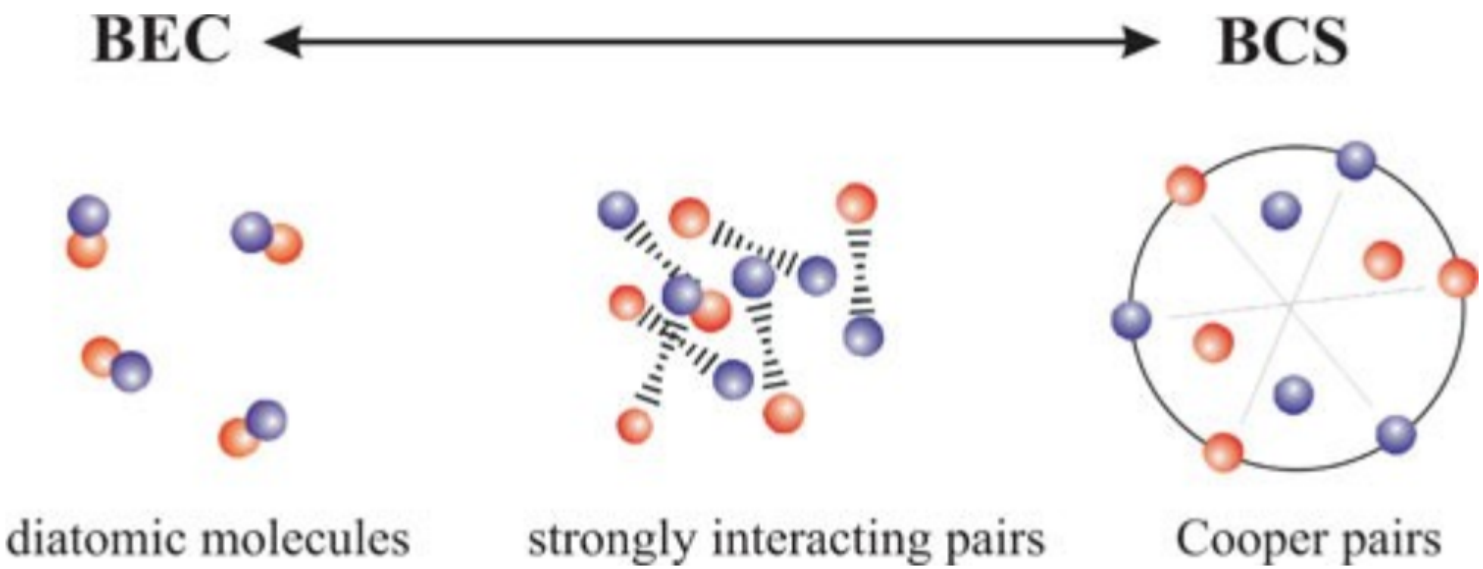
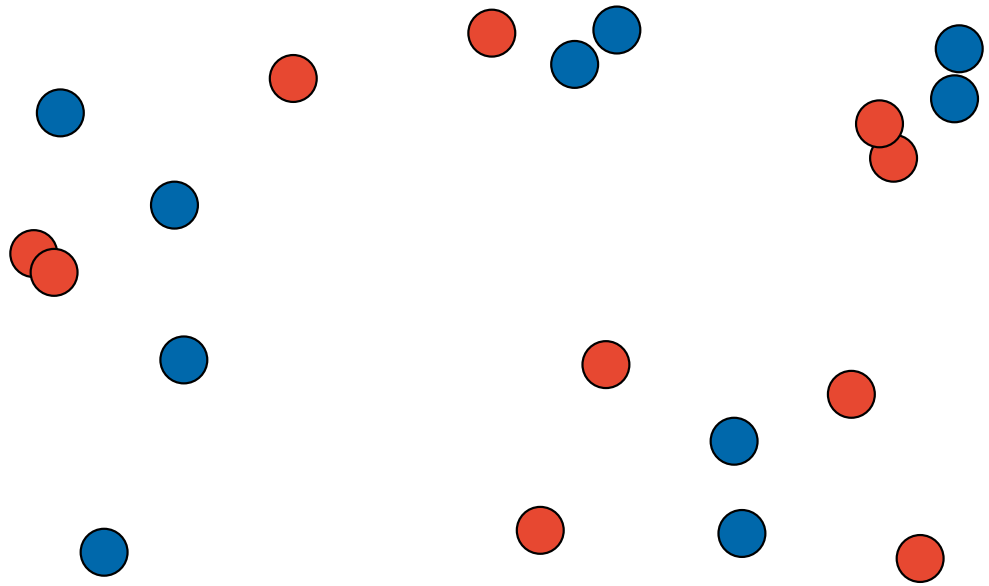
Course on ultracold Fermi gases

Fabrice Gerbier, Raphael Lopes, Sylvain Nascimbene, Aurélien Perrin

M2 ICFP 2021/2022



Attractive Fermi gases



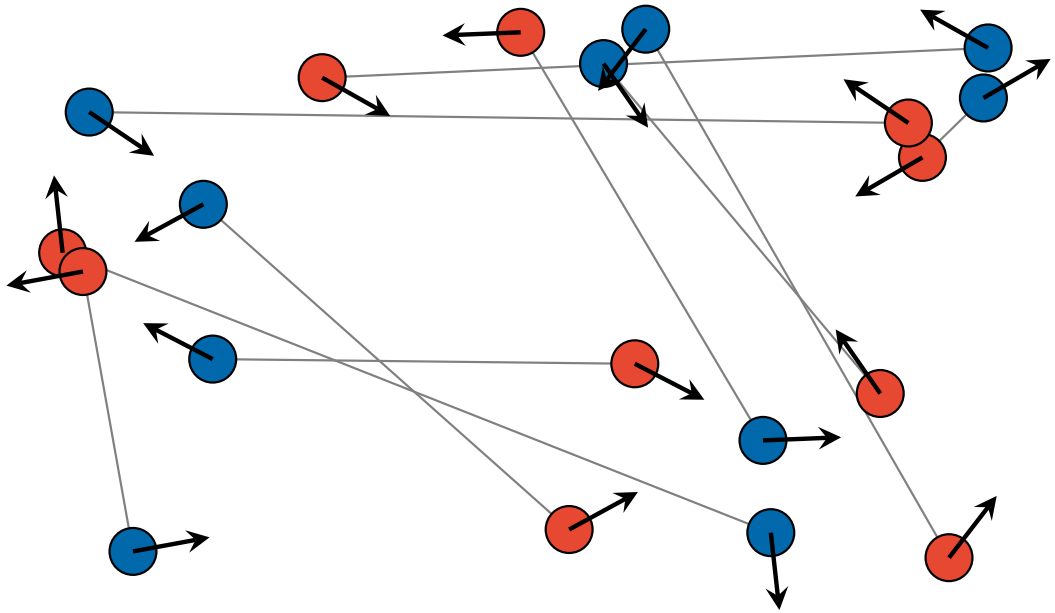
We consider a mixture of two spin component fermionic atoms. We assume attractive interactions between different spin components.

What is the many-body ground state?

Weak attraction: Cooper pairing

First scenario: weak attraction

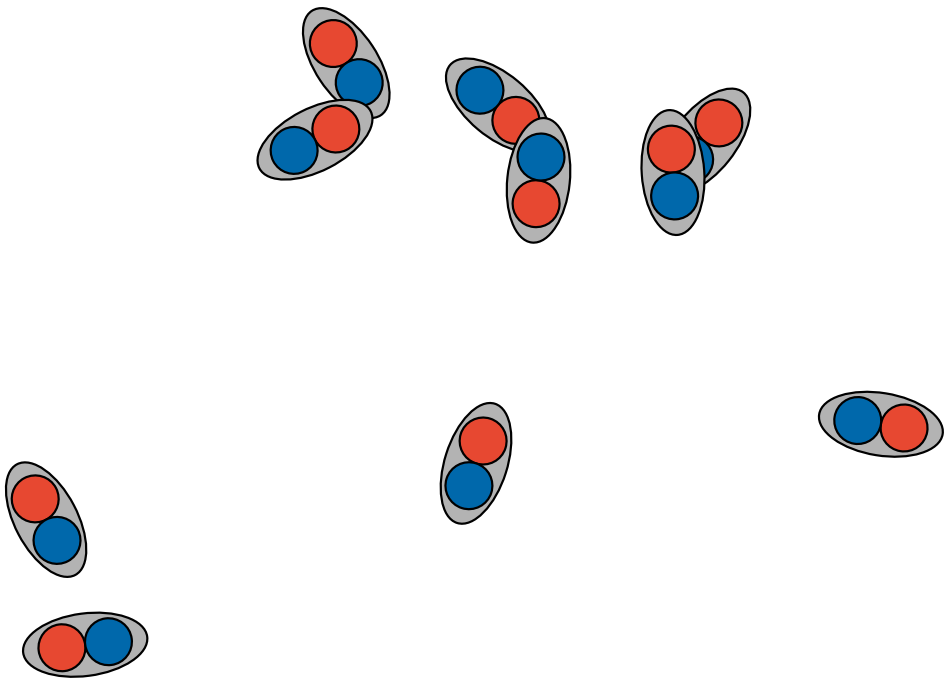
- Cooper pairing between atoms of opposite spin and momentum (around k_F)
- Many-body ground state: BCS superfluid



Strong attraction: molecular BEC

Second scenario: strong attraction

- Pairing into deeply bound molecules
- Molecules form a Bose-Einstein condensate



BEC/BCS crossover in atomic gases

- Tunability of interactions from weak to strong pairing
- Cooper pairing: size $\gg$ inter-particle spacing (loose pair)
- Molecular pairing: size $\ll$ inter-particle spacing (strongly bound pair)
- Intermediate regime: complex correlations

Experimental realization of an ancient proposal from Leggett (1980)

JOURNAL DE PHYSIQUE *Colloque C7, supplément au n° 7, Tome 41, juillet 1980, page* C7-19

COOPER PAIRING IN SPIN-POLARIZED FERMI SYSTEMS

A.J. Leggett

School of Mathematical and Physical Sciences, University of Sussex, Brighton, Sussex, BN1 9QH, Grande Bretagne

Abstract.- I discuss the phenomenon of Cooper pairing in strongly spin-polarized Fermi systems, such as $^3He\uparrow$ and $D\uparrow$, with particular attention to the questions (1) what are the conditions for Cooper pairing to occur, and at what temperatures is this likely to happen for the systems of practical interest ? (2) what is the relationship between Cooper pairs and diatomic molecules ? (3) what are the qualitatively new phenomena we expect in a Cooper-paired system, and what new physics is likely to emerge if the phenomena occurs in spin-polarized systems ?

Remains a challenge for many-body theories!

Outline of the course on Fermi gases

1. Reaching the strongly-interacting regime
   - Cold collisions, scattering length, pseudo-potential
   - Feshbach resonance
   - Collisional stability: bosons vs fermions
2. Theory of the BEC-BCS crossover
   - Cooper instability
   - Mean-field theory of the BEC-BCS crossover ($T = 0$)
   - Superfluid critical temperature
   - Short-range correlations: the contact
   - The unitary Fermi gas
3. Experimental characterization of the BEC-BCS crossover
   - Thermodynamics of the unitary Fermi gas
   - Ground state in the BEC-BCS crossover
   - Measurement of the contact
   - Excitations and superfluid gap
   - Superfluidity
   - Spin imbalance
4. Open questions

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Cold collisions: reminder

We consider two atoms of same mass m interacting via a potential $V(\mathbf{r})$, assumed to be

- spherically symmetric: $V(\mathbf{r}) = V(r)$,
- decaying fast enough at long distances.

A stationnary state of energy E is described by a wavefunction $\psi(\mathbf{r})$ obeying

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}),$$

where $\mu = m/2$ is the reduced mass.

A *scattering state* corresponds to the superposition of

- an incoming plane wave e^{ikz}, with $E = \hbar^2 k^2/2\mu$,
- a scattered wave behaving for large r as $f_k(\theta, \phi)e^{ikr}/r$.

Partial waves

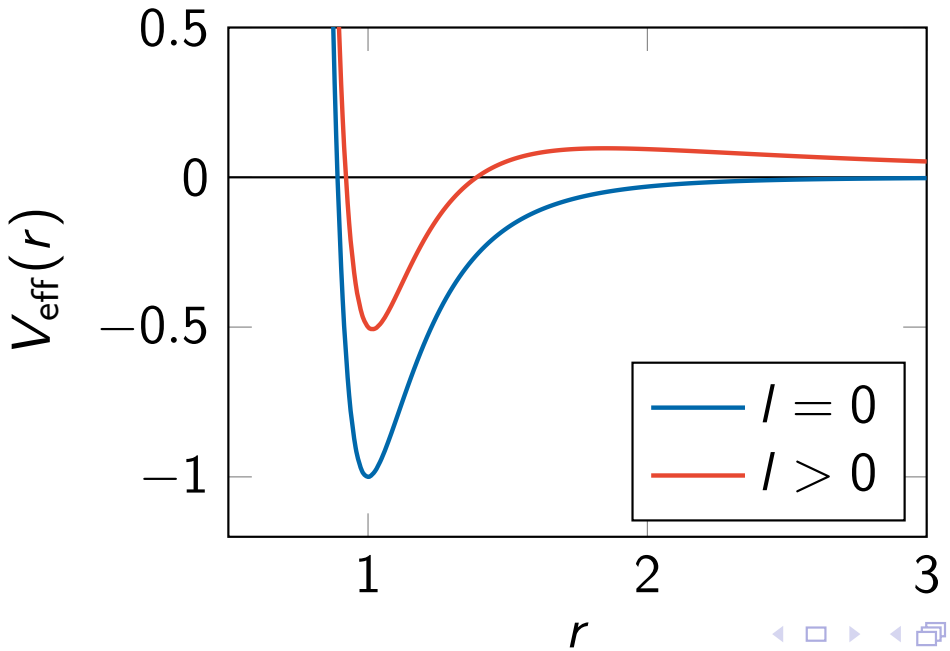
The potential is spherically symmetric $\rightarrow$ partial waves behave independently.
One writes

$$e^{ikz} = \sum_l i^l \sqrt{4\pi(2l+1)}\, j_l(kr)\, Y_l^0(\theta).$$

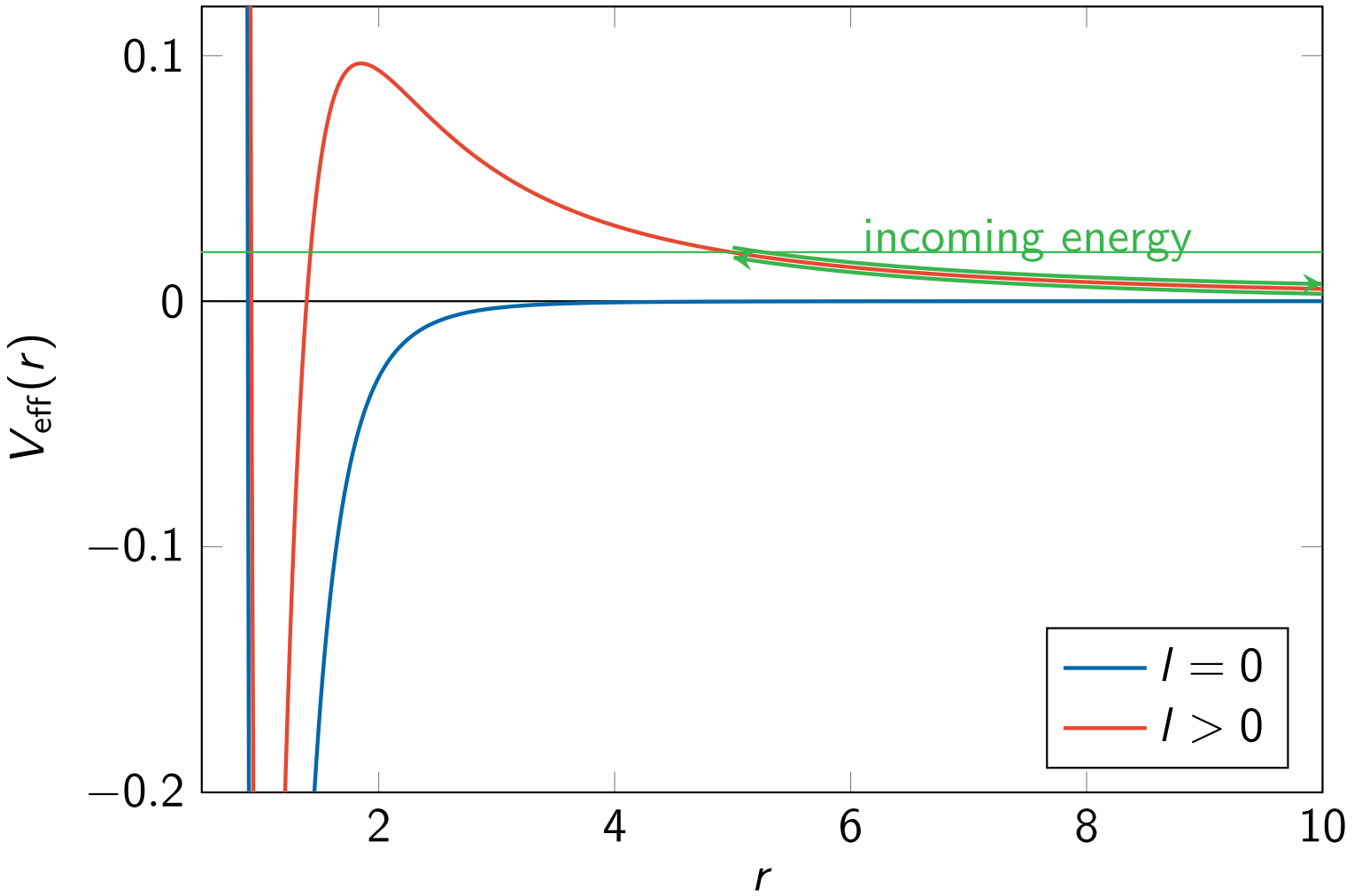
A partial wave $\phi_l(\mathbf{r})$ of angular momentum l obeys

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)\right] u_l(r) = E\, u_l(r), \quad \phi_l(\mathbf{r}) = \frac{1}{r} u_l(r)\, Y_l^0(\theta).$$

effective potential with a centrifugal barrier for $l > 0$.



Restriction to s-wave scattering at low energy



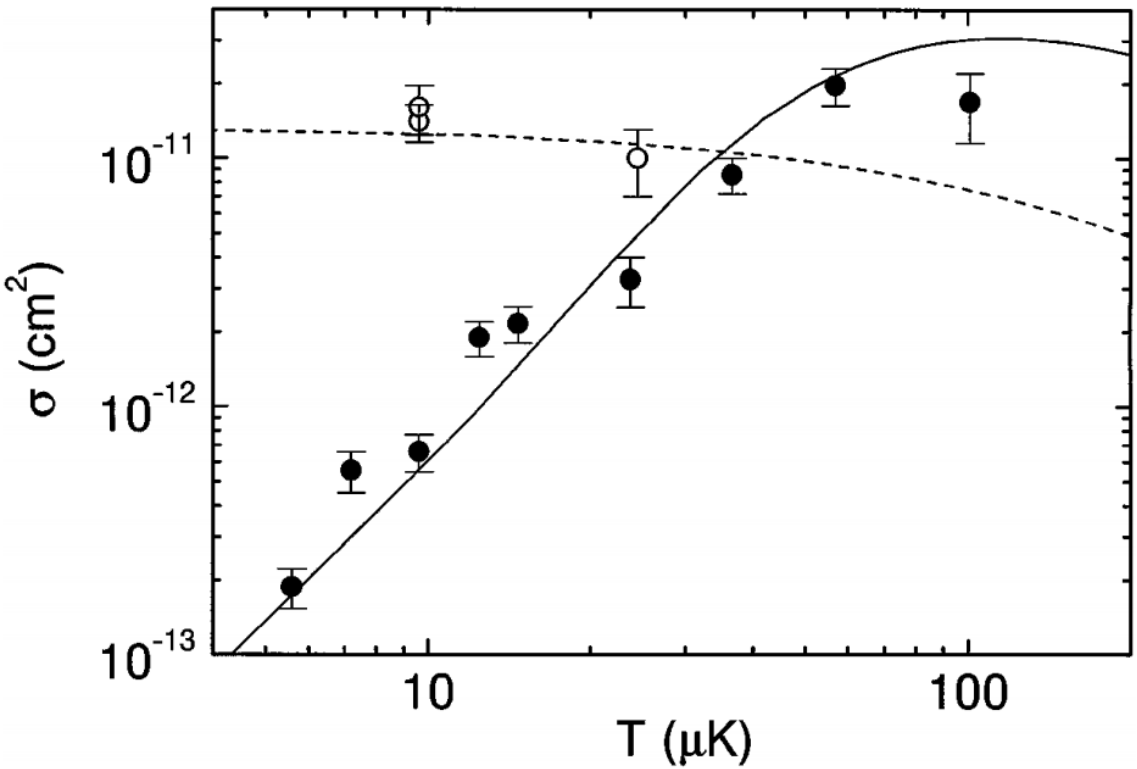
For $l > 0$ the atoms cannot approach each other $\rightarrow$ no effect of the interaction potential.

Cold collisions between identical particles

The relative wavefunction of identical particles (with same internal spin) must be properly symmetrized depending on the quantum statistics:

- bosons: symmetric wavefunction $\rightarrow$ only even-l partial waves
- fermions: antisymmetric wavefunction $\rightarrow$ only odd-l partial waves

Identical fermions do not interact in the cold collisions regime.



- ● polarized fermions
- ○ unpolarized fermions

De Marco et al, PRL 1999

Scattering length

Low-energy scattering: scattering length a

- Scattering restricted to s wave

$$\psi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kz} + \frac{f_k \mathrm{e}^{\mathrm{i}kr}}{r}.$$

- Low-energy limit

$$\psi(\mathbf{r}) = 1 - \frac{a}{r}, \quad a = -\lim_{k \to 0} f_k.$$

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Typical values

At long range, van der Waals interactions

$$V(r) = -\frac{C_6}{r^6}.$$

Typical length and energy scales

$$R_{\mathrm{vdW}} = \frac{1}{2}\left(\frac{2\mu C_6}{\hbar^2}\right)^{1/4} \sim 100\, a_0 = 5\,\mathrm{nm},$$

$$E_{\mathrm{vdW}} = \frac{\hbar^2}{2\mu R_{\mathrm{vdW}}^2} \sim k_{\mathrm{B}} \times 1\,\mathrm{mK} \quad (^6\mathrm{Li}:\ 30\,\mathrm{mK},\ ^{40}\mathrm{K}:\ 1\,\mathrm{mK},\ ^{87}\mathrm{Rb}:\ 0.3\,\mathrm{mK}).$$

R_{vdW} sets the typical values of scattering length a.

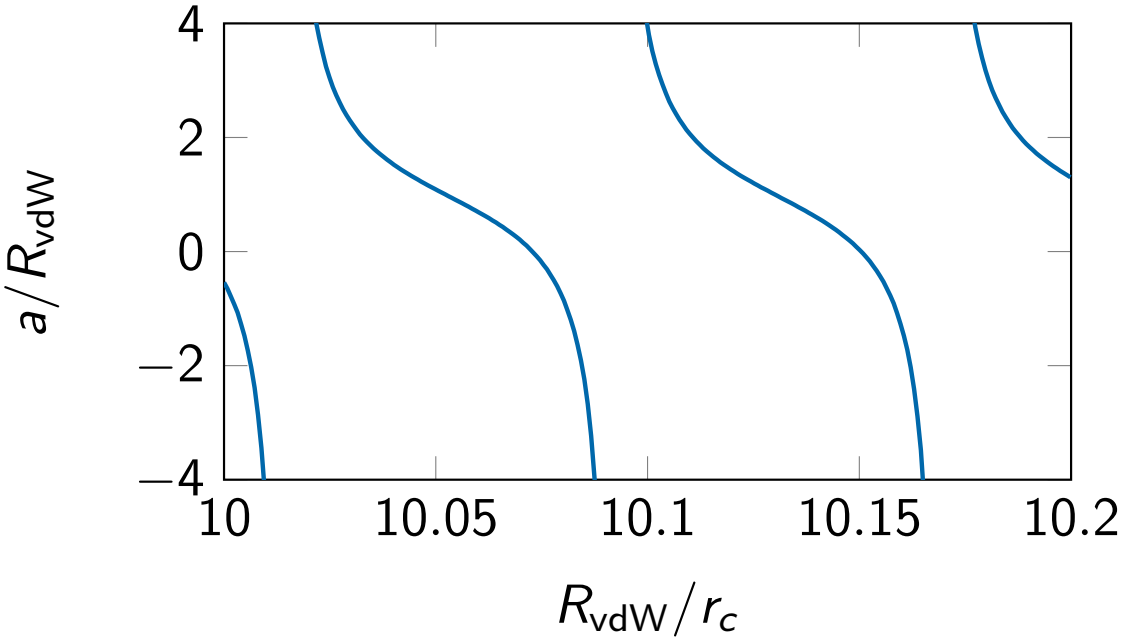
Scattering length in a toy model

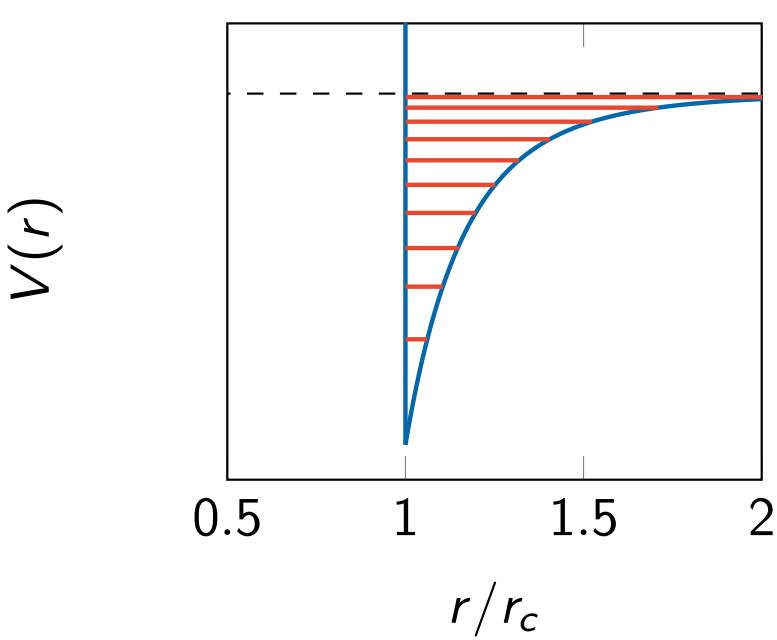
We use a toy model defined by a truncated Van der Waals potential

$$V(r) = +\infty, \quad r < r_c,$$
$$= -\frac{C_6}{r^6}, \quad r > r_c.$$

We find the scattering length

$$a \simeq 0.96\, R_{\mathrm{vdW}} \left[1 - \tan\left(\frac{2R_{\mathrm{vdW}}^2}{r_c^2} - \frac{3\pi}{8}\right)\right].$$





A shape resonance occurs each time a bound state disappears.

The pseudo-potential

At zero energy, the wavefunction takes the expression for large r

$$\psi(r) = 1 - \frac{a}{r}, \tag{1}$$

where a is the s-wave scattering length. The scattering length contains all the relevant information on interactions!

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$$\rightarrow \text{`pseudo potential'} \quad V(r)\psi(\mathbf{r}) = \frac{4\pi\hbar^2 a}{m}\delta(r)\frac{\partial[r\psi(\mathbf{r})]}{\partial r}.$$

This potential is convenient (simple expression, exactly zero range).

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Proof: checking that (1) solves the Schrödinger equation

$$0 = -\frac{\hbar^2}{2\mu}\Delta\psi + V(r)\psi.$$

- (1) satisfies $\Delta\psi = 4\pi a\,\delta(r)$,
- (1) satisfies $\partial_r(r\psi) = 1$.

Scattering amplitude in momentum space

We consider an eigenstate of energy E, and decompose its wavefunction as

$$\psi(\mathbf{r}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \tilde{\psi}(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{r}},$$

$$E\tilde{\psi}(\mathbf{k}) = \frac{\hbar^2 k^2}{m}\tilde{\psi}(\mathbf{k}) + \frac{4\pi\hbar^2 a}{m}\int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\Lambda(\mathbf{k}')\tilde{\psi}(\mathbf{k}'),$$

where the scattering amplitude involves the function $\Lambda(\mathbf{k})$, defined as

$$\delta(\mathbf{r})\frac{\partial}{\partial r}(r\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{r}}) = \delta(\mathbf{r})\Lambda(\mathbf{k}).$$

The function $\Lambda(k)$ is a generalized distribution with

$$\Lambda(k) = 1 \quad \text{for} \quad k < \infty,$$

$$\int \mathrm{d}^3 k \frac{\Lambda(\mathbf{k})}{k^2} = 0.$$

(last equation: results from the action on the wavefunction $\psi(\mathbf{r}) = 1/r$)

Bound state of the pseudo-potential

Schrödinger equation for a (molecular) bound state

$$E_{\mathrm{m}}\tilde{\psi}_{\mathrm{m}}(\mathbf{k}) = \frac{\hbar^2 k^2}{m}\tilde{\psi}_{\mathrm{m}}(\mathbf{k}) + \underbrace{\frac{4\pi\hbar^2 a}{m}\int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\Lambda(\mathbf{k}')\tilde{\psi}_{\mathrm{m}}(\mathbf{k}')}_{A},$$

giving the amplitude

$$\tilde{\psi}_{\mathrm{m}}(\mathbf{k}) = -\frac{A}{\frac{\hbar^2 k^2}{m} - E_{\mathrm{m}}}.$$

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giving the amplitude

$$\tilde{\psi}_{\mathrm{m}}(\mathbf{k}) = -\frac{A}{\frac{\hbar^2 k^2}{m} - E_{\mathrm{m}}}.$$

We then use the identity

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\Lambda(\mathbf{k})}{k^2+\alpha} = -\frac{\sqrt{\alpha}}{4\pi}$$

to obtain the relation

$$A = Aa\sqrt{\frac{-mE_{\mathrm{m}}}{\hbar^2}}.$$

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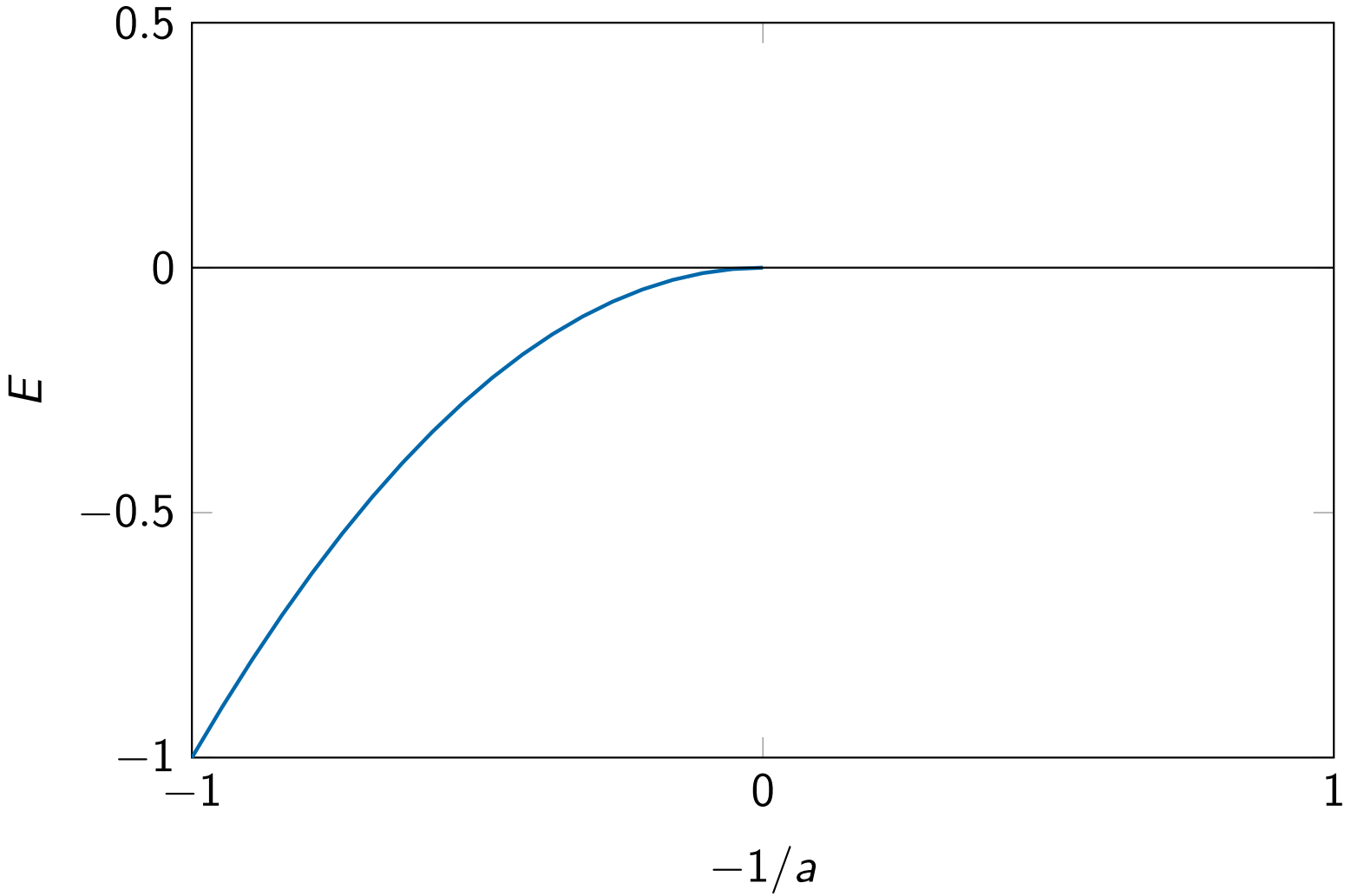
to obtain the relation

$$A = Aa\sqrt{\frac{-mE_{\mathrm{m}}}{\hbar^2}}.$$

We obtain the final result

$$E_{\mathrm{m}} = -\frac{\hbar^2}{ma^2}, \quad \tilde{\psi}_{\mathrm{m}}(\mathbf{k}) \propto \frac{1}{1+(ka)^2}, \quad \psi_{\mathrm{m}}(\mathbf{r}) = \frac{1}{\sqrt{2\pi a}}\frac{\mathrm{e}^{-r/a}}{r}.$$

Binding energy versus scattering length

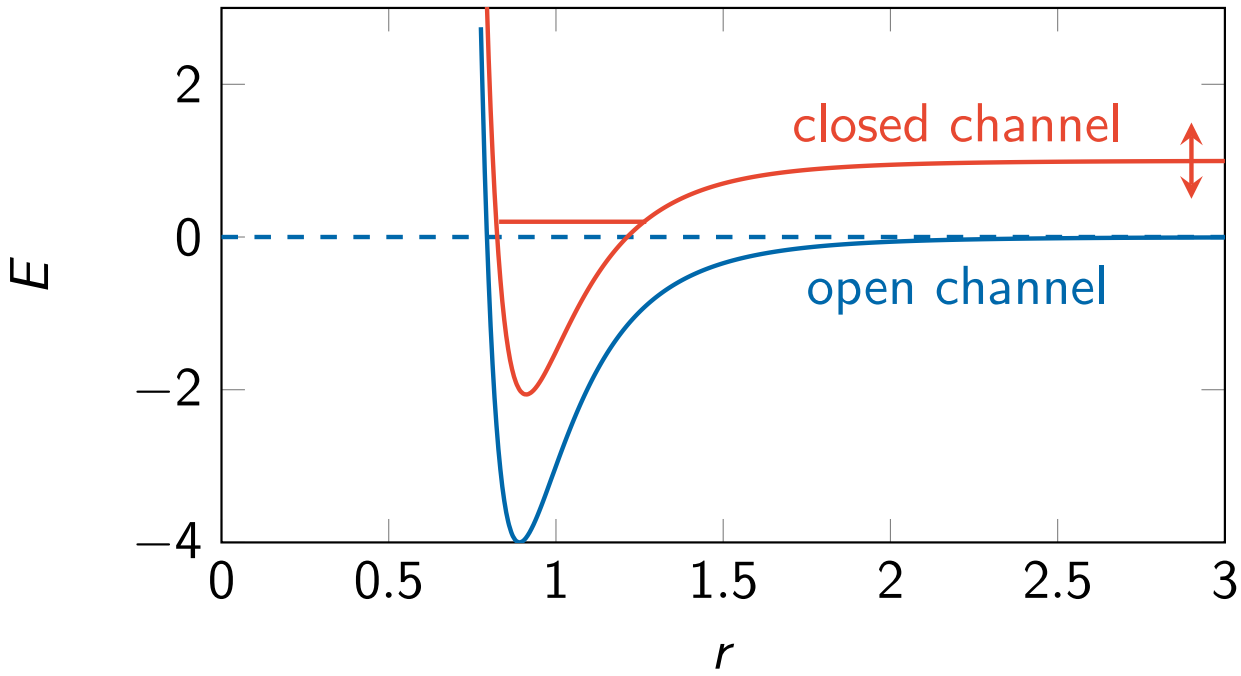


Outline

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Feshbach resonance

The scattering length can be varied by tuning the scattering state close to a bound state with different internal structure.



The energy difference between closed and open channels is tuned using a magnetic field.

Virtual population in the closed channel modifies the scattering phase shifts.

Feshbach resonance model

A simplified model:

- no interactions in the open channel $|\mathrm{o}\rangle$: $H_\mathrm{o} = -(\hbar^2/2\mu)\Delta$,
- a molecular state of wavefunction $\phi_\mathrm{m}(\mathbf{r})$ in the closed channel $|\mathrm{c}\rangle$: $H_\mathrm{c} = \hbar\delta\,|\phi_\mathrm{m}\rangle\langle\phi_\mathrm{m}|$,
- a position-independent coupling $\hbar\Omega$.

We look for a scattering state

$$|\psi\rangle = [\mathrm{e}^{\mathrm{i}kz} + \psi_\mathrm{sc}(\mathbf{r})]\,|\mathrm{o}\rangle + \alpha\phi_\mathrm{m}(\mathbf{r})\,|\mathrm{c}\rangle\,.$$

This problem can be solved exactly, with

$$\alpha = \frac{\Omega\tilde{\phi}_\mathrm{m}(k=0)}{\delta_0 - \delta}, \quad \delta_0 = \frac{m\Omega^2}{2\pi^2\hbar}\int_0^\infty \mathrm{d}k[\tilde{\phi}_\mathrm{m}(k)]^2.$$

The closed channel is populated resonantly, with a resonance pole at $\delta = \delta_0$.

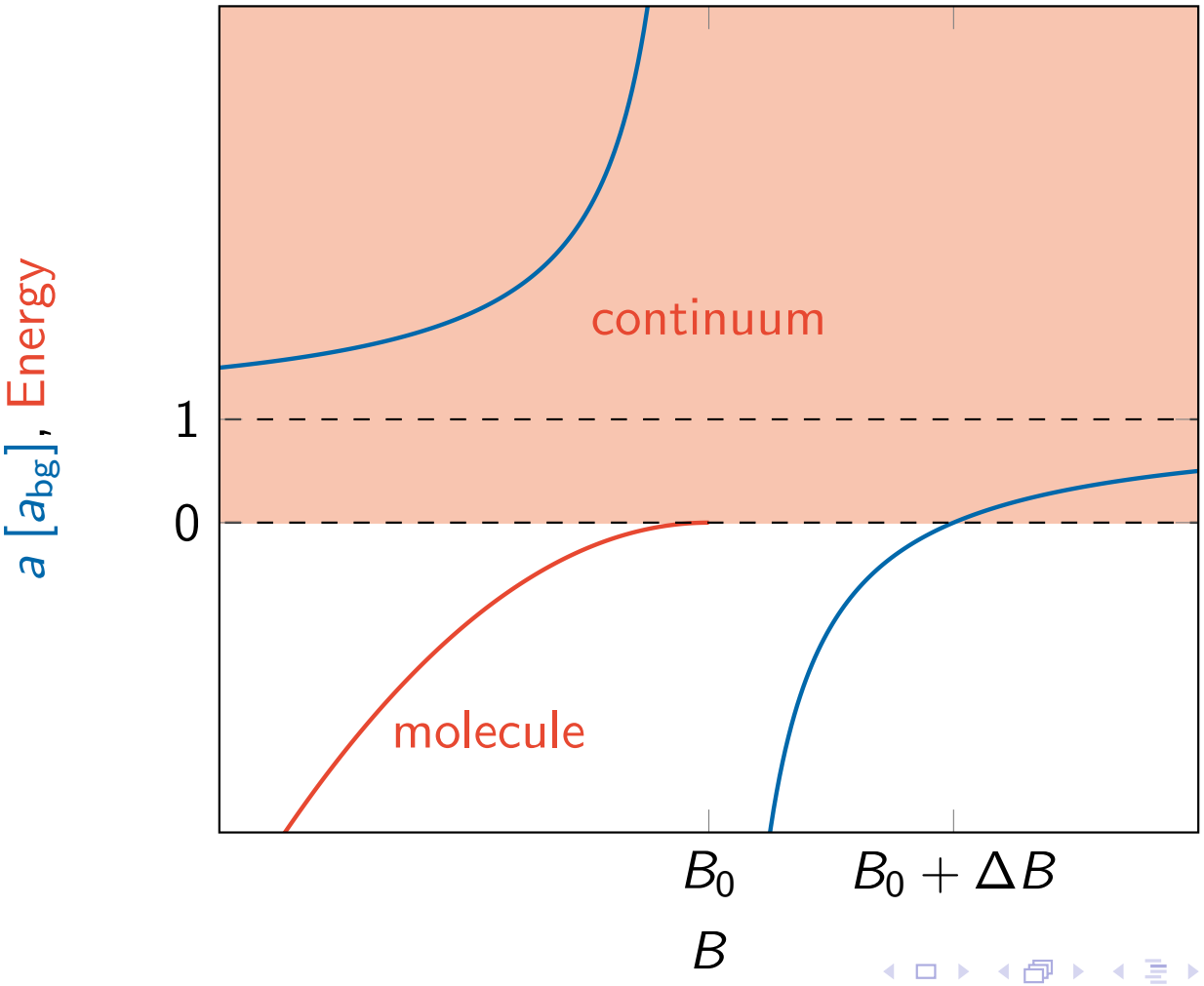
The scattered wave allows to compute the scattering length

$$a = -\frac{m\Omega^2\tilde{\phi}_\mathrm{m}^2(k=0)}{4\pi\hbar(\delta - \delta_0)}.$$

Feshbach resonance

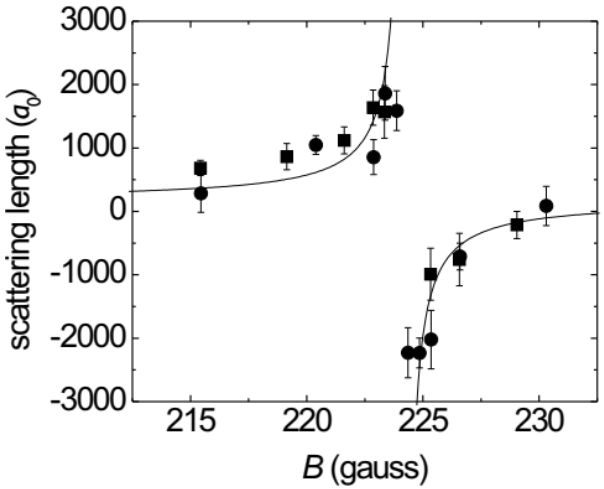
A Feshbach resonance corresponds to a resonant variation

$$a = a_{\mathrm{bg}} \left(1 - \frac{\Delta B}{B - B_0}\right).$$

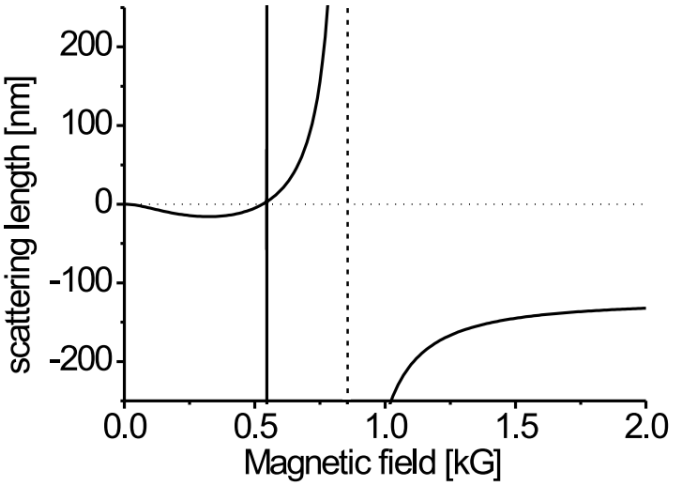


Examples of Feshbach resonances

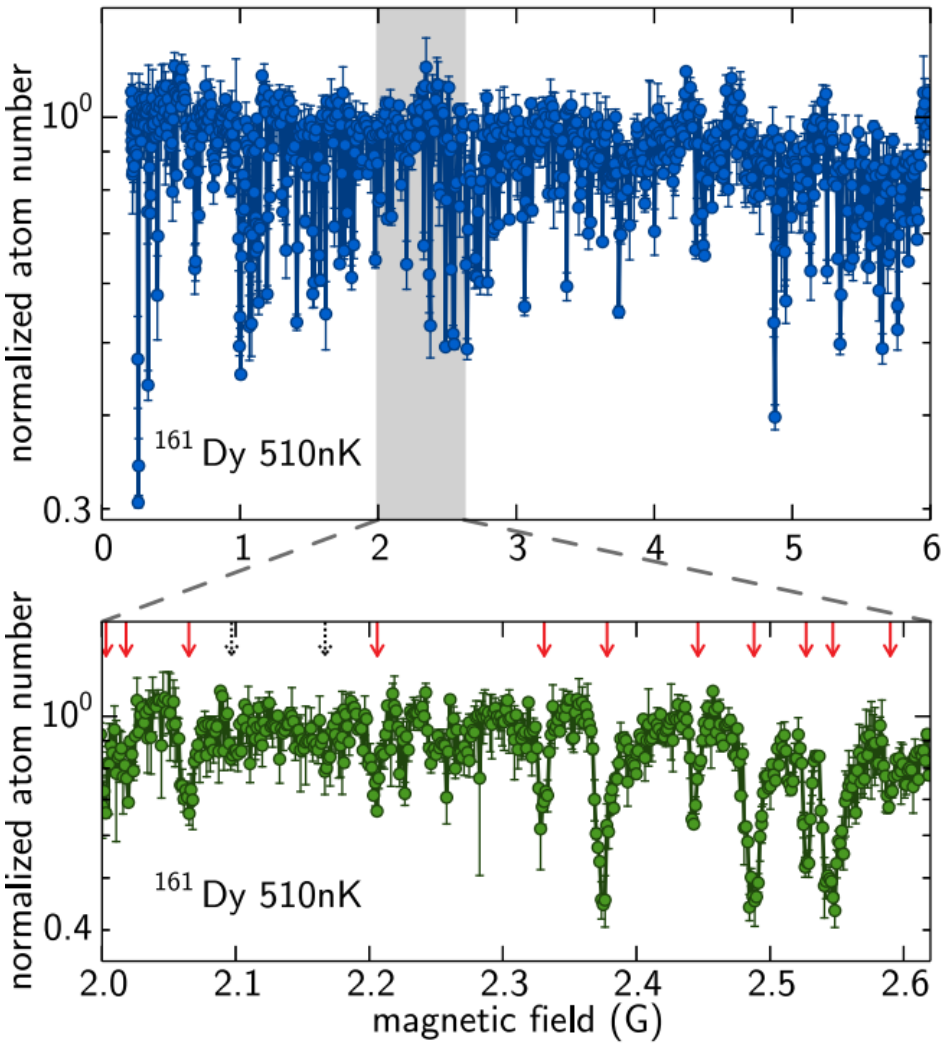
^{40}K



^{6}Li



^{161}Dy



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Towards a strongly interacting regime

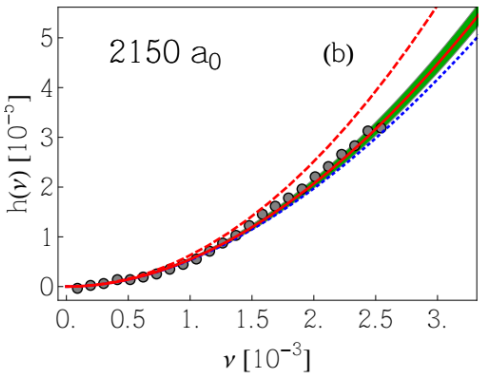
In most cases a BEC is well described within mean field (classical field described by the Gross-Pitaevskii equation)

First quantum corrections

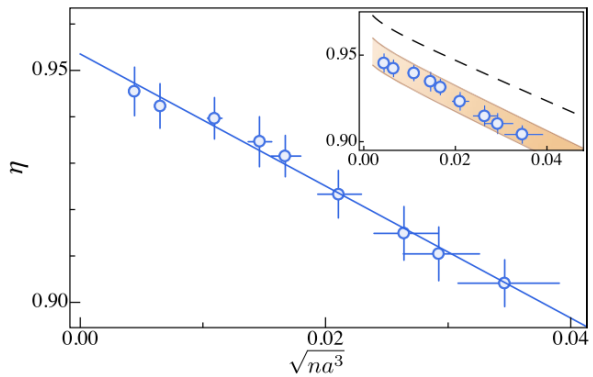
- Lee-Huang-Yang energy correction

$$\frac{E}{V} = \frac{1}{2}\frac{4\pi\hbar^2 a}{m} n^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{na^3}\right).$$

- Quantum depletion $$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}}\sqrt{na^3}.$$



Navon et al, PRL 2011

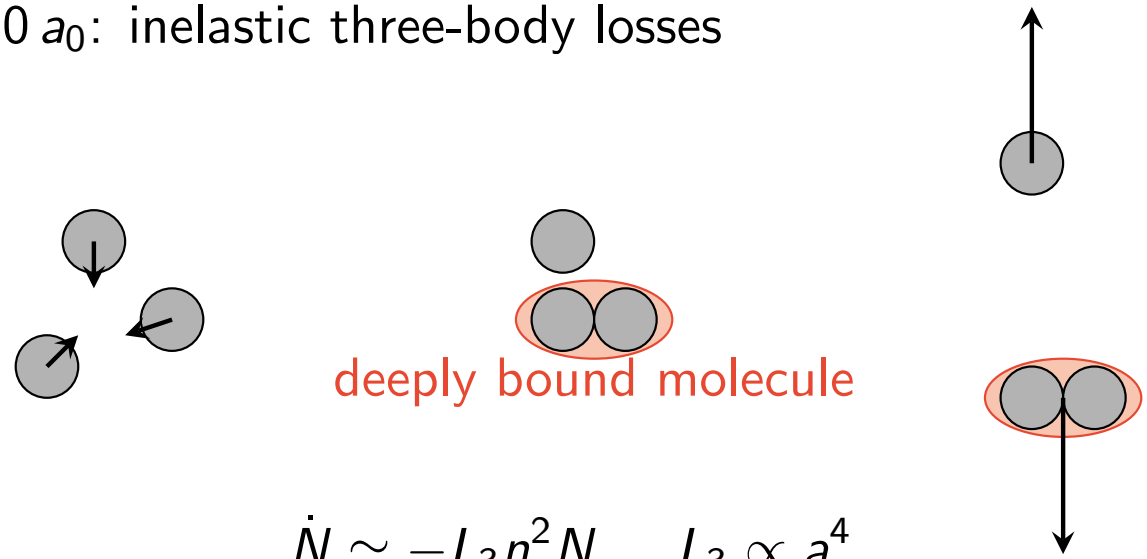


Lopes et al, PRL 2017

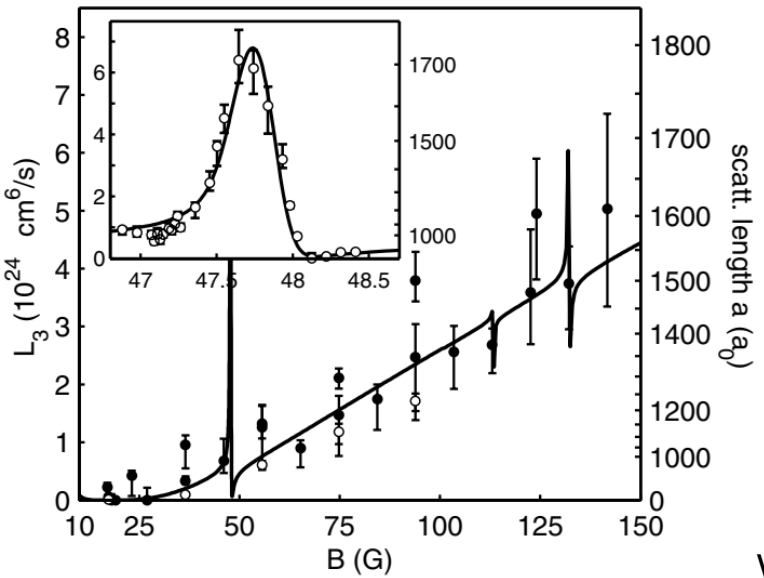
The strongly interacting regime corresponds to $na^3 \sim 1$.
Standard regime: $na^3 \sim 10^{-5}$ ($n = 1 \times 10^{14}\,\mathrm{cm}^{-3}$, $a = 100 a_0$)

Three-body recombination in Bose gases

Regime $a \gtrsim 1000\, a_0$: inelastic three-body losses



$$\dot{N} \simeq -L_3 n^2 N, \quad L_3 \propto a^4.$$



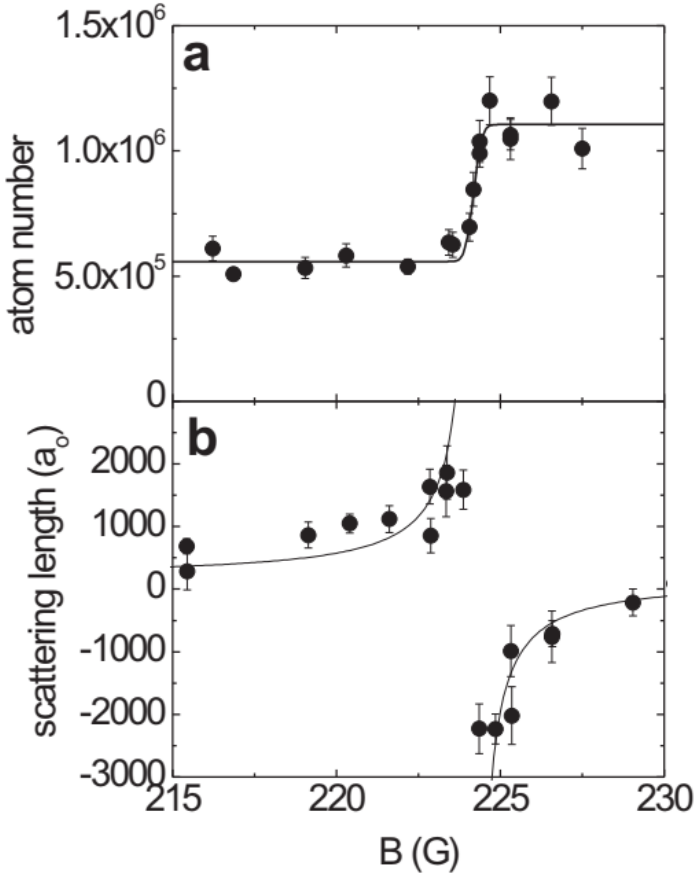
Weber et al, PRL 2003

Three-body recombination in Fermi gases

- Reminder: 2-body collisions between identical fermions are inhibited.
- For a mixture of 2 spins $\uparrow$ and $\downarrow$: inhibition of dimer relaxation with

$$\alpha_{dd} \propto (R_{vdW}/a)^{2.55}, \quad \alpha_{ad} \propto (R_{vdW}/a)^{3.33} \quad \text{(Petrov et al, PRL 2004)}.$$

The gas is stable in the strongly-interacting regime!



Regal et al, Nature 2003

Outline

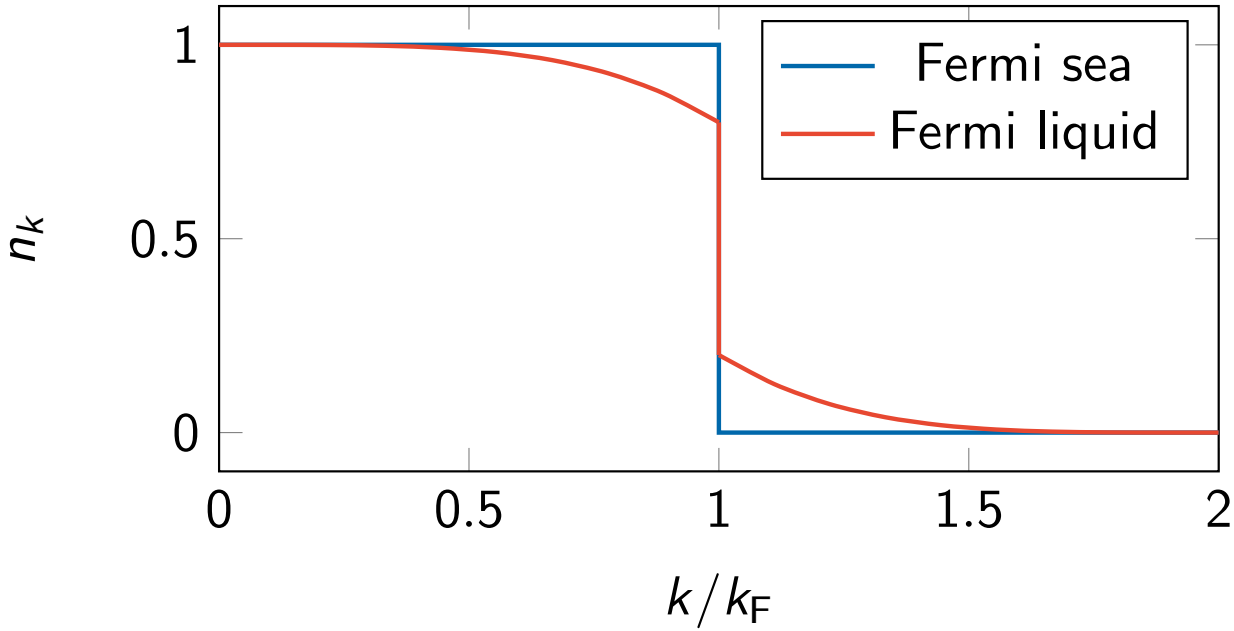
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The attractive Fermi gas: possible scenarios

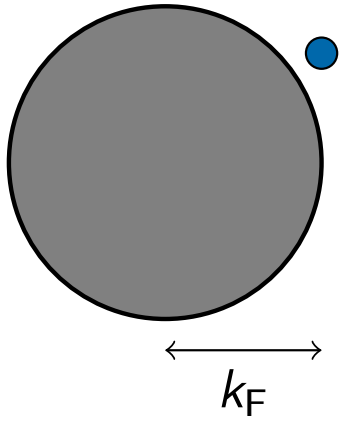
- absence of interactions: Fermi sea at $T = 0$.
- scenario 1: interactions $\rightarrow$ Fermi liquid



- scenario 2: interactions $\rightarrow$ Cooper pairing and superfluidity always wins at $T = 0$ but $T_c \sim T_F e^{-\pi/(2k_F|a|)}$

The Cooper instability

Cooper problem: two-body problem with Pauli blocking for $k < k_F$.



We look for a bound state of energy $E = 2E_F + \epsilon$, with $\epsilon < 0$.
We write the bound state as before

$$(2E_F + \epsilon)\tilde{\psi}(\mathbf{k}) = \frac{\hbar^2 k^2}{m}\tilde{\psi}(\mathbf{k}) + \underbrace{\frac{4\pi\hbar^2 a}{m}\int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\Lambda(\mathbf{k}')\tilde{\psi}(\mathbf{k}')}_{A},$$

albeit with the restriction $k, k' \geq k_F$.

Binding energy for the Cooper problem

We write the wavefunction as

$$\tilde{\psi}(\mathbf{k}) = -\frac{mA}{\hbar^2}\frac{1}{k^2 - k_F^2 - m\epsilon/\hbar^2}.$$

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We integrate over $k > k_F$ to obtain the relation

$$A = -4\pi A a \int_{k_F}^{\infty} \frac{d^3k}{(2\pi)^3}\frac{\Lambda(\mathbf{k})}{k^2 - k_F^2 - m\epsilon/\hbar^2}.$$

The divergence at ∞ can be removed by adding the term

$$0 = \int_0^{\infty} \frac{d^3k}{(2\pi)^3}\frac{\Lambda(\mathbf{k})}{k^2}.$$

We then write

$$\frac{1}{4\pi a} = \underbrace{\int_0^{k_F} \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}}_{k_F/(2\pi^2)} + \underbrace{\int_{k_F}^{\infty} \frac{d^3k}{(2\pi)^3}\left(\frac{1}{k^2} - \frac{1}{k^2 - k_F^2 - m\epsilon/\hbar^2}\right)}_{k_F \log(|\epsilon|/8E_F)/(4\pi^2)},$$

leading to the binding energy

$$\epsilon = -\frac{8}{e^2}E_F e^{\pi/(k_F a)}.$$

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BCS variational ansatz

- Many-body Hamiltonian

$$H = \sum_{\sigma=\uparrow,\downarrow,\mathbf{k}} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + g \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q},\uparrow} c^\dagger_{\mathbf{k}'-\mathbf{q},\downarrow} c_{\mathbf{k}',\downarrow} c_{\mathbf{k},\uparrow}$$

- Variational BCS ansatz: 'Glauber coherent state' of pairs

$$|\psi_{\mathrm{BCS}}\rangle \propto \exp(\gamma C^\dagger)\,|0\rangle\,,$$
$$\gamma C^\dagger = \sum_{\mathbf{k}} \Gamma_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow}.$$

  The operator $C^\dagger$ creates a pair, but it does not correspond to a bosonic operator (except in the BEC regime).

Equivalent writing as

$$|\psi_{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} - v_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow})\,|0\rangle\,,$$
$$u_{\mathbf{k}} = \frac{1}{\sqrt{1+\Gamma^2_{\mathbf{k}}}}, \quad v_{\mathbf{k}} = \frac{-\Gamma_{\mathbf{k}}}{\sqrt{1+\Gamma_{\mathbf{k}^2}}}.$$

BCS Hamiltonian

One calculates the expectation values

$$\langle c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}\rangle = v^2_{\mathbf{k}},$$
$$\langle c_{\mathbf{k},\uparrow} c_{-\mathbf{k},\downarrow}\rangle = -u_{\mathbf{k}} v_{\mathbf{k}}.$$

The quartic interaction terms can be replaced by the quadratic expression (Wick contraction theorem, exact for all BCS ansatz)

$$\begin{aligned} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} =& \langle c^\dagger_{\mathbf{k},\uparrow} c_{\mathbf{k},\uparrow}\rangle c^\dagger_{-\mathbf{k},\downarrow} c_{-\mathbf{k},\downarrow} + \langle c^\dagger_{\mathbf{k},\downarrow} c_{\mathbf{k},\downarrow}\rangle c^\dagger_{-\mathbf{k},\uparrow} c_{-\mathbf{k},\uparrow} \\ & \text{(Hartree mean field)} \\ & + \langle c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow}\rangle c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} + \langle c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow}\rangle c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} \\ & \text{(pairing)} \end{aligned}$$

BCS Hamiltonian

$$H = \sum_{\sigma=\uparrow,\downarrow,\mathbf{k}} \left(\frac{\hbar^2 k^2}{2m} + g n_{-\sigma} - \mu \right) c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + \Delta (c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} + \text{h.c.})$$

where one introduces the superfluid gap $\Delta = g \sum_k \langle c_{\mathbf{k},\uparrow} c_{-\mathbf{k},\downarrow}\rangle = -g \sum_k u_{\mathbf{k}} v_{\mathbf{k}}$

Ground state of the BCS Hamiltonian

BCS Hamiltonian is quadratic and can be written as ('Nambu spinor' notation)

$$H = \frac{1}{2}\sum_{k,\sigma}\begin{pmatrix} c^\dagger_{\mathbf{k},\uparrow} & c_{-\mathbf{k},\downarrow}\end{pmatrix}\begin{pmatrix} \frac{\hbar^2 k^2}{2m} + gn_{-\sigma} - \mu & \sigma\Delta \\ \sigma\Delta^* & -(\frac{\hbar^2 k^2}{2m} + gn_\sigma - \mu)\end{pmatrix}\begin{pmatrix} c_{\mathbf{k},\uparrow} \\ c^\dagger_{-\mathbf{k},\downarrow}\end{pmatrix}.$$

Each 2×2 matrix can be diagonalized (Bogoliubov transformation) as

$$H = E_0 + \sum_{\mathbf{k},\sigma}\epsilon_k b^\dagger_{\mathbf{k},\sigma} b_{\mathbf{k},\sigma},$$

with

$$\epsilon_k = \left[\left(\frac{\hbar^2 k^2}{2m} + \frac{gn}{2} - \mu\right)^2 + \Delta^2\right]^{1/2},$$

$$b_{\mathbf{k},\uparrow} = u^*_k c_{\mathbf{k},\uparrow} + v^*_k c^\dagger_{\mathbf{k},\downarrow},$$

$$b_{\mathbf{k},\downarrow} = u^*_k c_{\mathbf{k},\downarrow} - v^*_k c^\dagger_{\mathbf{k},\uparrow},$$

$$(u_k + \mathrm{i}v_k)^2 = \frac{\frac{\hbar^2 k^2}{2m} + \frac{gn}{2} - \mu + \mathrm{i}\Delta}{\epsilon_k}.$$

BCS equations

The dispersion relation ϵ_k uses the density n and gap Δ as parameters. The latter can be deduced from the u_k, v_k, leading to the self-consistency equations

$$\begin{aligned} n &= 2\sum_k v_k^2 \\ &= \int \frac{d^3k}{(2\pi)^3}\left[1 - \frac{\frac{\hbar^2k^2}{2m} + \frac{4\pi\hbar^2 a}{m}\frac{n}{2} - \mu}{\epsilon_k}\right], \\ \Delta &= -g\sum_k u_{\mathbf{k}} v_{\mathbf{k}} \\ &= \frac{4\pi\hbar^2 a}{m}\Delta\int \frac{d^3k}{(2\pi)^3}\frac{\Lambda(k)}{2\epsilon_k}. \end{aligned}$$

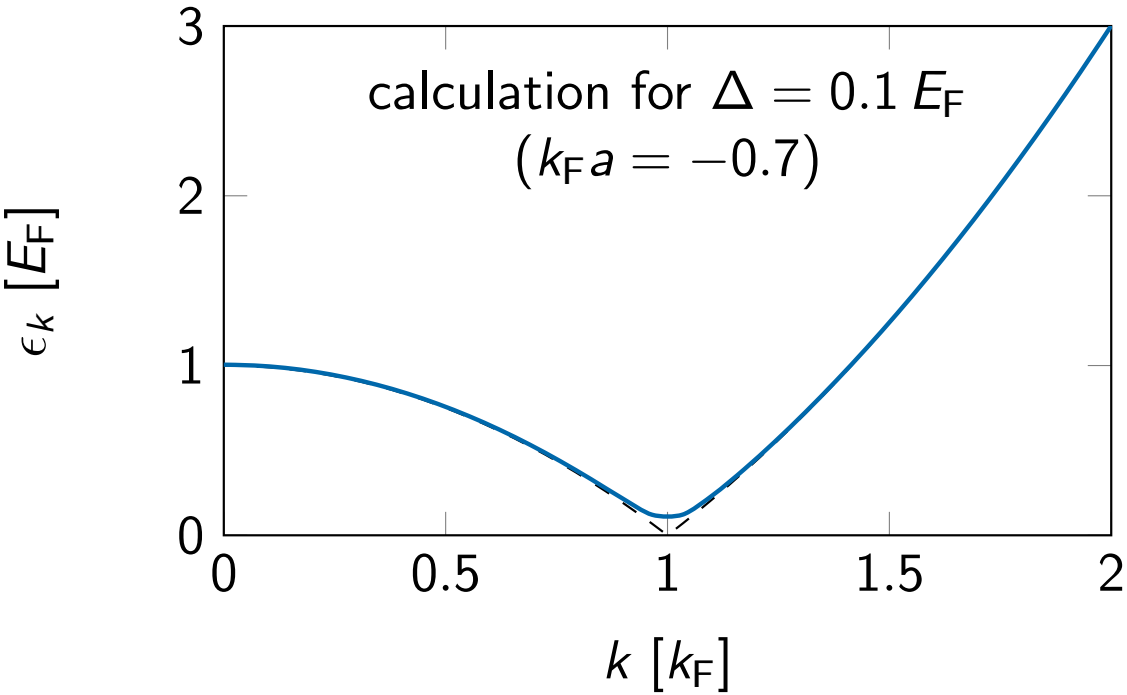
These two equations allow to determine the chemical potential μ and gap Δ for given n and a.

BCS prediction in the 'BCS limit'

'BCS limit' means $k_F a \to 0^-$. One finds

$$\mu \simeq E_F \quad (+ \text{ Hartree shift } gn/2\text{: beyond renormalized mean field}),$$

$$\Delta = \frac{8}{e^2} E_F e^{-\pi/(2k_F|a|)}.$$



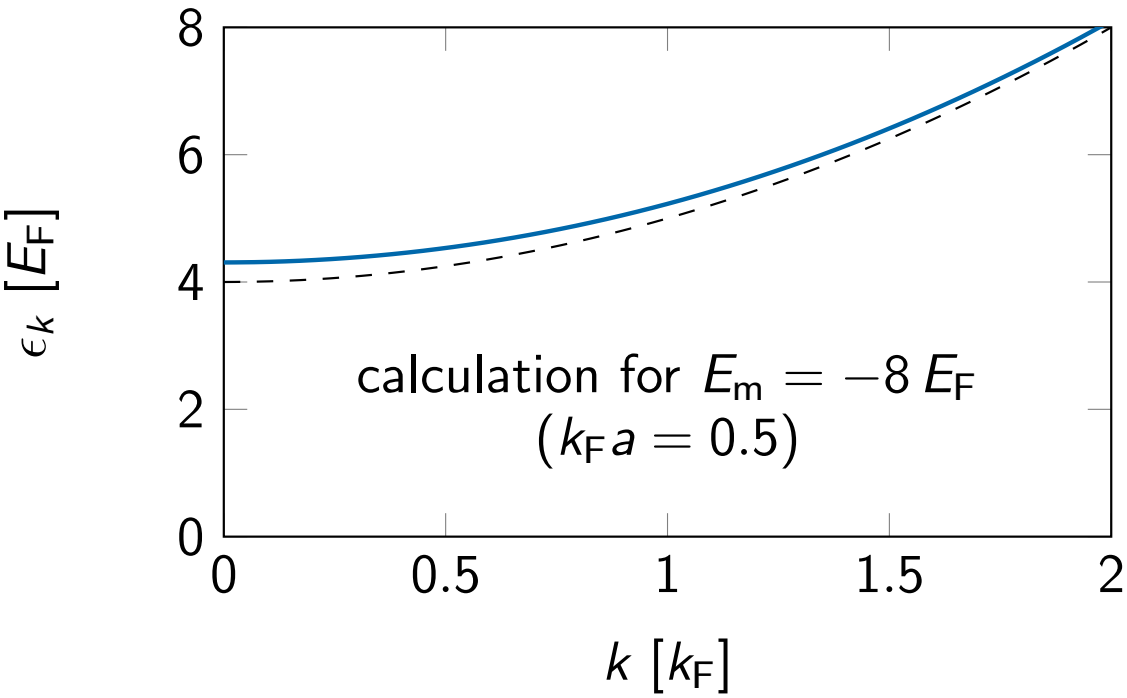
The gap Δ is the minimal quasiparticle energy: breaking a pair costs 2Δ.

BCS prediction in the 'BEC limit'

'BEC limit' means $k_F a \to 0^+$. One finds

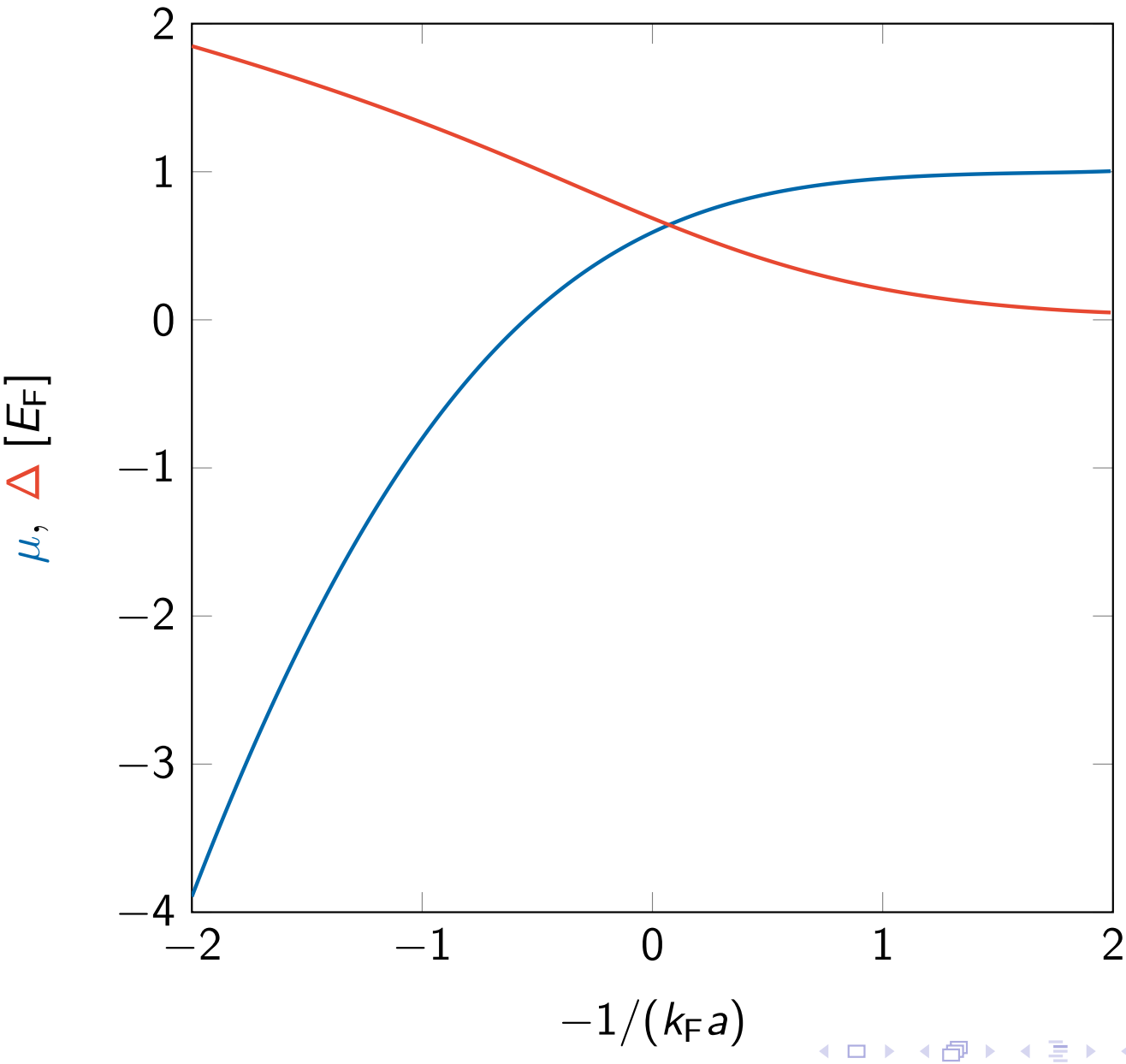
$$\mu = -\frac{\hbar^2}{2ma^2} = E_m/2,$$

$$\Delta = \frac{2}{\sqrt{3\pi}}(k_F a)^{3/2}\frac{\hbar^2}{ma^2}.$$



The gap Δ is **not** the energy required to break a pair.

BCS prediction for all interactions



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Finite temperature BCS theory

Standard thermodynamics applied to Bogoliubov excitations

$$\rightarrow \Delta\left(\frac{T}{T_F}, \frac{1}{k_F a}\right).$$

The gap cancels for a critical temperature $T_c(1/k_F a)$.

- BCS limit

$$T_c = \frac{e^\gamma}{\pi}\Delta(T=0) \simeq 0.57\Delta(T=0).$$

- BEC limit

$$T_c^* \sim |E_m| : \text{ molecule formation temperature}$$

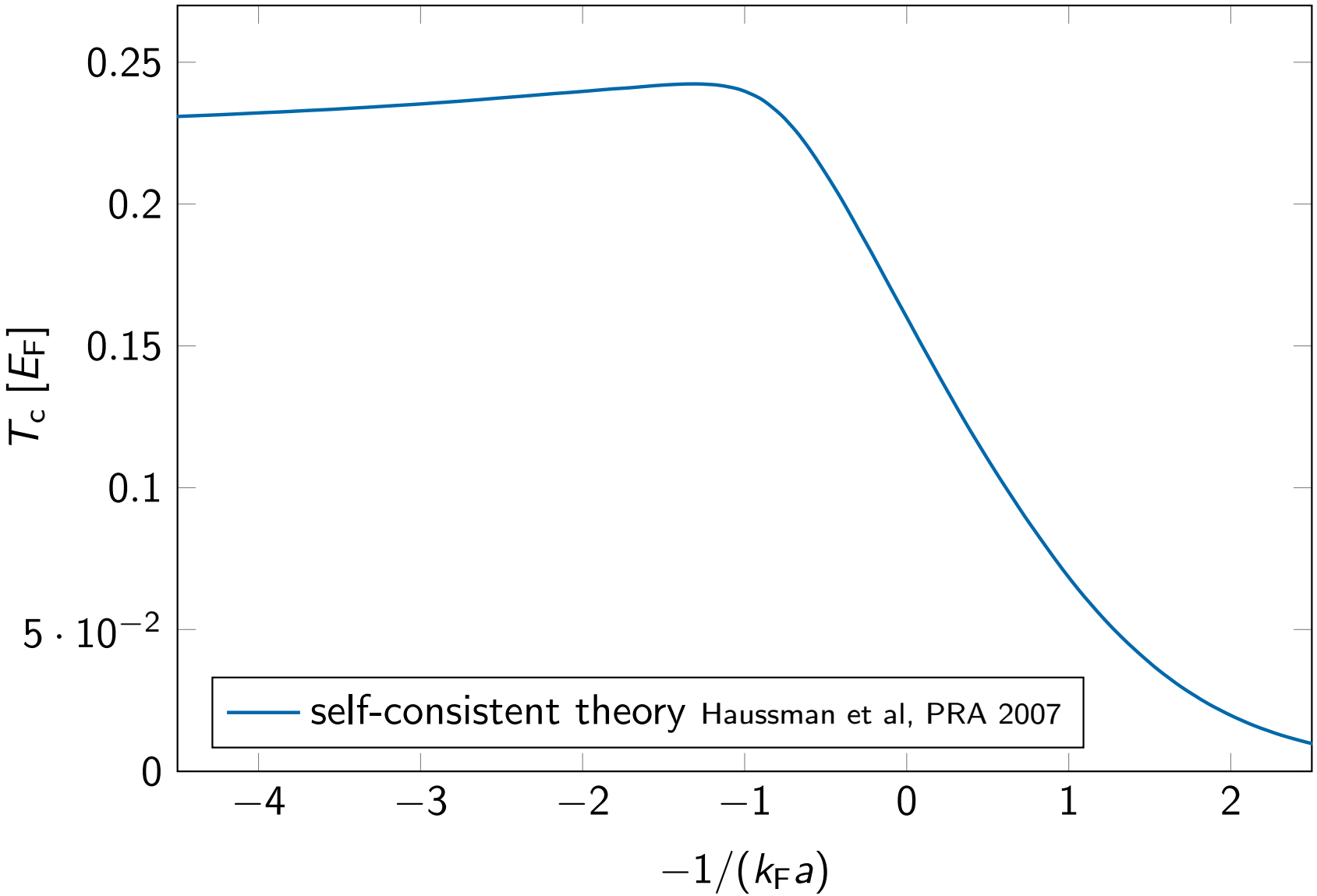
The actual superfluid transition is the molecular BEC transition, at

$$T_c = \frac{\pi\hbar^2}{m}\left(\frac{n}{2\zeta(\frac{3}{2})}\right)^{2/3} \simeq 0.22\ T_F.$$

(requires including pairs with non-zero momentum)

BCS prediction for the superfluid critical temperature

Numerical simulations are notoriously difficult and the problem remains unsolved!



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Short-range correlations and Tan's contact

In a many-body system, the wavefunction diverges when two particles approach each other

$$\psi(\mathbf{r}_i) \underset{\mathbf{r}_1 \to \mathbf{r}_2}{\sim} A\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \mathbf{r}_3, \ldots, \mathbf{r}_N\right)\left(\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{1}{a}\right).$$

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Consequence on the two-body correlation function

$$\langle n_\uparrow(\mathbf{R}-\mathbf{r}/2) n_\downarrow(\mathbf{R}+\mathbf{r}/2)\rangle = \frac{C(\mathbf{R})}{16\pi^2}\left(\frac{1}{r^2}-\frac{2}{ar}\right)+O(1).$$

The quantity $C(r)$ is the contact coefficient.

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The quantity $C(r)$ is the contact coefficient.
It is related to other physical quantities

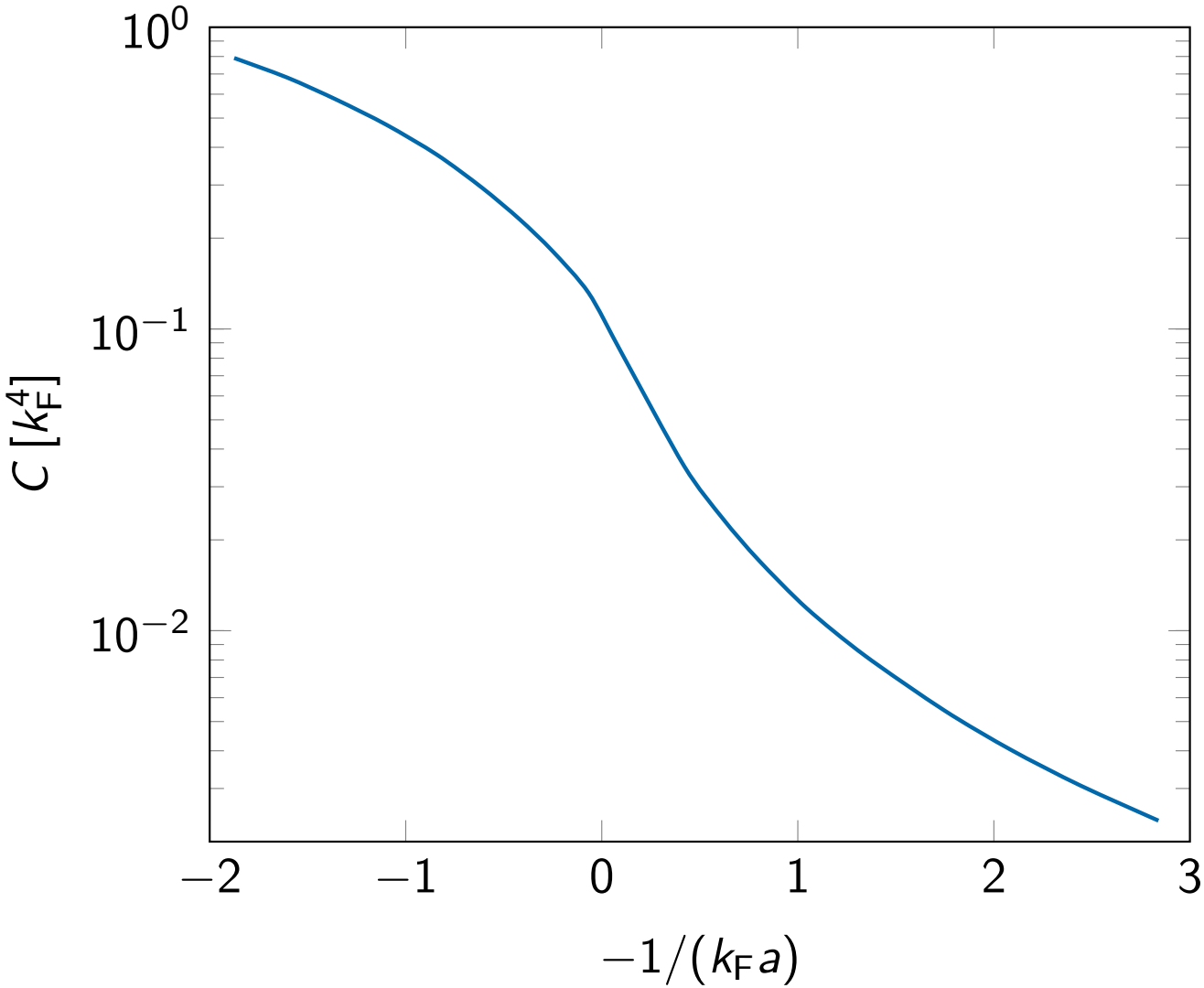
- Tail of the momentum distribution

$$C = \lim_{k\to\infty} k^4 n(k).$$

- Derivative of the energy with respect to a

$$C = -\frac{4\pi m}{\hbar^2}\frac{\partial(\langle E\rangle)}{\partial(1/a)}.$$

Contact in the BEC-BCS crossover



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Scale invariance of the unitary Fermi gas

We remind the pair scattering amplitude at unitarity ($a = \infty$)

$$f_k = -\frac{1}{\mathrm{i}k}.$$

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Spatial scaling by a factor λ leads to scalings

momenta	$\mathbf{k} \to \mathbf{k}/\lambda$
kinetic energy	$E_{\mathrm{kin}} \to E_{\mathrm{kin}}/\lambda^2$
density	$n \to n/\lambda^3$
scattering amplitudes	$f_k \to \lambda f_k$
interaction energy	$E_{\mathrm{int}} \to E_{\mathrm{int}}/\lambda^2$.

Kinetic and interaction energies have the same scalings!

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\end{aligned}$$

Kinetic and interaction energies have the same scalings!

Consequence: structure of many-body eigenstates

$$\begin{aligned}
\text{energy} &\quad E \to E/\lambda^2 \\
\text{wavefunction} &\quad \psi(\mathbf{r}_i) \to \psi(\mathbf{r}_i/\lambda)/\lambda^{3N/2}.
\end{aligned}$$

Universal thermodynamics of the unitary Fermi gas

We write the free energy as

$$F = N f(n, T).$$

Scale invariance imposes the relation

$$f(n/\lambda^3, T/\lambda^2) = f(n, T)/\lambda^2.$$

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Consequences

1. Zero temperature

$$E/N \propto n^{2/3}, \quad \text{i.e.} \quad E = \xi \frac{3}{5} N E_{\mathrm{F}}.$$

One needs to determine the Bersch parameter ξ only!

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$$f(n, T) = E_{\mathrm{F}}\, g\left(\frac{k_{\mathrm{B}} T}{E_{\mathrm{F}}}\right).$$

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3. Virial theorem

$$PV = \frac{2}{3} E$$

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Testing the scale invariance

We remind the virial theorem

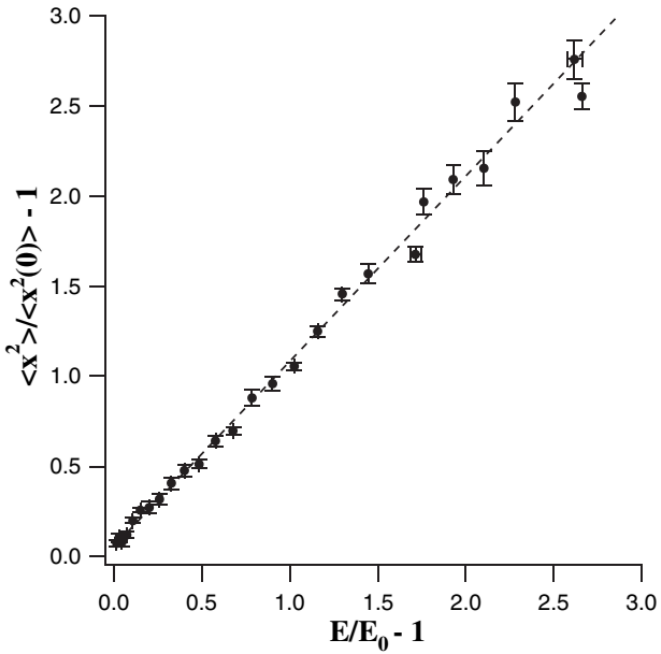
$$PV = \frac{2}{3}E. \tag{2}$$

Mechanical equilibrium for a trapped gas

$$\nabla P(\mathbf{r}) + n(\mathbf{r})\nabla U(\mathbf{r}) = 0. \tag{3}$$

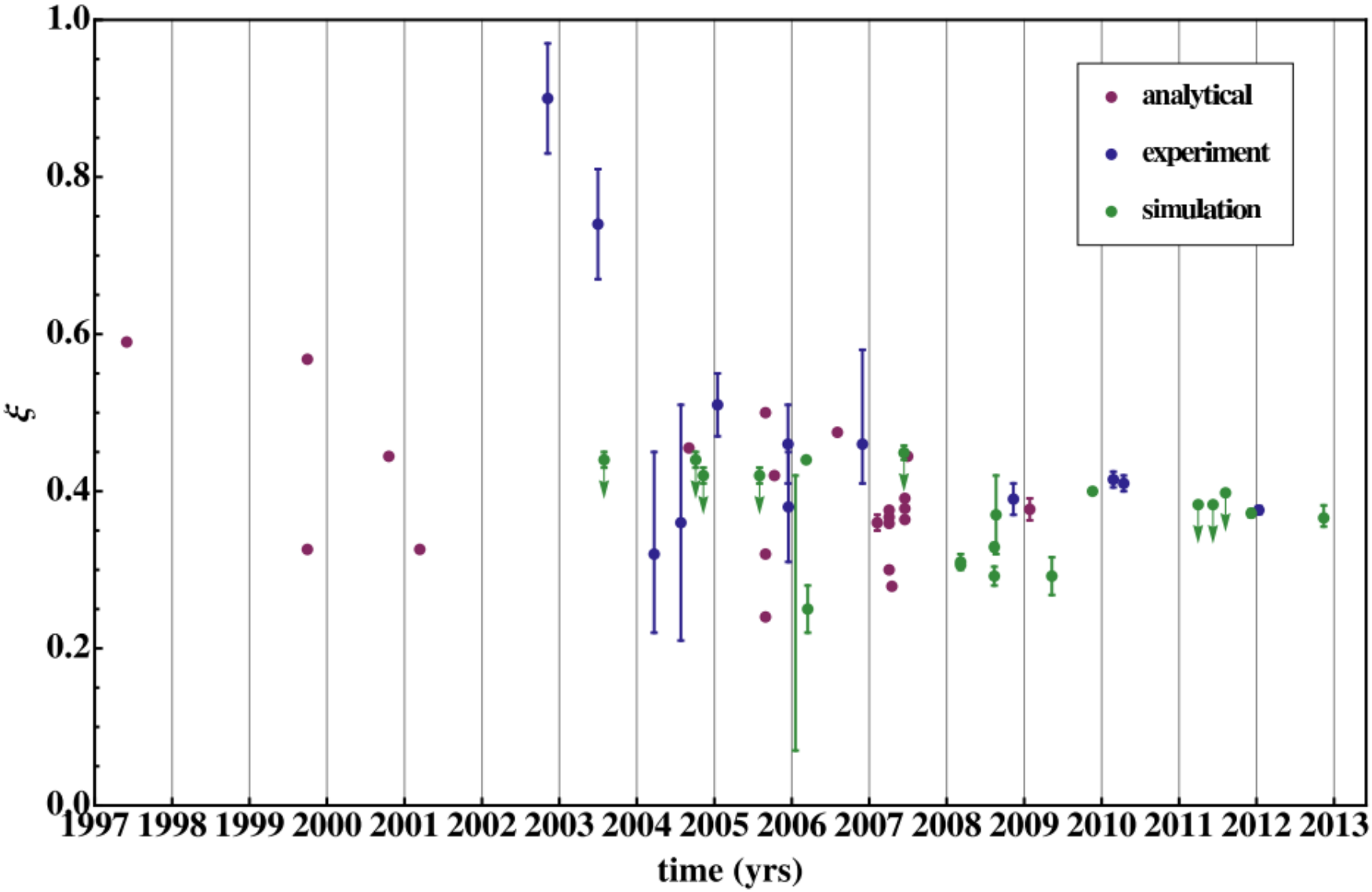
For a harmonic trap, (2) and (3) imply

$$N\langle U\rangle = \frac{E}{2}$$



Thomas et al, PRL 2005

Ground state energy for $a = \infty$: history



from Endres et al, PRA 2013

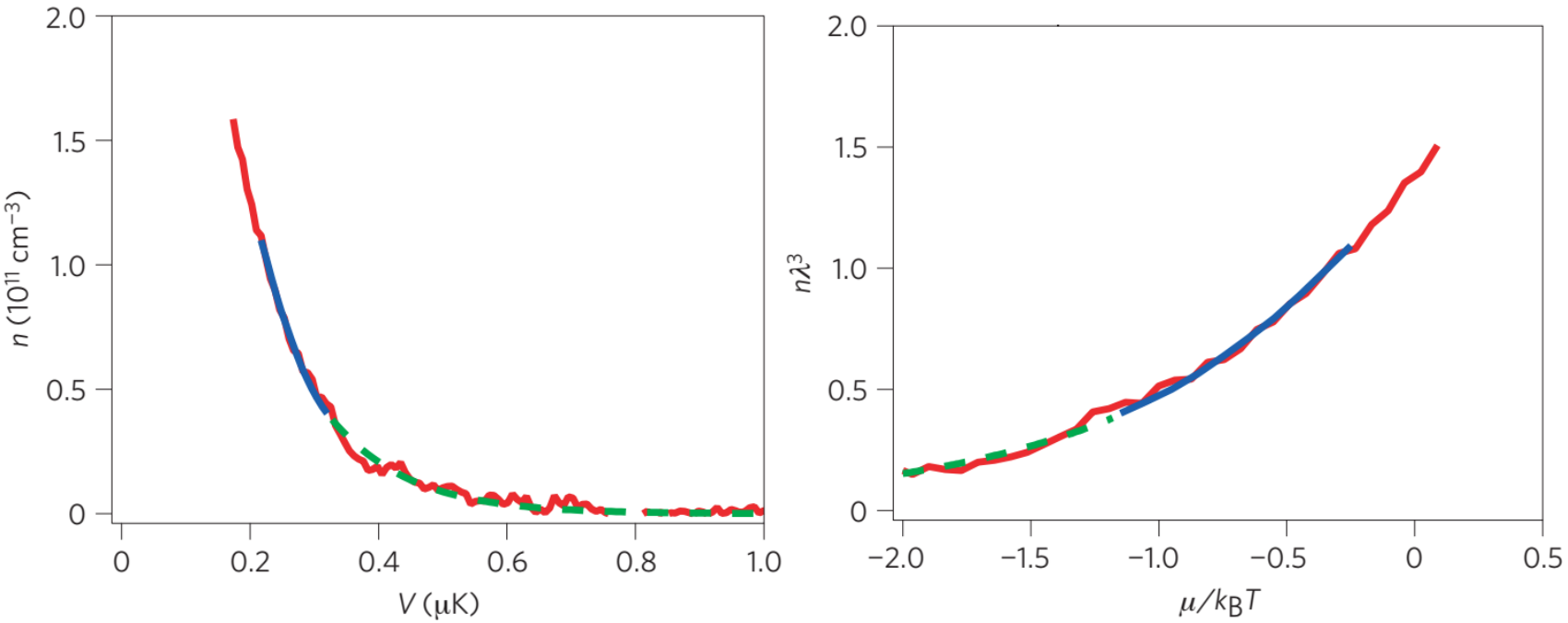
Equation of state measurement

The gas is usually held in a weak harmonic trap $V(\mathbf{r})$ $\rightarrow$ smooth density variations.
Local density approximation: the gas locally behaves like a homogenous system.
Hydrostatic pressure balance at equilibrium

$$\nabla T = 0$$

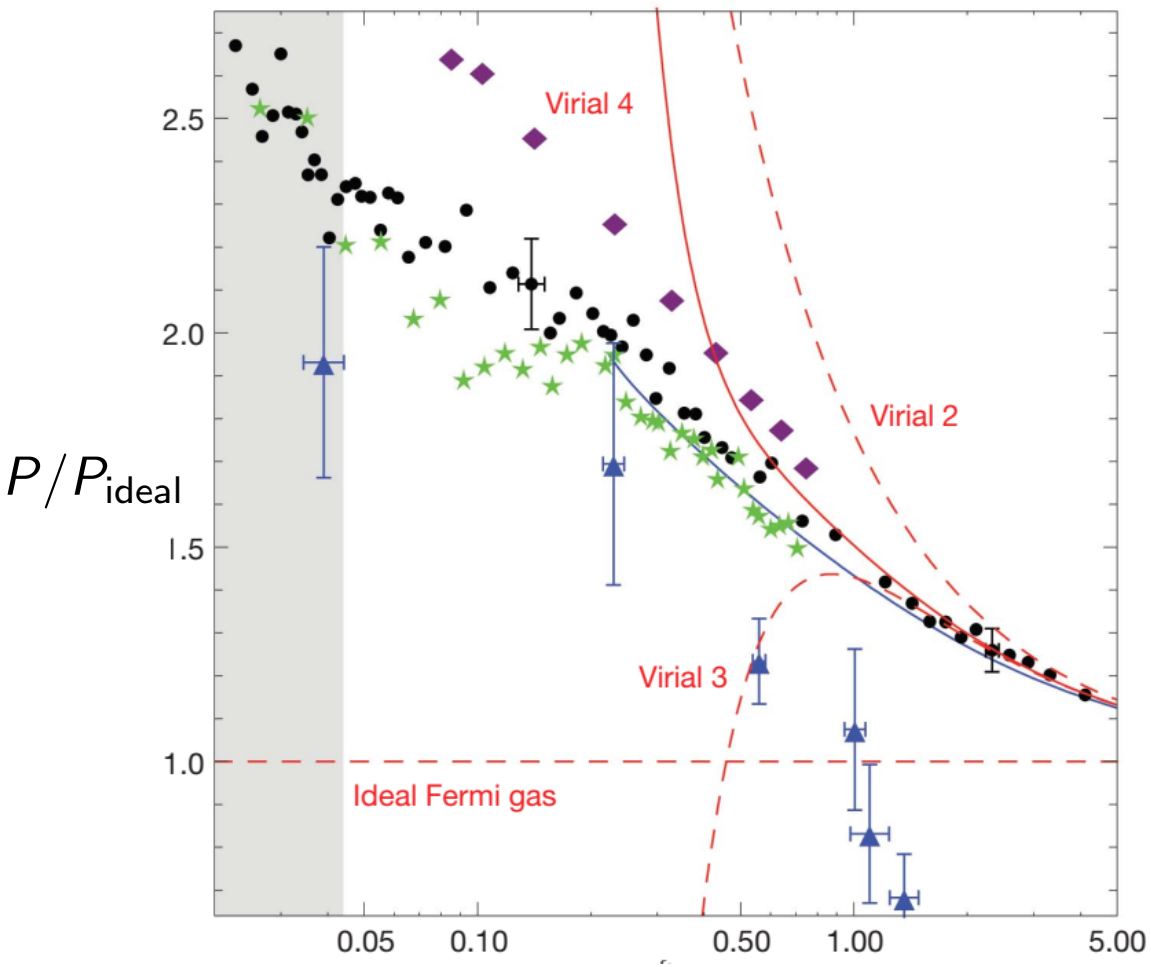
$$\nabla P = n \nabla \mu = -n \nabla V$$

$\rightarrow$ T uniform, $\mu(\mathbf{r}) = \mu_0 - V(\mathbf{r})$.



van Houcke et al, Nature Phys. 2012

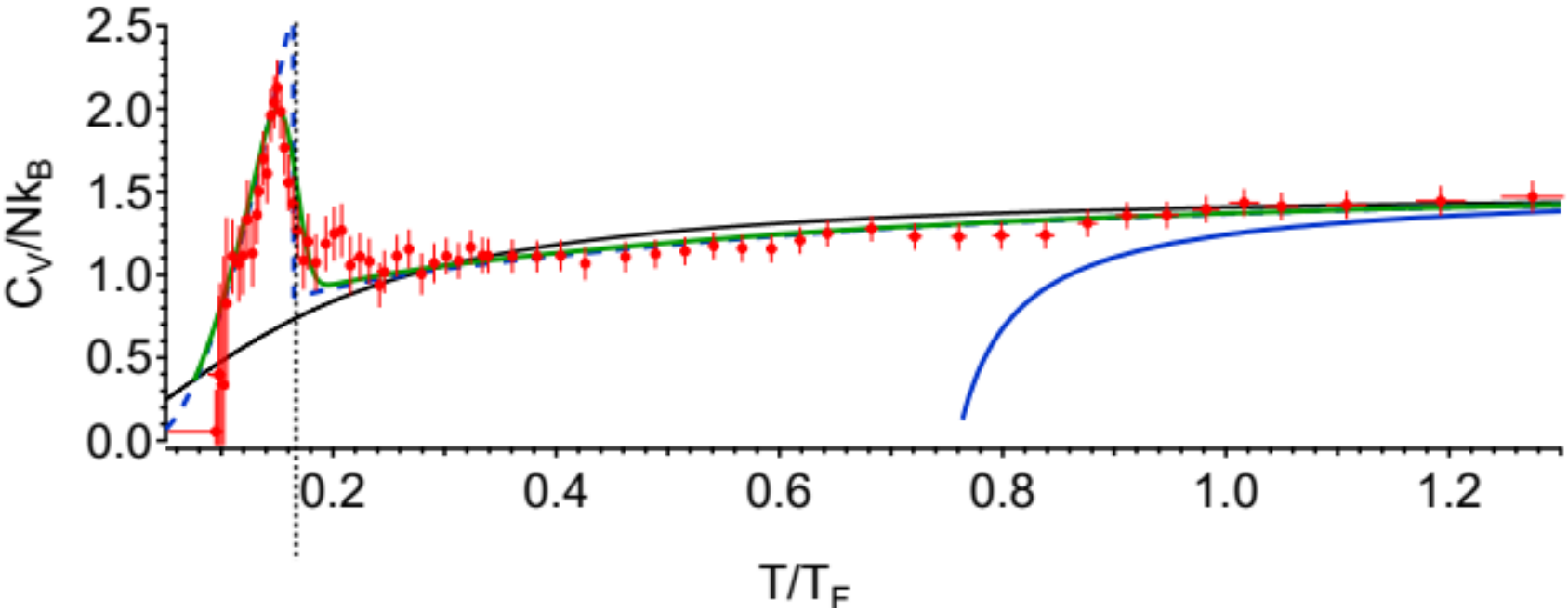
Equation of state in the normal phase



SN et al, Nature 2010

- Allows discriminating between various theories
- Determination of virial coefficients $\rightarrow$ resolution of few-body problems

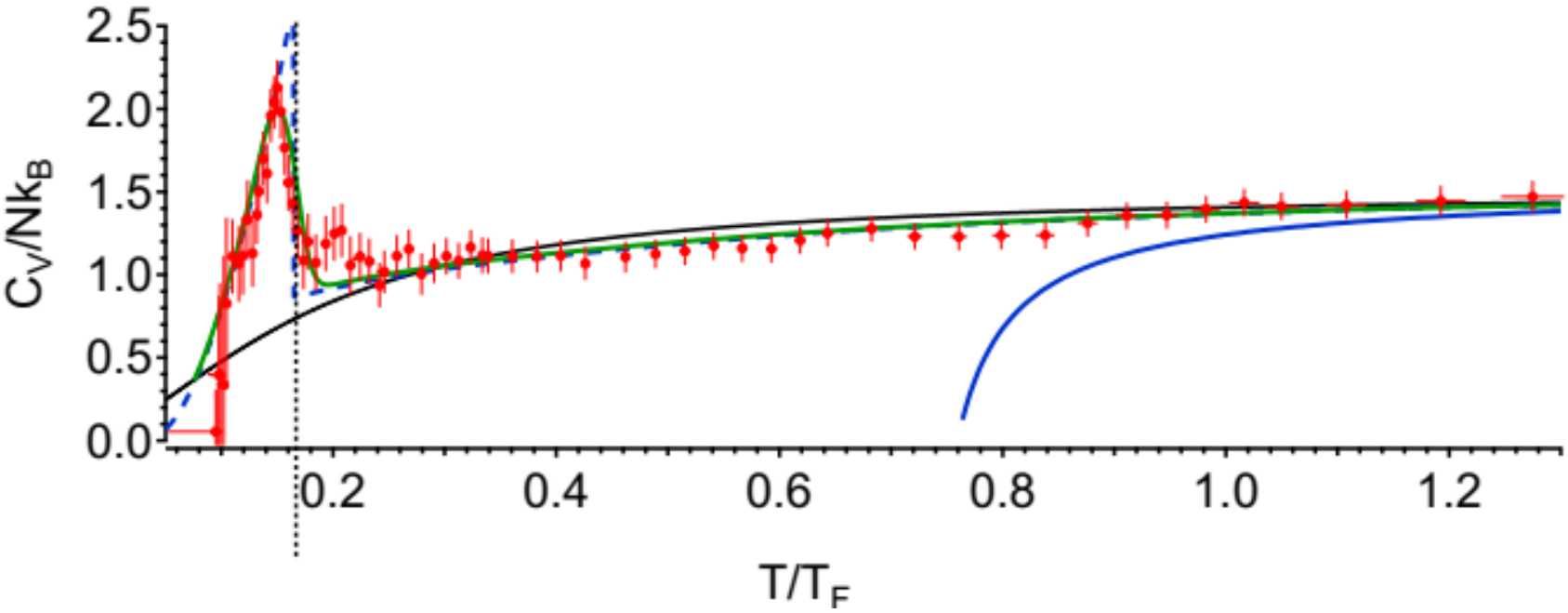
Superfluid λ transition



Ku et al, Science 2012

- Reminiscent of the superfluid transition in ^{4}He.
- So far no theory can predict this behavior!

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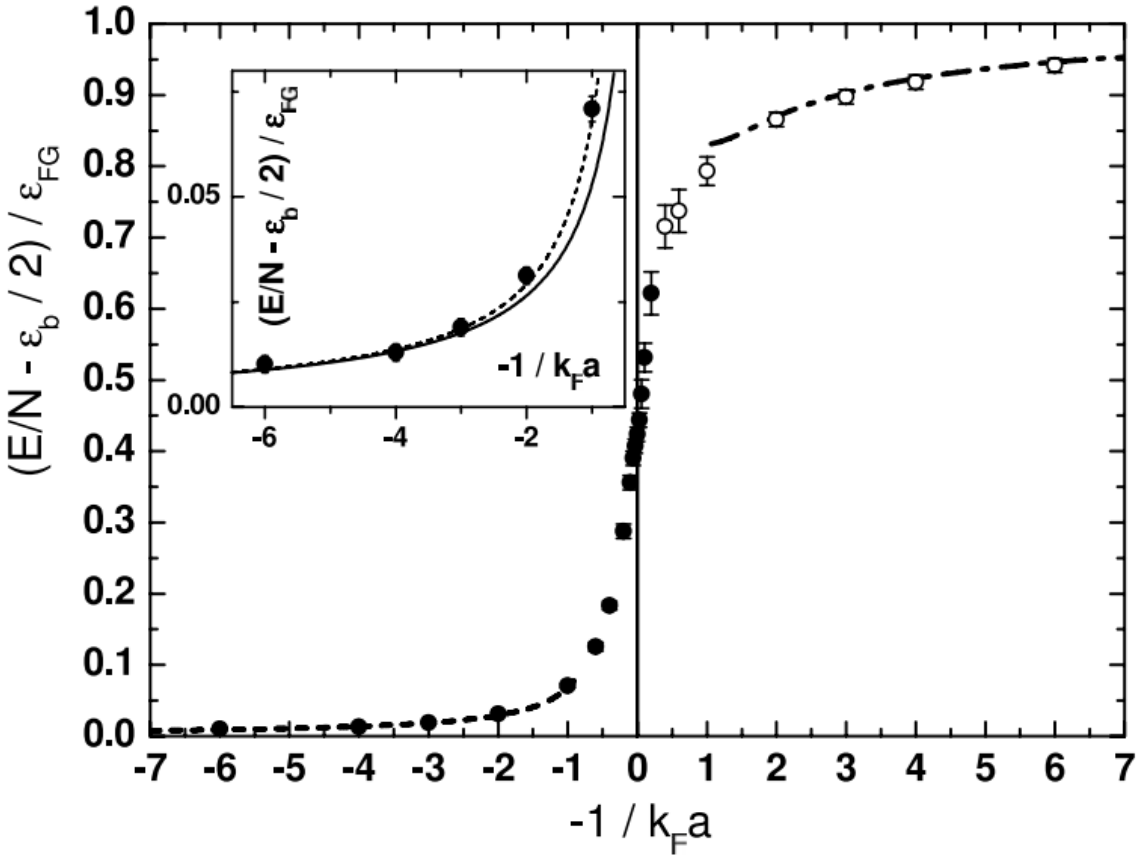
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Ground state energy in the BEC-BCS crossover

The equation of state can be written as

$$E = \frac{3}{5} N E_F (1/k_F a).$$

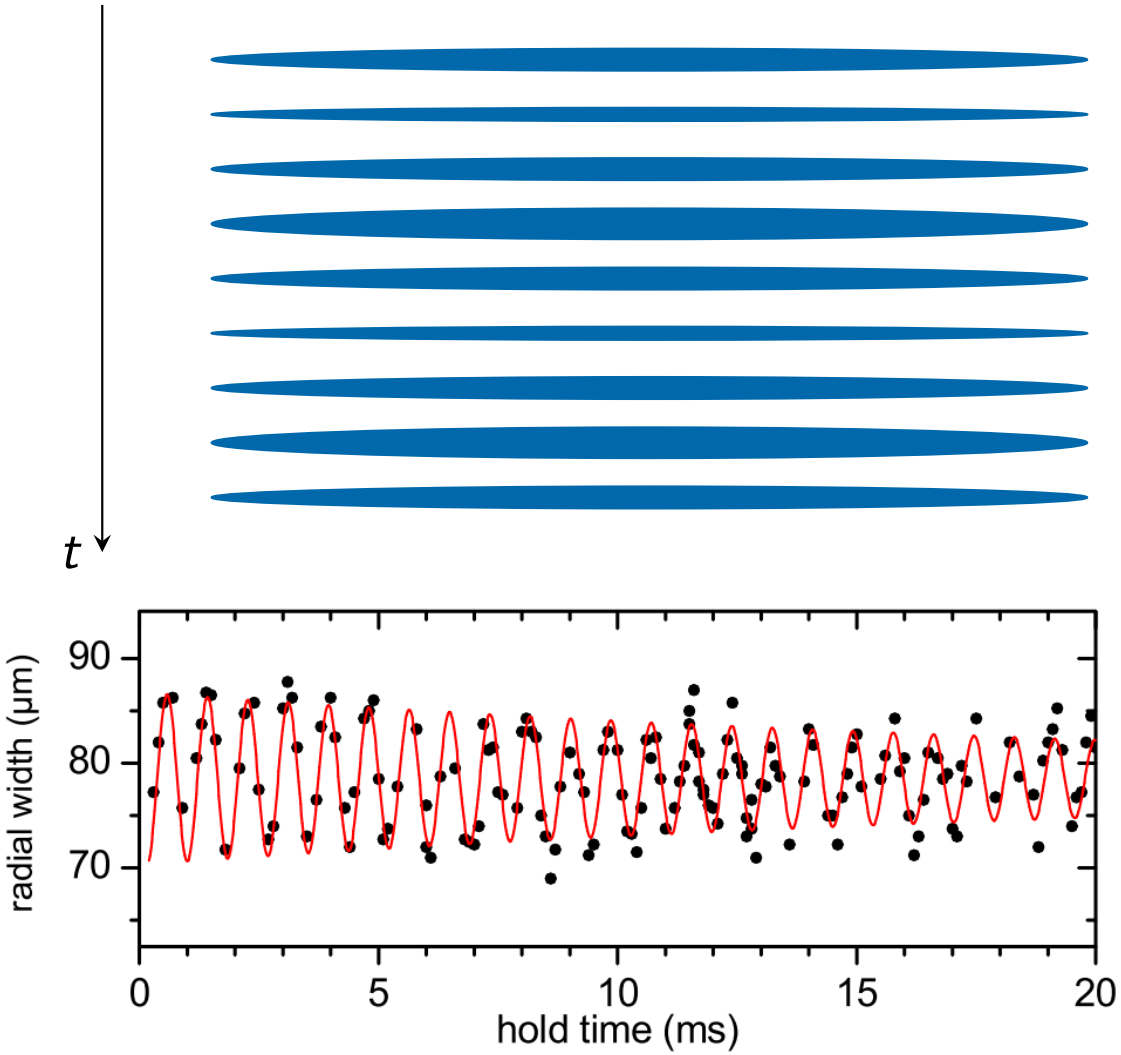


Astrakharchik et al, PRL 2004

Mean-field theory is only qualitatively correct. Variational Monte Carlo quite accurate.

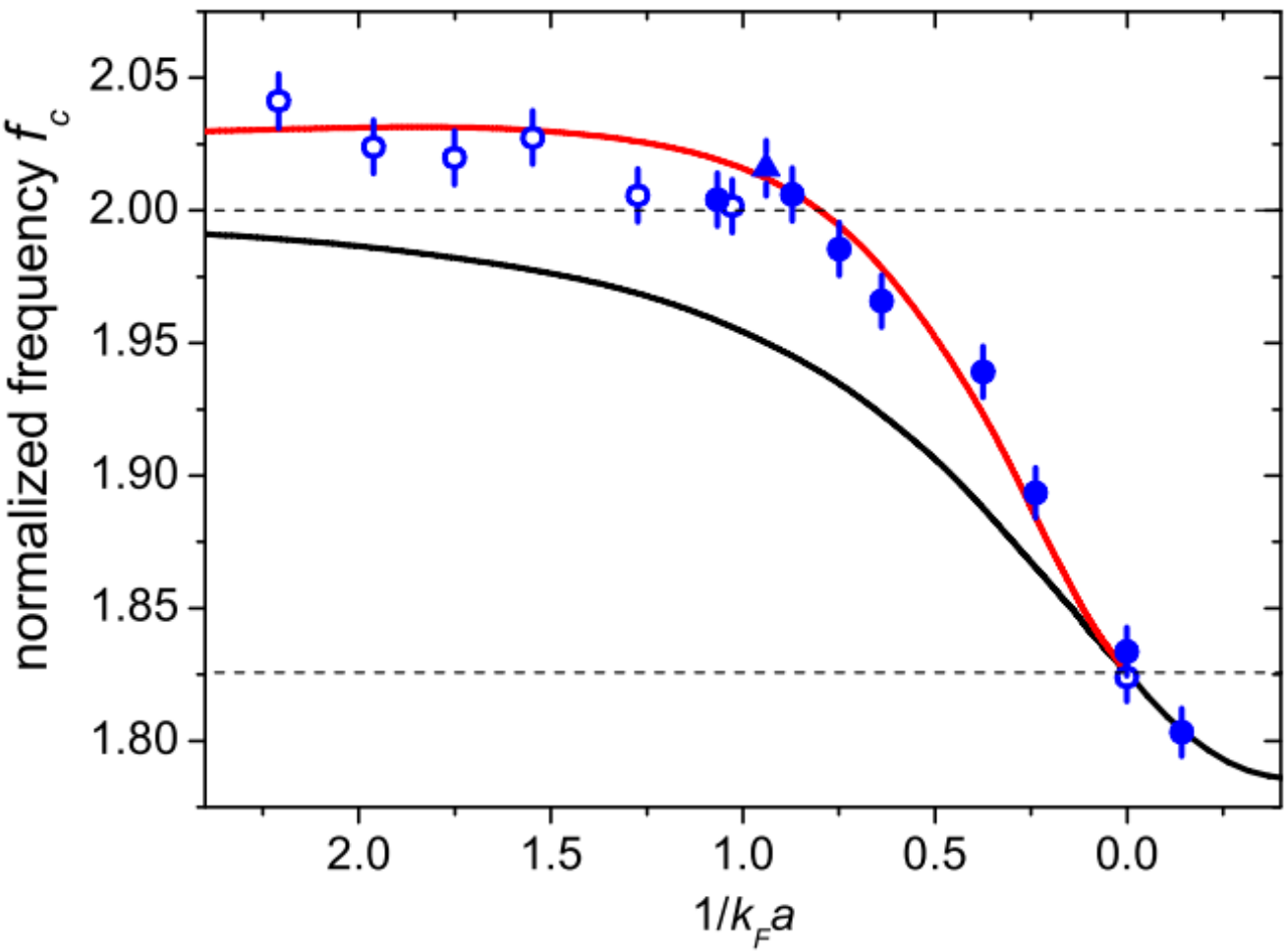
Collective mode measurements

Collective mode frequencies are sensitive to the equation of state.



Altmeyer et al, PRL 2007

Collective mode measurements



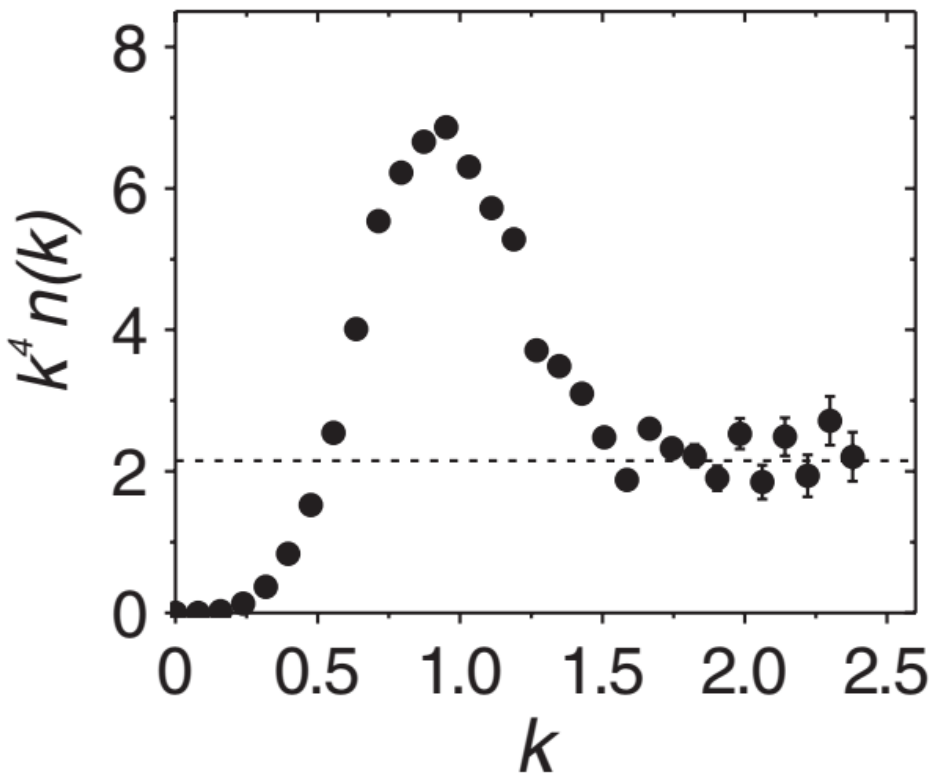
Altmeyer et al, PRL 2007

Discriminates between mean-field (black) and Monte Carlo simulations (red).

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Contact from the tails of the momentum distribution

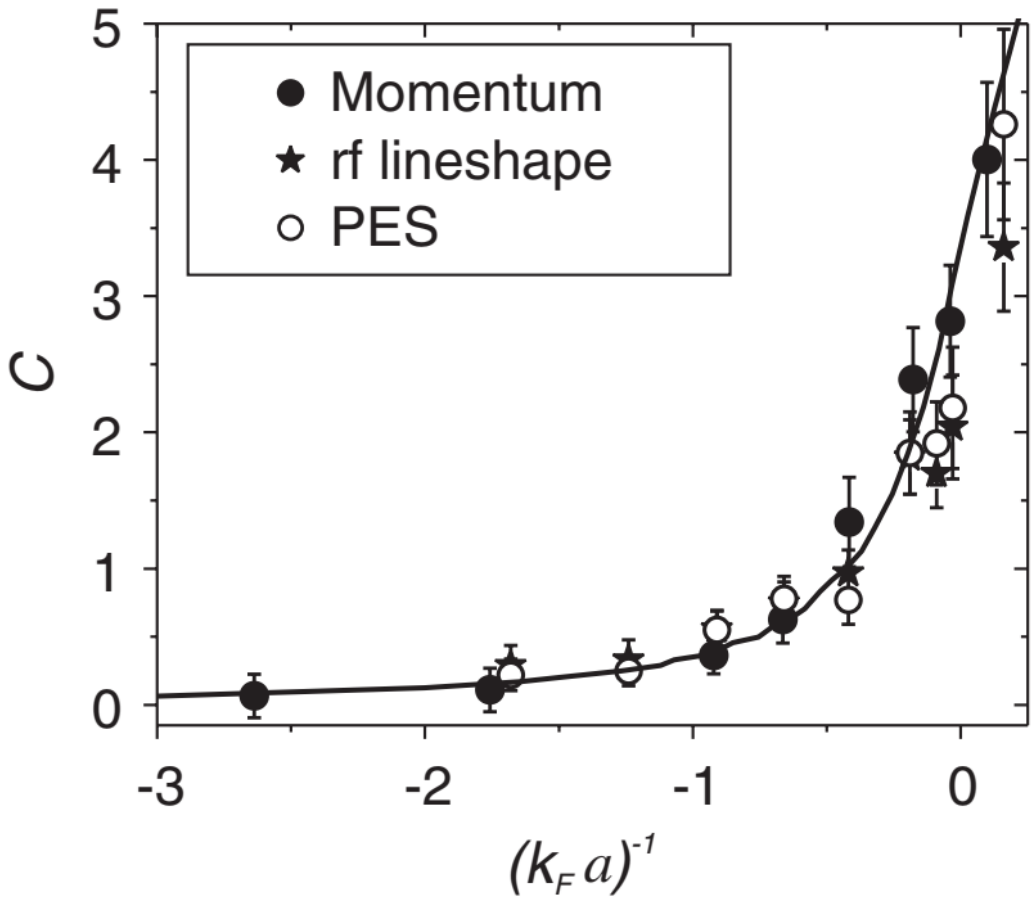


Stewart et al, PRL 2010

Other observables

- Derivative of the equation of state with respect to a (Navon et al, Science 2010)
- Tails of photoemission spectroscopy spectra (Stewart et al, PRL 2010)
- Pair density correlations (Kuhnle et al, PRL 2010)

Link between the contact and the equation of state



Stewart et al, PRL 2010

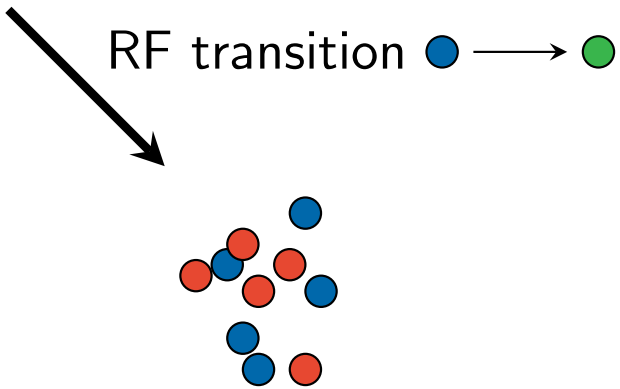
At finite temperature, measurements still wait for a theoretical understanding.

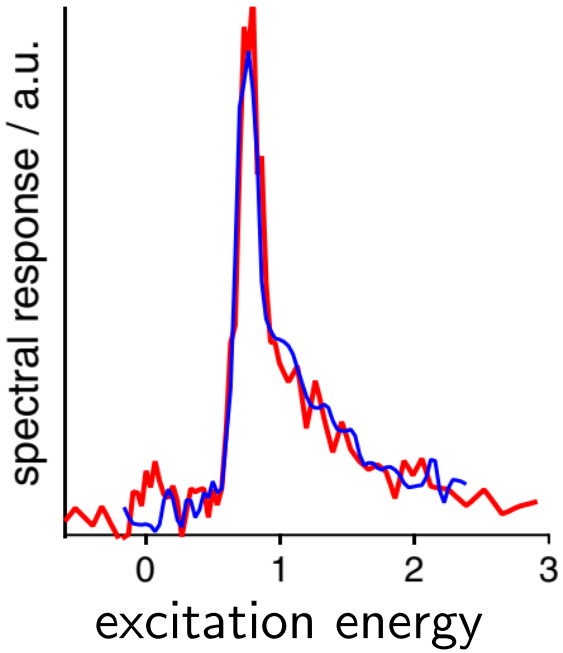
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Superfluid gap measurement

Spectroscopy by transfering a spin state to an empty level.





Schirotzek et al, PRL 2008

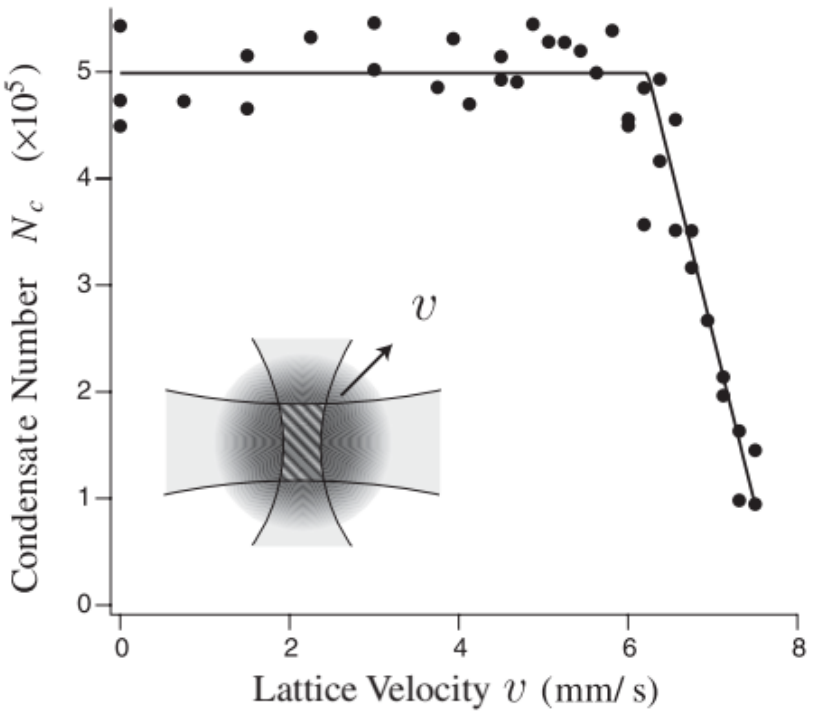
Energy gap at unitarity $\Delta = 0.44\, E_F$.

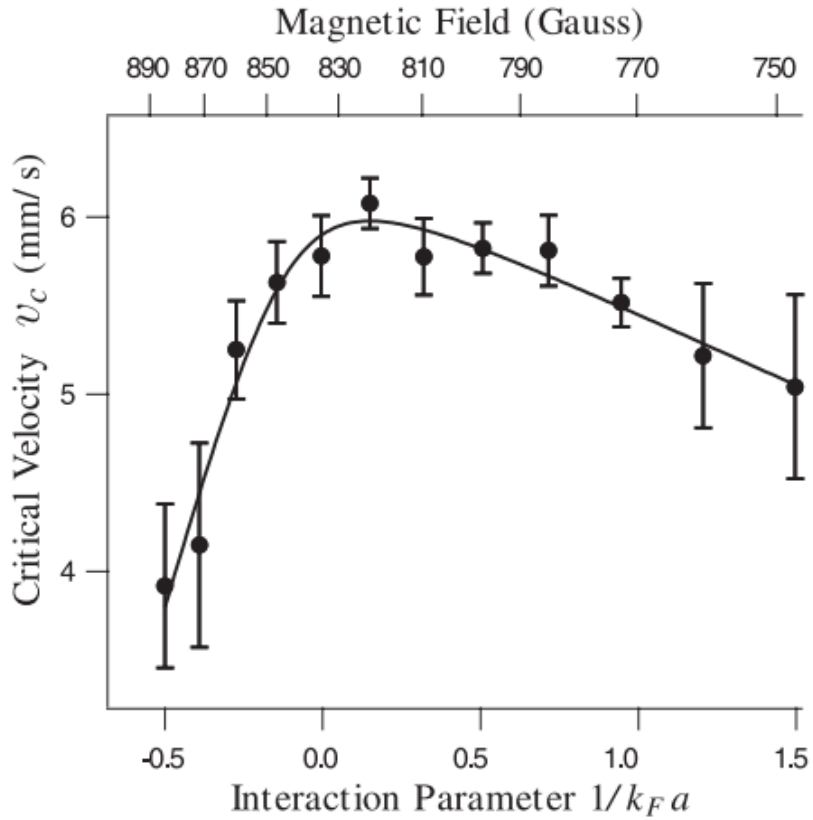
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Critical velocity

A moving perturbation of speed v_c does not heat for $v < v_c$.





Miller et al, PRL 2007

Quantized vortices

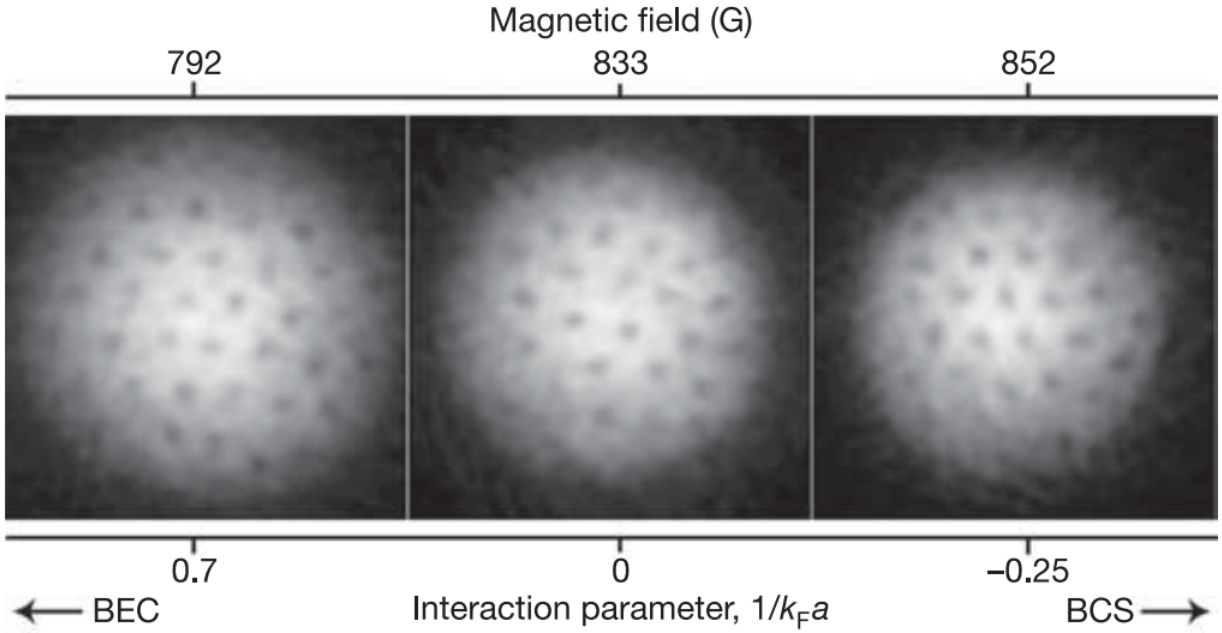
In a superfluid system described by a classical field $\psi = \sqrt{n}e^{i\theta}$, the velocity reads

$$\mathbf{v} = \frac{\hbar}{2m}\nabla\theta \quad \text{(pair mass } 2m\text{)}.$$

Hence on a closed contour $\mathcal{C}$ one has

$$\oint_{\mathcal{C}} \mathbf{v}\cdot d\mathbf{l} = p\,\frac{h}{2m}, \qquad p \in \mathbb{Z}.$$

The vorticity $\nabla \times \mathbf{v}$ is concentrated in singular vortex lines.



Zwierlein et al, Nature 2005

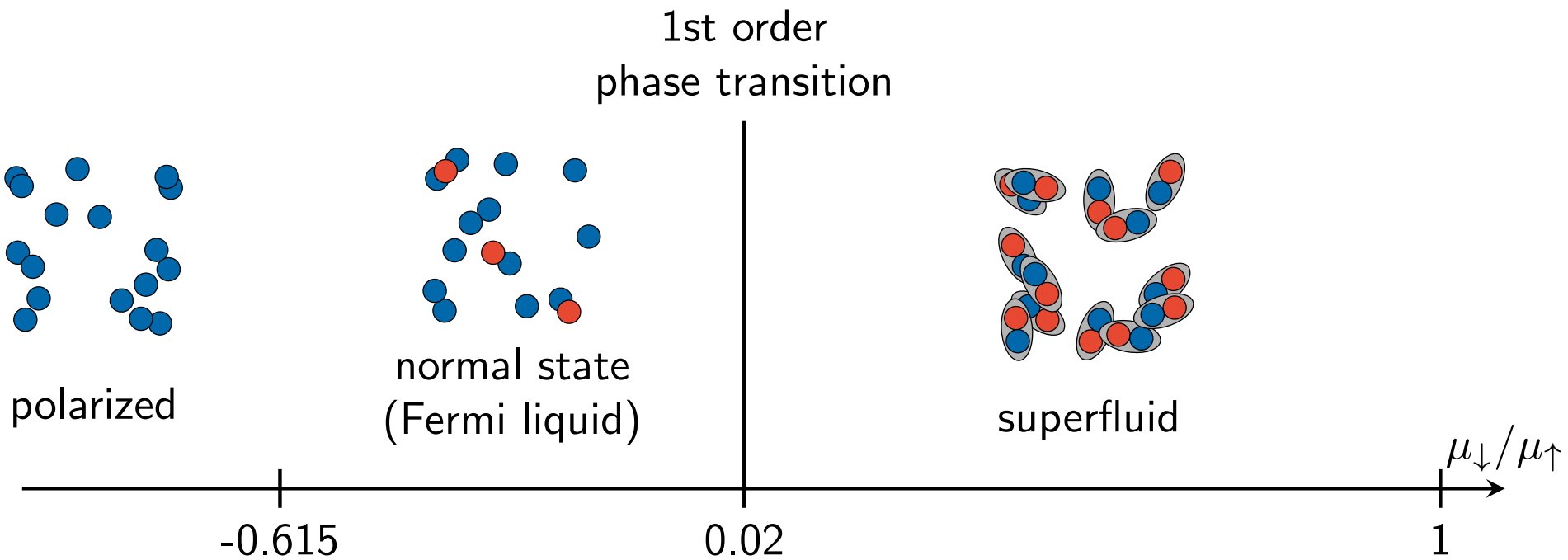
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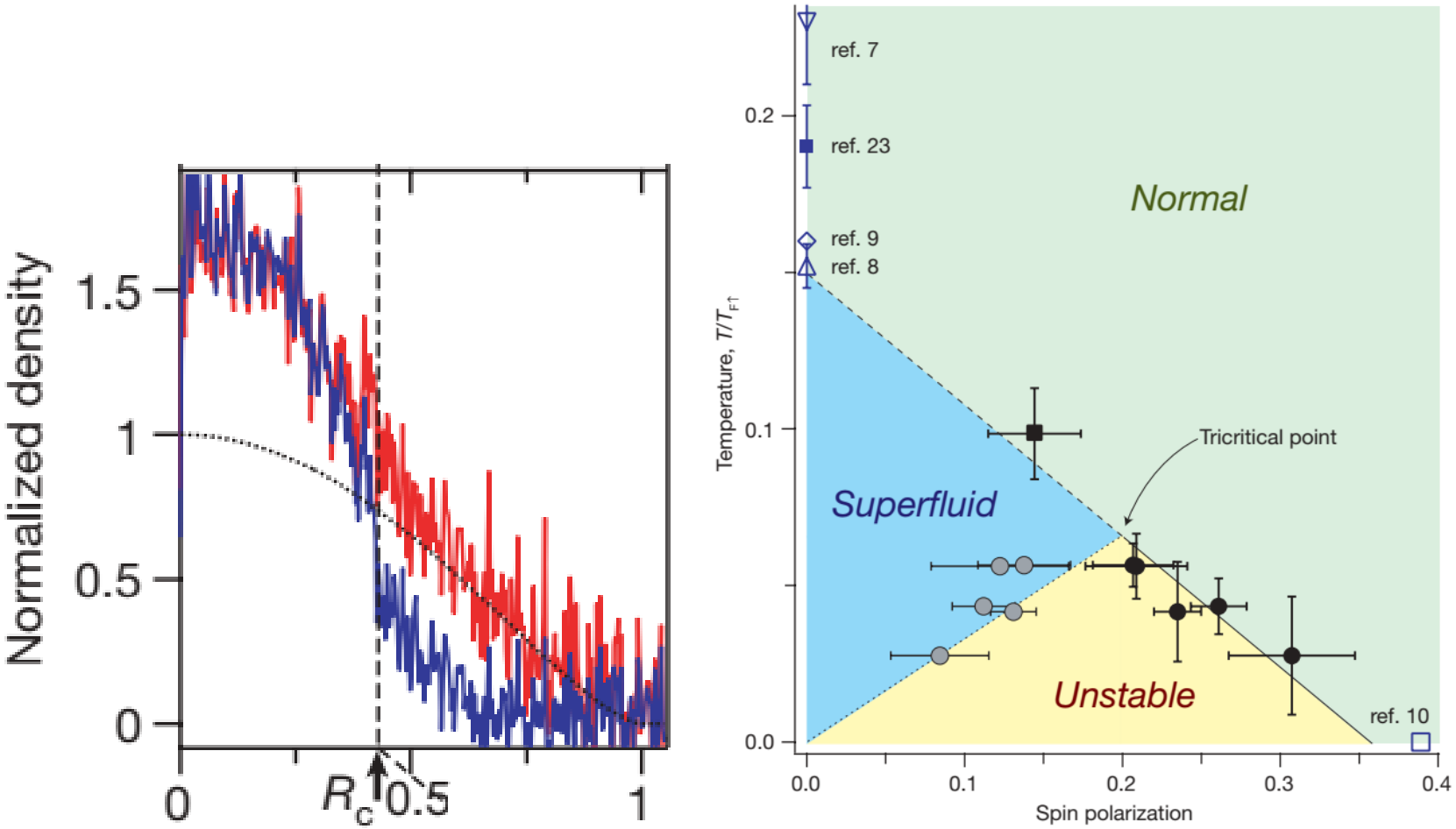
Spin imbalanced Fermi gases

What happens if one prepares $N_\uparrow \neq N_\downarrow$?

- related to the effect of a Zeeman field $h = (\mu_\uparrow - \mu_\downarrow)/2$
  $\rightarrow$ pair breaking for $h > h_c = \Delta/\sqrt{2}$ (Chandrasekhar-Clogston limit)
- not accessible in superconductors (Meissner effect)
- phase diagram at unitarity



Spin imbalanced Fermi gases



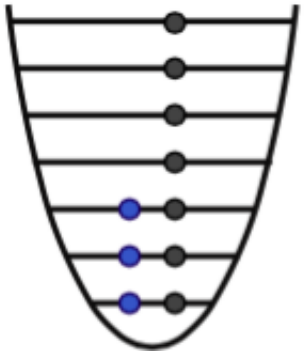
Shin et al, Nature 2008

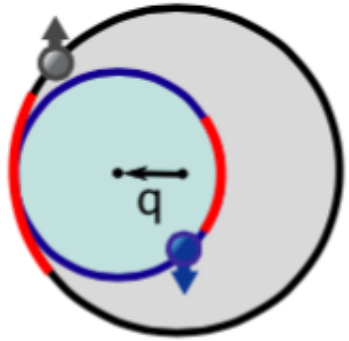
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Ultracold Fermi gases: open questions

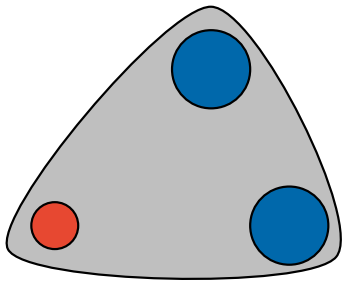
- Exotic superfluidity in weakly imbalances gases
  $\rightarrow$ Cooper pairing at finite center-of-mass momentum





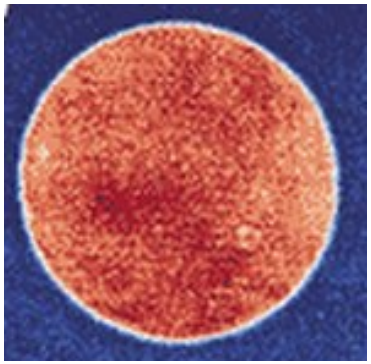
Ultracold Fermi gases: open questions

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- Mixtures of different species
  $\rightarrow$ mass imbalance
  $\rightarrow$ occurence of 3-body bound states (Efimov trimers)



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- Low dimensions
  $\rightarrow$ 2D: BKT topological phase transition
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- Low dimensions
  $\rightarrow$ 2D: BKT topological phase transition
  $\rightarrow$ 1D: Luttinger liquids
- Artificial gauge fields
  $\rightarrow$ quantum Hall effect, topological insulators

