particle in the state $|u\rangle$ acting on any bosonic state $|\Psi\rangle$:

$$\hat{c}^{\dagger}_{|u\rangle}|\Psi\rangle = \sqrt{\hat{N}}S\left(|u\rangle \otimes |\Psi\rangle\right) \tag{2}$$

   where S is the operator that symmetrizes and $\hat{N}$ the operator of the total number of particles.

   a) Show that

$$\hat{U}\hat{c}^{\dagger}_{|u\rangle}\hat{U}^{\dagger} = \hat{c}^{\dagger}_{\hat{U}|u\rangle} \tag{3}$$

   b) Deduce the transformations $\hat{U}\hat{a}^{\dagger}\hat{U}^{\dagger}$ and $\hat{U}\hat{b}^{\dagger}\hat{U}^{\dagger}$ of the operators $\hat{a}^{\dagger}$ and $\hat{b}^{\dagger}$.

   c) Recalculate the previous results of question 2 for two bosons.

4. We send $2N$ identical bosons on the BS, N from the channel a and N from the channel b. For a perfect BS calculate the probability that all bosons exit from the same channel. How does this probability vary in the limit $N \to +\infty$? Compare with the case of distinguishable particles.
   We recall the Stirling's approximation :

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \quad \text{when } n \to +\infty$$