Lecture 5: Bose-Einstein condensates

ICFP Master Course

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Master ICFP, Ultracold atoms

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1. Refresher on second-quantization

2. The Gross-Pitaevskii equation
   - Static properties of a weakly-interacting BEC

3. Superfluid hydrodynamics and collective modes
   - Superfluid hydrodynamics
   - Collective modes: low-energy excitations
   - Superfluid vortices
   - Scaling solutions in a harmonic trap

4. Coherence
   - Interference fringes and first-order coherence
   - Measurement of the coherence length
   - Penrose-Onsager criterion
   - Spontaneous symmetry breaking
   - Higher order correlation functions : second and third order coherences

Refresher : second quantization

Standard quantum-mechanical approach ("First quantized description") :
N-particle states:

- $N-$particle Hilbert space $\mathcal{F}_N$ built by direct product of N single-particle Hilbert spaces $\mathcal{F}_i$,
- $N-$particle states labeled by N quantum numbers $\{\nu_j\}_{k=1,\cdots N}$, corresponding to the single-particle orbitals occupied by each particle

$$|\Psi_N\rangle = |1:\nu_1, 2:\nu_2, \cdots, N:\nu_N\rangle,$$

- symmetrization (or anti-symmetrization) of the physical states for bosons (fermions).

Operators :

- One-body operators, *e.g.* kinetic energy or density ...

$$\hat{K} = \sum_{i=1}^{N} \frac{\hat{\boldsymbol{p}}_i^2}{2M}, \qquad \hat{n} = \sum_{i=1}^{N} \delta(\boldsymbol{r} - \hat{\boldsymbol{r}}_i), \cdots$$

- Two-body operators, *e.g.* interaction energy ,

$$\hat{H}_{\text{int}} = \sum_{i \neq j} V_{\text{int}}(\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j)$$

- ...

Refresher : second quantization

Relabeling of physical states (Fock basis) :
N-particle states:

- $N-$particle states labeled by the occupation numbers n_λ of the different single-particle orbitals

$$|\Psi_N\rangle = |n_1, n_2, \cdots, n_\lambda, \cdots\rangle,$$

with $\sum_\lambda n_\lambda = N$. Same state, but different labelling.

- bosons : $0 \leq n_\lambda \leq +\infty$,
- fermions : $n_\lambda = 0, 1$ (Pauli principle).

Creation and annihilation operators :

- consider an extended Hilbert space

$$\mathcal{F} = \oplus_{N=1}^{\infty} \mathcal{F}_N$$

- Define creation and annihilation operators

$$\begin{aligned}
\text{Bosons :}\quad \hat{b}_\nu^\dagger |\Psi_N\rangle &= \sqrt{n_\nu + 1}\,|n_1, n_2, \cdots, n_\nu + 1, \cdots\rangle \\
\hat{b}_\nu |\Psi_N\rangle &= \sqrt{n_\nu}\,|n_1, n_2, \cdots, n_\nu - 1, \cdots\rangle, \\
\text{Fermions :}\quad \hat{c}_\nu^\dagger |\Psi_N\rangle &= (-1)^{\nu-1} |n_1, n_2, \cdots, n_1, \cdots\rangle \text{if } n_\nu = 0, = 0 \text{ otherwise} \\
\hat{c}_\nu |\Psi_N\rangle &= (-1)^{\nu-1} |n_1, n_2, \cdots, n_\nu = 0, \cdots\rangle \text{if } n_\nu = 1, = 0 \text{ otherwise.}
\end{aligned}$$

Refresher : second quantization

Commutation relations for bosons :

$$\text{Bosons} : [\hat{b}_\mu, \hat{b}_\nu^\dagger] = \delta_{\mu\nu}, [\hat{b}_\mu, \hat{b}_\nu] = 0$$

Anti-commutation relations for fermions :

$$\text{Fermions} : \{\hat{c}_\mu, \hat{c}_\nu^\dagger\} = \delta_{\mu\nu}, \{\hat{c}_\mu, \hat{c}_\nu\} = 0$$

Construction of Fock states from the vacuum state (only element in $\mathcal{F}_0$):

$$|\Psi_N\rangle = \prod_{\lambda, n_\lambda \neq 0} \frac{1}{\sqrt{n_\lambda!}} \left(\hat{b}^\dagger\right)^{n_\lambda} |\text{vac}\rangle,$$

Expression of single-and two-body operators :

$$\hat{O}^{(1)} = \sum_{n=1}^{N} \hat{o}^{(1)}(n) \rightarrow \hat{O}^{(1)} = \sum_{\mu,\nu} \langle\mu|\hat{o}^{(1)}|\nu\rangle \hat{a}_\mu^\dagger \hat{a}_\nu$$

$$\hat{O}^{(2)} = \sum_{n \neq m} \hat{o}^{(2)}(n,m) \rightarrow \hat{O}^{(2)} = \sum_{\mu,\nu,\mu'\nu'} \langle\mu',\nu'|\hat{o}^{(2)}|\mu,\nu\rangle \hat{a}_{\mu'}^\dagger, \hat{a}_{\nu'}^\dagger, \hat{a}_\mu \hat{a}_\nu$$

$\hat{a}_\mu$: generic notation for bosonic or fermionic operators

Field operators

Definition :

$$\hat{\Psi}(\boldsymbol{r}) = \sum_{\mu} \phi_{\mu}(\boldsymbol{r}) \hat{a}_{\mu},$$

$$\hat{a}_{\mu} = \int d^3\boldsymbol{r} \phi^*_{\mu}(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r})$$

One and two body operators :

$$\hat{n} = \hat{\Psi}^{\dagger}(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r})$$

$$\hat{K} = \int d^3\boldsymbol{r} \hat{\Psi}^{\dagger}(\boldsymbol{r}) \left(-\frac{\hbar^2}{2M} \Delta \right) \hat{\Psi}(\boldsymbol{r})$$

$$\hat{V}_{\text{int}} = \int d^3\boldsymbol{r} d^3\boldsymbol{r}' \hat{\Psi}^{\dagger}(\boldsymbol{r}) \hat{\Psi}^{\dagger}(\boldsymbol{r}') V_{\text{int}}(\boldsymbol{r} - \boldsymbol{r}') \hat{\Psi}(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}')$$

Heisenberg equation of motion : $i\hbar \frac{\partial \hat{\Psi}}{\partial t} = [\hat{H}, \hat{\Psi}]$

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = -\frac{\hbar^2}{2M} \Delta \hat{\Psi} + V_{\text{ext}} \hat{\Psi} + g \hat{\Psi}^{\dagger} \hat{\Psi} \hat{\Psi}$$

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Mean-field theory of interacting bosons at $T = 0$

Many-body Hamiltonian :

$$\hat{H} = \underbrace{\sum_{i=1}^{N} \frac{\hat{p}_i^2}{2M}}_{E_{\mathrm{kin}}} + \underbrace{\sum_{i=1}^{N} U(\hat{r}_i)}_{E_{\mathrm{pot}}} + \underbrace{\sum_{i \neq j} V_{\mathrm{int}}(\hat{r}_i - \hat{r}_j)}_{E_{\mathrm{int}}}$$

For simplicity we use the pseudopotential approach and replace

$$V_{\mathrm{int}} \longrightarrow V_{\mathrm{pseudo}} = g\delta(\boldsymbol{r}),$$

with $g = \frac{4\pi\hbar^2 a}{M}$ and a the scattering length.

Gross-Pitaevskii theory: first and second-quantized versions

The Gross-Pitaevskii theory **assumes** that the many-body wavefunction at $T = 0$ keeps the same structure as for an ideal gas, *i.e.* all bosons condense in the same single-particle orbital state ϕ,

$$\Psi(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_N) = \phi(\boldsymbol{r}_1) \times \cdots \times \phi(\boldsymbol{r}_N)$$

Macroscopic occupation of a single-particle orbital ϕ (*not* the ground state of the trap potential U).

Product ansatz in second-quantization:

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} \left(\hat{a}_\phi^\dagger\right)^N |\mathrm{vac}\rangle$$

where $\hat{a}_\phi^\dagger$ creates a particle in the orbital $\phi(\boldsymbol{r})$.

Gross-Pitaesvkii energy functional

Hamiltonian in second quantized form :

$$\begin{aligned}\hat{H} &= \int d^3\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r})\left(-\frac{\hbar^2}{2M}\Delta + U(\boldsymbol{r})\right)\hat{\Psi}(\boldsymbol{r}) \\ &\quad + \frac{1}{2}\int d^3\boldsymbol{r}d^3\boldsymbol{r}'\,\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r}')V_{\text{int}}(\boldsymbol{r}-\boldsymbol{r}')\hat{\Psi}(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r}'), \\ &= \int d^3\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r})\left(-\frac{\hbar^2}{2M}\Delta + U(\boldsymbol{r})\right)\hat{\Psi}(\boldsymbol{r}) + \frac{g}{2}\int d^3\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})\end{aligned}$$

Energy expectation value for Ψ_N (effectively $\hat{\Psi} \to \phi(\boldsymbol{r})\hat{a}_\phi$):

$$\begin{aligned}E_N(\phi,\phi^*) &= \langle\Psi_N|\hat{H}|\Psi_N\rangle \\ &= \int d^3\boldsymbol{r}\left(-\frac{\hbar^2}{2M}\phi^*\Delta\phi + U(\boldsymbol{r})|\phi|^2\right) \times \langle\hat{a}_\phi^\dagger\hat{a}_\phi\rangle + \frac{g}{2}\int d^3\boldsymbol{r}\,|\phi|^4 \times \langle\hat{a}_\phi^\dagger\hat{a}_\phi^\dagger\hat{a}_\phi\hat{a}_\phi\rangle\end{aligned}$$

This can be viewed as a functional

$$E_N(\phi,\phi^*) = \int d^3\boldsymbol{r}\left(-\frac{\hbar^2}{2M}N\phi^*\Delta\phi + U(\boldsymbol{r})N|\phi|^2 + \frac{g}{2}N(N-1)|\phi|^4\right)$$

with ϕ, ϕ^* considered as **independent variables**.

Gross-Pitaesvkii equation

$$E_N(\phi,\phi^*) = \int d^3\boldsymbol{r}\left(-\frac{\hbar^2}{2M}N\phi^*\Delta\phi + U(\boldsymbol{r})N|\phi|^2 + \frac{g}{2}N(N-1)|\phi|^4\right)$$

Variational method:

Find the "best" product ansatz, *i.e.* the one that gives the closest energy to the true the ground state wavefunction.

Minimization of E_N wrt ϕ, ϕ^* under the constraint $\langle\phi|\phi\rangle = 1$:

$$\frac{\partial}{\partial\phi^*}\left(E_N(\phi,\phi^*) - \mu\int d^3\boldsymbol{r}\,|\phi|^2\right) = 0$$

One finally obtains the (time-independent) Gross-Pitaevskii equation

$$\mu\phi = -\frac{\hbar^2}{2M}\Delta\phi + U(\boldsymbol{r})\phi + g(N-1)|\phi|^2\phi$$

Mean-field approach: each boson behaves as a "non-interacting particle" in the effective potential $U_{\text{eff}}(\boldsymbol{r}) = U(\boldsymbol{r}) + gN|\phi|^2$ due to a) the trap and b) the average interaction with all other bosons.

At this stage, μ is a Lagrange multiplier introduced to enforce the constraint.

Gross-Pitaevskii equation: general properties

One commonly introduces the condensate wavefunction $\psi = \sqrt{N}\phi$ (normalized to N), which obeys :

$$\text{Gross-Pitaevskii equation : } \mu\psi = -\frac{\hbar^2}{2M}\Delta\psi + U(\boldsymbol{r})\psi + g|\psi|^2\psi$$

Particle density :

$$n_0(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^2.$$

Interpretation of μ:

$$\mu = \int d^3\boldsymbol{r}\left(-\frac{\hbar^2}{2M}\psi^*\Delta\psi + U(\boldsymbol{r})|\psi|^2 + g|\psi|^4\right) = \left.\frac{\partial E}{\partial N}\right|_{S=0}$$

Thermodynamic definition of the chemical potential (constant entropy $S = 0$ since $T = 0$).

$$\mu = E_{\text{kin}} + E_{\text{pot}} + 2E_{\text{int}} \neq \frac{E}{N} = E_{\text{kin}} + E_{\text{pot}} + E_{\text{int}}$$

NB : $E_{\text{kin}} = -\int d^3\boldsymbol{r}\,\frac{\hbar^2}{2M}\psi^*\Delta\psi = \int d^3\boldsymbol{r}\,\frac{\hbar^2}{2M}\,|\boldsymbol{\nabla}\psi|^2$

Virial theorem :

$$2E_{\text{kin}} - 2E_{\text{pot}} + 3E_{\text{int}} = 0$$

Gross-Pitaevskii equation in a box for repulsive interactions ($g > 0$)

Box of finite size L along x, infinite along y, z. GP equation reads

$$\mu\psi = -\frac{\hbar^2}{2M}\frac{d^2\psi}{dx^2} + g|\psi|^2\psi$$

Far from the walls at $x = 0, L$:

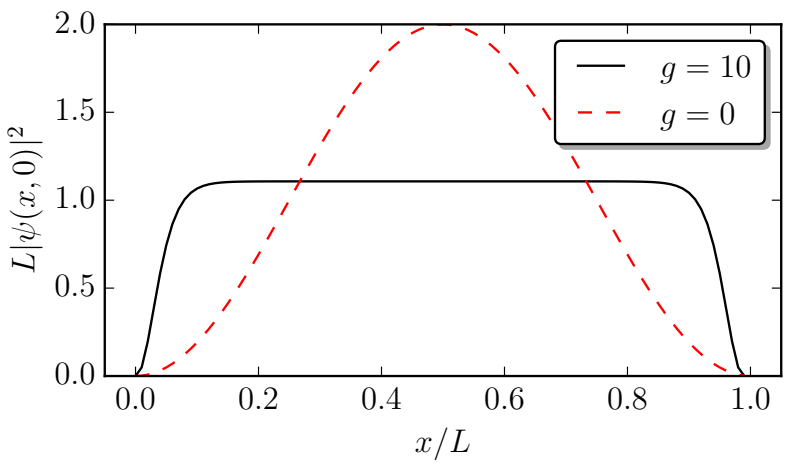
$$\text{uniform density } n_0 = \frac{N}{V}, \qquad \mu = gn_0, \qquad \psi = \sqrt{\frac{\mu}{g}}e^{i\theta}$$

Characteristic length ζ : $\frac{\hbar^2}{2M\zeta^2} = gn_0$

Healing length :

$$\zeta = \sqrt{\frac{\hbar^2}{2Mgn_0}}$$

$$\left.\begin{aligned} \eta &= \tfrac{x}{\zeta} \\ \psi &= \sqrt{n_0 f(\eta)}e^{i\theta} \end{aligned}\right\} \Longrightarrow \quad f = -f'' + f^3$$



- Sharp perturbations are "healed" on a length scale ζ.

BEC in a harmonic trap for repulsive interactions ($g > 0$)

Order-of-magnitude estimates for a cloud of N atoms, of typical size R, in a spherical trap of frequency ω :

$$E_{\rm kin} \sim N\frac{\hbar^2}{MR^2}, \qquad E_{\rm pot} \sim NM\omega^2 R^2, \qquad E_{\rm int} \sim \frac{gN^2}{R^3}$$

- kinetic and interaction energies minimized by increasing the cloud radius R
- at the cost of potential energy which increases with R

For large enough N, the interaction term dominates (if R grows slowly enough with N).

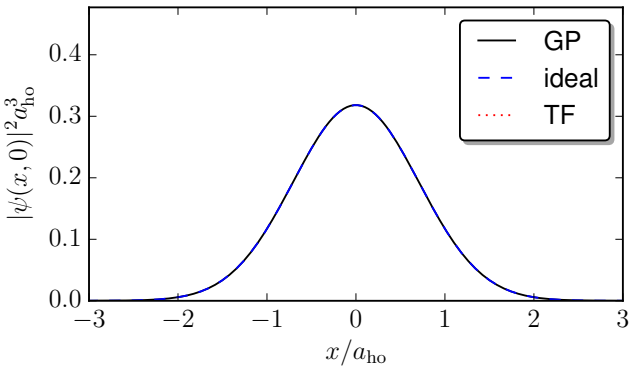
We introduce a parameter $b = \frac{R}{a_{\rm ho}}$, with $a_{\rm ho} = \sqrt{\frac{\hbar}{M\omega}}$ the harmonic oscillator length, to quantify deviations from the non-interacting ground state:

$$\frac{E_{\rm kin}}{N\hbar\omega} \sim \frac{1}{b^2}, \qquad \frac{E_{\rm pot}}{N\hbar\omega} \sim b^2, \qquad \frac{E_{\rm int}}{N\hbar\omega} \sim \frac{Na}{a_{\rm ho}}\frac{1}{b^3}$$

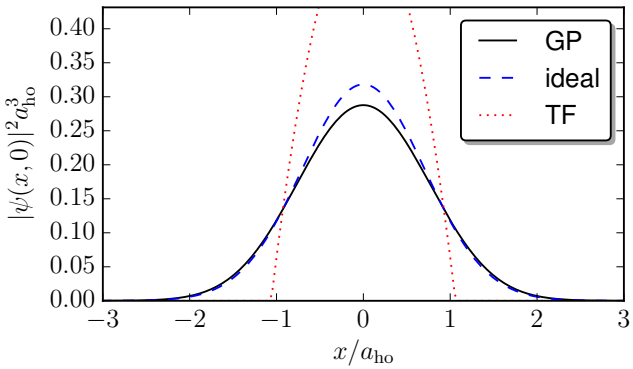
The parameter $\chi = \frac{Na}{a_{\rm ho}}$ defines the atom number around which repulsive interactions start to dominate.

Thomas-Fermi limit for repulsive interactions : numerics

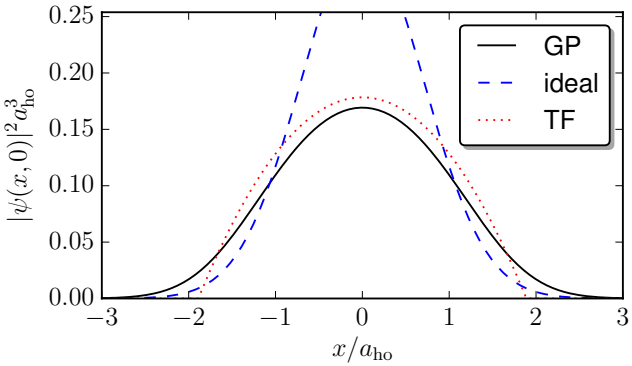
$\chi = 0$



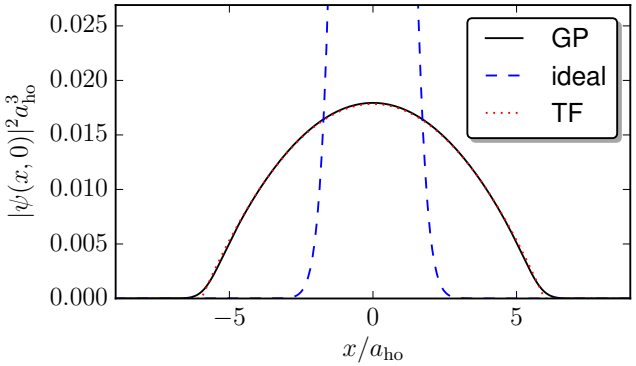
$\chi = 1$



$\chi = 10$



$\chi = 1000$



For small interactions (or $N \to 0$), the condensate forms in the trap ground state with $R = a_{\rm ho} = \sqrt{\frac{\hbar}{M\omega}}$.

Increasing g from 0 to its actual value, the condensate wavefunction changes from the Gaussian ground state of the harmonic oscillator to a flatter profile.

Thomas-Fermi approximation for large, repulsive interactions

Strong repulsive interactions : $\chi = \frac{Na}{a_{\rm ho}} \gg 1$ (cloud size $\gg$ healing length).

Kinetic energy neglected : equilibrium density profile determined by the balance between potential and kinetic energy terms

$$n(\boldsymbol{r}) = \begin{cases} \frac{\mu - U(\boldsymbol{r})}{g}, & \mu \geq U(\boldsymbol{r}), \\ 0, & \mu \leq U(\boldsymbol{r}), \end{cases}, \text{ or } n(\boldsymbol{r}) = \begin{cases} n_{0m}\left[1 - \left(\frac{r}{R_{\rm TF}}\right)^2\right], & r \leq R_{\rm TF}, \\ 0, & r \geq R_{\rm TF}. \end{cases}$$

Thomas-Fermi radius : $R_{\rm TF} = \sqrt{\frac{2\mu_{\rm TF}}{M\omega^2}}$, peak density $n_{0m} = \frac{\mu}{g}$

From the normalization condition $\int d^3\boldsymbol{r}|\Psi|^2 = N$, one finds the chemical potential :

$$\mu_{\rm TF} = \frac{\hbar\omega}{2}(15\chi)^{2/5} \gg \hbar\omega$$

Note the scaling with atom number, $\mu_{\rm TF} \propto N^{2/5}$ and thus $R_{\rm TF} \propto N^{1/5}$.

- Energetics in the TF limit :

$$E_{\rm pot} = \frac{3}{7}\mu N \qquad E_{\rm int} = \frac{2}{7}\mu N.$$

- Anisotropic potential :$U(\boldsymbol{r}) = \frac{1}{2}M\omega_\perp^2(x^2+y^2) + \frac{1}{2}M\omega_z^2 z^2$

  same results in the TF limit, with the replacement $\omega \to \overline{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$

Thomas-Fermi limit -II

First note that the Thomas-Fermi approximation is the same as the semi-classical/local density approximation introduced earlier for ideal gases:

$$\text{Uniform gas}: n = \frac{\mu}{g} \implies \text{Trapped gas in the LDA}: n(\boldsymbol{r}) = \frac{\mu_{\text{loc}}}{g} = \frac{\mu - U(\boldsymbol{r})}{g}.$$

The TF profile can also be derived from the condition of hydrostatic equilibrium,

$$\frac{1}{n}\boldsymbol{\nabla} P = -\boldsymbol{\nabla} U \implies \boldsymbol{\nabla} P + n\boldsymbol{\nabla} U = 0$$

P: pressure, n: density, U: external potential

For a uniform BEC at $T = 0$, the energy is given by $E = \frac{gN^2}{2V}$. The pressure is $P = \frac{\partial E}{\partial V} = \frac{gn^2}{2}$.

$$\boldsymbol{\nabla} P = gn\boldsymbol{\nabla} n \implies \boldsymbol{\nabla}\left(gn + U\right) = 0$$

This recovers the Thomas-Fermi profile $n(\boldsymbol{r})$.

Alternative point of view: Gibbs-Duhem relation

$$-V dP + N d\mu (+S dT) = 0 \implies \boldsymbol{\nabla} P = n\boldsymbol{\nabla}\mu_{\text{loc}} = -n\boldsymbol{\nabla} U$$

Comparison with experiments

Density profile (*in-situ* dispersive imaging) of a Sodium BEC:

Dalfovo *et al.*, RMP 1998; Hau *et al.*, PRA 1998

Typical numbers (Na BEC):

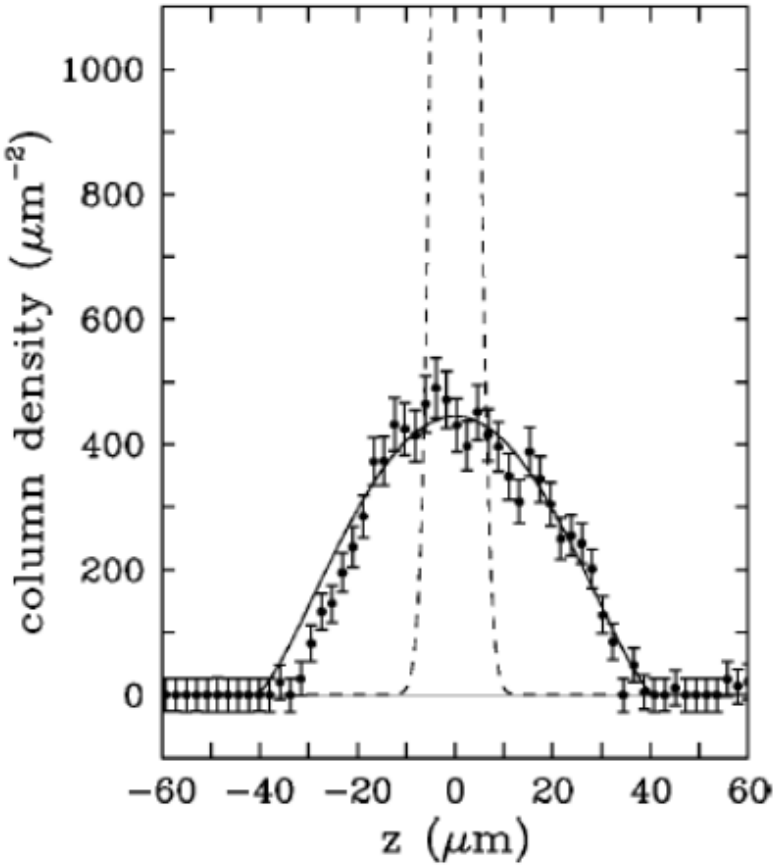
$N \sim 10^6$,

$\overline{\omega}/2\pi = 15\,\text{Hz}$ (after decompression),

Chemical potential :
$\mu \approx h \times 300\,\text{Hz} \approx k_B \times 30\,\text{nK}$

Peak density : $n \sim 2 \cdot 10^{13}\,\text{at/cm}^3$,

Typical sizes : $R_{\text{TF}} \sim 37\,\mu\text{m}$, $\zeta \sim 800\,\text{nm}$



Common numbers : $\mu/h \sim$ kHz, $n \sim 10^{14}\,\text{at/cm}^3$, $R_{\text{TF}} \sim 10\,\mu\text{m}$, $\zeta \sim 100\,\text{nm}$
These number can change by factors up to 10 depending on atomic species and experimental details.

NB $1\,\text{kHz} \leftrightarrow 50\,\text{nK}$

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Time-dependent variational approach

Mean-field Lagrangian :

$$\mathcal{L}[\psi,\psi^*,\dot{\psi},\dot{\psi}^*] = \int d^3\boldsymbol{r}\,\frac{i\hbar}{2}\left(\psi^*\dot{\psi}-\dot{\psi}^*\psi\right) - \left(-\frac{\hbar^2}{2M}\psi^*\Delta\psi + U(\boldsymbol{r})|\psi|^2 + \frac{g}{2}|\psi|^4\right)$$

Euler-Lagrange equations :

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = \frac{\partial\mathcal{L}}{\partial\psi^*} \implies i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2M}\Delta\psi + U(\boldsymbol{r})\psi + g|\psi|^2\psi$$

Time-dependent Gross-Pitaevskii equation

Equilibrium solution : $\psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})e^{-i\frac{\mu t}{\hbar}}$

Hydrodynamic equations

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2M}\Delta\psi + U(\boldsymbol{r})\psi + g|\psi|^2\psi$$

Insert $\psi = \sqrt{n}e^{i\theta}$ and separate real and imaginary parts :

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\left(n\frac{\hbar}{M}\boldsymbol{\nabla}\theta\right) = 0 \qquad -\hbar\frac{\partial\theta}{\partial t} = -\frac{\hbar^2}{2M}\frac{\Delta\sqrt{n}}{\sqrt{n}} + \frac{\hbar^2}{2M}\boldsymbol{\nabla}^2\theta + U + gn$$

Superfluid velocity : $\boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\theta$

Superfluid hydrodynamics :

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\left(n\boldsymbol{v}_s\right) = 0 \qquad M\frac{\partial}{\partial t}\boldsymbol{v}_s = -\boldsymbol{\nabla}\left(\mu_{\mathrm{loc}} + \frac{1}{2}M\boldsymbol{v}_s^2\right)$$

$$\mu_{\mathrm{loc}} = -\frac{\hbar^2}{2M}\frac{\Delta\sqrt{n}}{\sqrt{n}} + gn + U(\boldsymbol{r})$$

Continuity and Euler equations for a perfect fluid with density n and velocity field $\boldsymbol{v}_s$.

Uniform gas : response to small perturbations

Uniform gas : Condensate wavefunction : $\Psi = \sqrt{n_0} e^{i\theta}$. Equilibrium : $\mu = g n_0$.

We consider small deviations from equilibrium, $\Psi = \sqrt{n_0 + \delta n(\boldsymbol{r}, t)} e^{i\delta\theta(\boldsymbol{r},t) - i\frac{\mu t}{\hbar}}$.

The *linearized* hydrodynamics equations reduce to

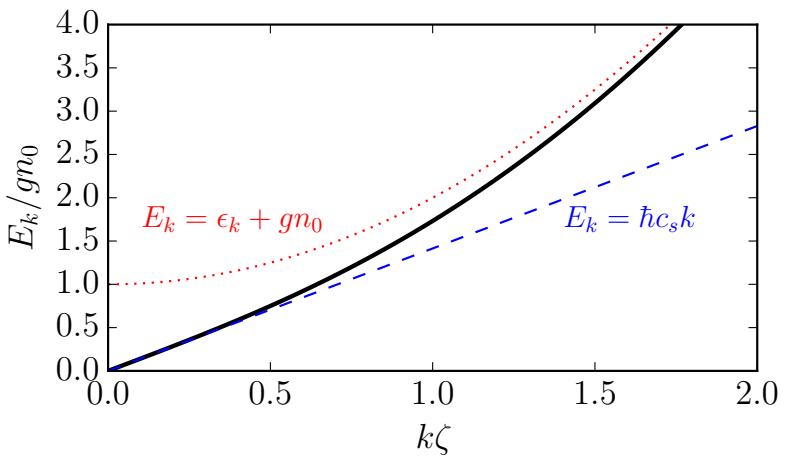
$$\frac{\partial \delta n}{\partial t} + \frac{\hbar n_0}{M} \boldsymbol{\nabla}^2 \delta\theta = 0 \qquad -\hbar \frac{\partial}{\partial t} \delta\theta = \left(g - \frac{\hbar^2}{4M} \frac{\boldsymbol{\nabla}^2}{n_0} \right) \delta n$$

We decompose δn, $\delta\theta$ in normal modes (plane waves):

$$\left.\begin{array}{ll} \delta n & \propto \sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \frac{E_{\boldsymbol{k}} t}{\hbar})} \\ \delta\theta & \propto \sum_{\boldsymbol{k}} \delta\theta_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \frac{E_{\boldsymbol{k}} t}{\hbar})} \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} -iE_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} & = \frac{\hbar^2 k^2 n_0}{M} \delta\theta_{\boldsymbol{k}}, \\ iE_{\boldsymbol{k}} \delta\theta_{\boldsymbol{k}} & = \left(g + \frac{\hbar^2 k^2}{4M n_0} \right) \delta n_{\boldsymbol{k}}. \end{array}\right.$$

Bogoliubov dispersion relation :

$$E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}} \left(\epsilon_{\boldsymbol{k}} + 2 g n_0 \right)},$$
$$\epsilon_{\boldsymbol{k}} = \frac{\hbar^2 \boldsymbol{k}^2}{2M}$$



Low-energy excitations :
sound waves (linear dispersion relation),
speed of sound $c_s = \sqrt{\frac{g n_0}{M}}$.

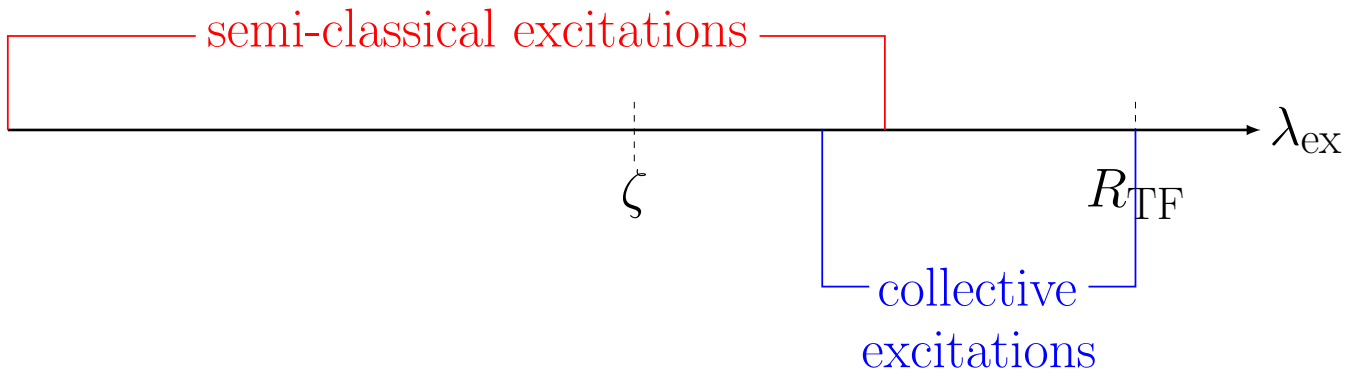
Note that the excitation energies are measured from the chemical potential.

Trapped gas in the Thomas-Fermi regime : classification of excitation modes

Same method for a trapped gas : $\Psi = \sqrt{n_0(\boldsymbol{r}) + \delta n(\boldsymbol{r}, t)} e^{i\delta\theta(\boldsymbol{r},t) - i\frac{\mu t}{\hbar}}$

$$\frac{\partial \delta n}{\partial t} + \boldsymbol{\nabla}\left(n_0(\boldsymbol{r})\frac{\hbar}{M}\boldsymbol{\nabla}\delta\theta\right) = 0 \qquad -\hbar\frac{\partial \delta\theta}{\partial t} = -\frac{\hbar^2}{4M}\frac{\Delta\delta n}{n_0(\boldsymbol{r})} + g\delta n$$

A given excitation mode with characteristic wavelength λ_{ex} can be classified on the following scale :

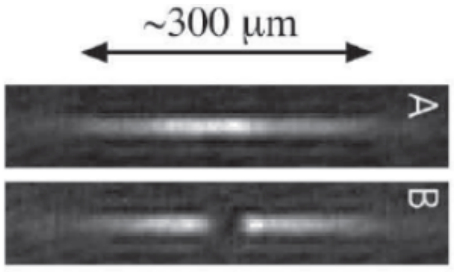


When $\lambda_{\mathrm{ex}} \ll R$, the excitations modes can be described in a semi-classical approximation, *i.e.* they have the same form as the excitations of a uniform gas at each point $\boldsymbol{r}$ with a slowly varying envelope due to spatial variations of the density.

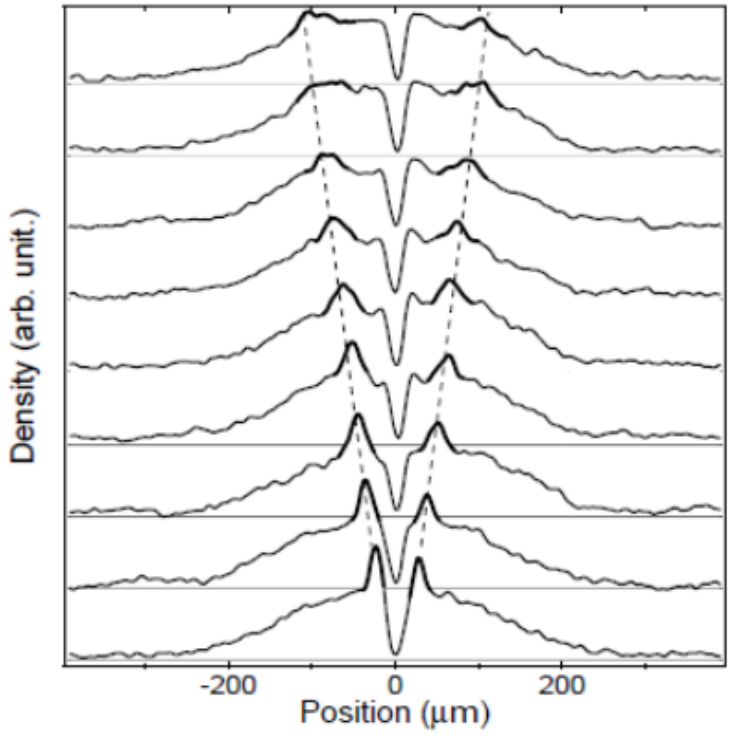
The excitations resemble sound waves when $\lambda_{\mathrm{ex}} \gg \zeta$ (excitation energy $\ll \mu$) and become closer to free particles in an effective potential $U_{\mathrm{eff}} = U(\boldsymbol{r}) + 2gn_0(\boldsymbol{r})$ when $\lambda_{\mathrm{ex}} \ll \zeta$ (excitation energy $\gg \mu$).

Observation of sound waves

- Propagation of a density perturbation
- cigar-shaped condensate "cut in half" by a repulsive light sheet



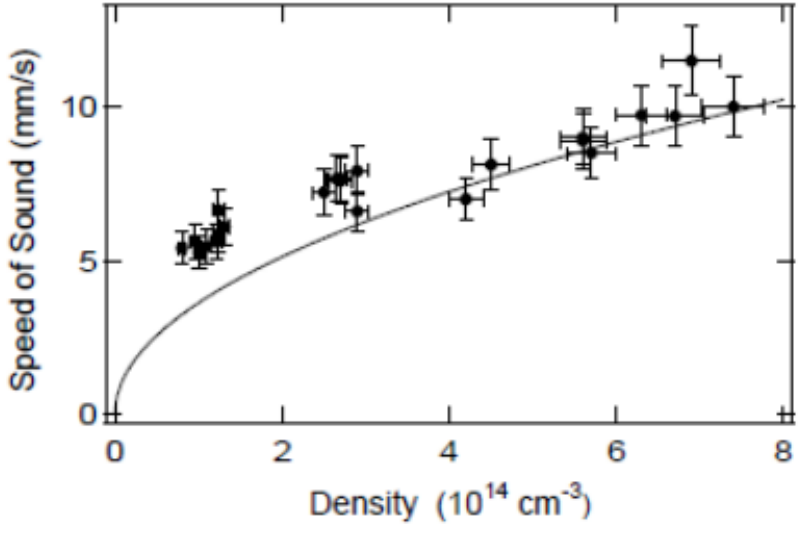
- time-resolved observation of the density after removing the light sheet using dispersive imaging



- Peak velocity in good agreement with the expected speed of sound,

$$c_s = \sqrt{\frac{g n_{0m}}{M}}.$$

- wave packet "dispersion" due to the decreasing density near the edges of the cloud $[c_s(R_{\mathrm{TF}} = 0)]$.



Collective modes of a trapped gas in the Thomas-Fermi regime

When $\lambda_{\mathrm{ex}} \sim R$, the semiclassical approximation no longer holds. The excitation modes "feel" the boundaries of the condensate and are closer to standing waves than to plane waves.

Density gradients can be neglected n the linearized hydrodynamics equations:

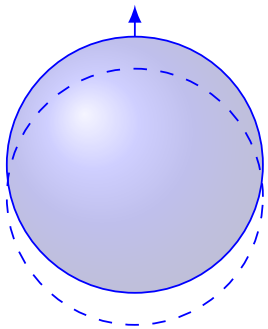
$$\frac{\partial \delta n}{\partial t} + \boldsymbol{\nabla}\left(n_0(\boldsymbol{r})\frac{\hbar}{M}\boldsymbol{\nabla}\delta\theta\right) = 0 \qquad -\hbar\frac{\partial \delta\theta}{\partial t} = g\delta n$$

$$\Longrightarrow \quad \frac{\partial^2 \delta n}{\partial t^2} - \boldsymbol{\nabla}\left(\frac{g n_0(\boldsymbol{r})}{M}\boldsymbol{\nabla}\delta n\right) = 0 \text{ (Stringari equation)}$$

In the TF regime, excitations are confined to the condensate volume. Solutions can be found as polynomials in x, y, z with the same symmetry as the trap potential.

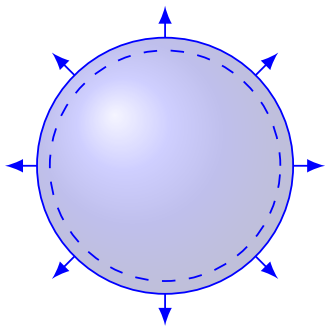
Dipole mode

$\delta n \propto y$



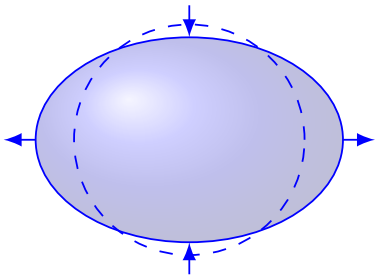
Radial breathing mode

$\delta n \propto x^2 + y^2$



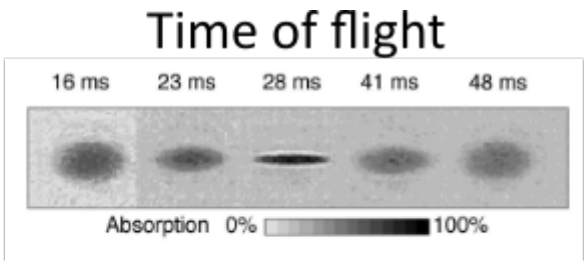
quadrupole mode

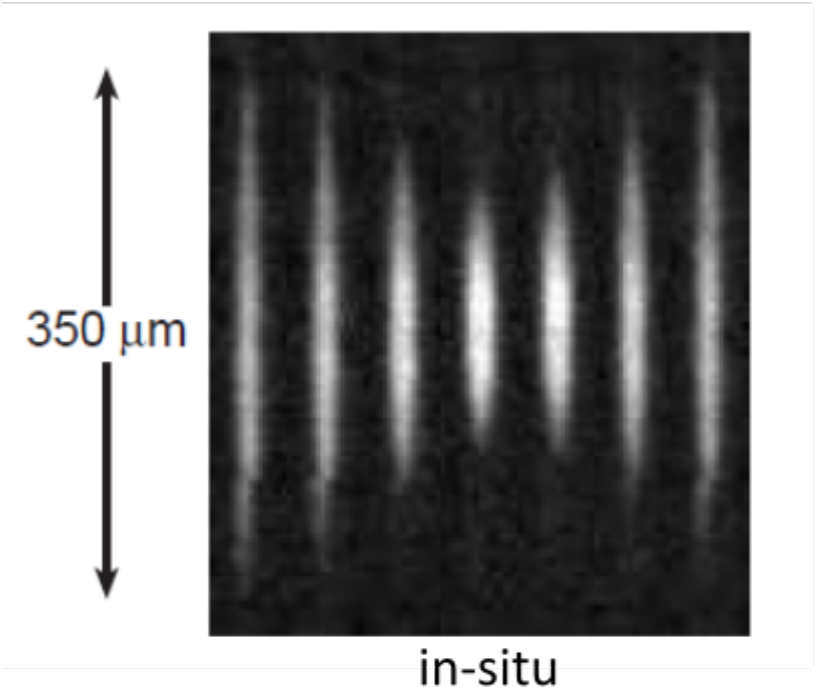
$\delta n \propto x^2 - y^2$



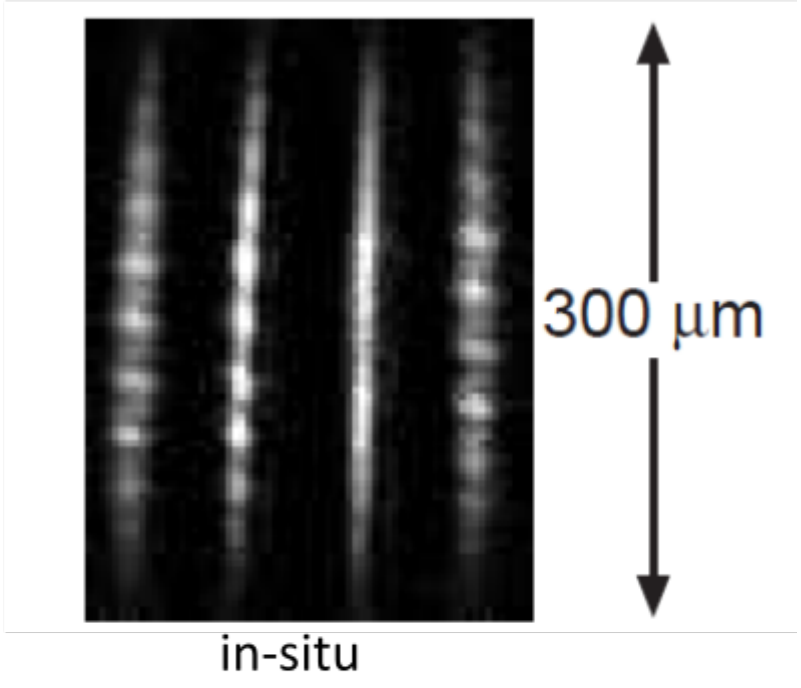
Experimentally, the frequency of these collective modes can be measured with high precision (typ. $1\,\%$ or better).

Examples of collective modes in trapped Sodium condensates





Quadrupole mode



High-lying axial mode

taken from W. Ketterle's lecture at Varenna Summer School, 1998

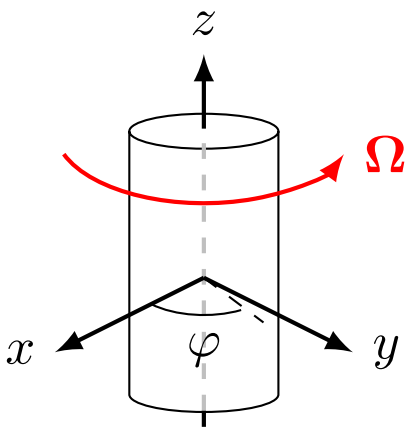
The rotating bucket experiment

BEC is a superfluid : dissipationless flow .
Interpretation of the phase θ of the condensate wavefunction :

$$\boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\theta \text{ : superfluid velocity}$$

Rotating bucket with cylindrical symmetry :

- a classical fluid will be "dragged along" and rotate with the container.
- velocity field of a "solid body" $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$, $\boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{\Omega}$.



What happens if we have a superfluid inside the bucket ?

$\boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\theta \implies$ Irrotational flow : $\boldsymbol{\nabla} \times \boldsymbol{v}_s = 0$

Superfluid stays at rest !

Quantized vortices

There is however another possibility : singularity of the phase θ (and of $\boldsymbol{v}_s$), which can only occur at a point where the density vanishes $n_0 = 0$.



In classical hydrodynamics, this corresponds to a vortex line:

$$\text{Velocity field : } \boldsymbol{v} = v_0 \boldsymbol{e}_\phi$$

$$\text{Circulation : } \oint \boldsymbol{v} \cdot \boldsymbol{dl} = 2\pi\rho v_0 = \mathcal{C}$$

$$\boldsymbol{\nabla} \times \boldsymbol{v}_s = \mathcal{C}\delta^{(2)}(\boldsymbol{\rho})\boldsymbol{e}_z$$

Quantization of circulation for a quantum fluid :

$$\mathcal{C} = \oint \boldsymbol{v}_s \cdot \boldsymbol{dl} = \frac{\hbar}{M} \oint \boldsymbol{\nabla}\theta \cdot \boldsymbol{dl} = \frac{\hbar}{M} \underbrace{[\theta(2\pi) - \theta(0)]}_{=2\pi \times \text{integer}} = s\frac{h}{M}, \qquad s \in \mathbb{Z}$$

Circulation must be quantized to keep the condensate wavefunctions single-valued.
s **: vortex charge.**

Velocity field far from the quantized vortex core : $\boldsymbol{v}_s = \frac{\hbar s}{M\rho}\boldsymbol{e}_\phi$

Quantized vortices-II

Quantization of circulation for a quantum fluid :

$$\mathcal{C} = \oint \boldsymbol{v}_s \cdot \boldsymbol{dl} = \frac{\hbar}{M} \oint \boldsymbol{\nabla}\theta \cdot \boldsymbol{dl} = \frac{\hbar}{M} \underbrace{(\theta(2\pi) - \theta(0))}_{=2\pi s, s\in\mathbb{Z}} = s\frac{h}{M}, \qquad s \in \mathbb{Z}$$

Circulation must be quantized to keep the condensate wavefunctions single-valued.
s **: vortex charge.**

Velocity field far from the quantized vortex core : $\boldsymbol{v}_s = \frac{\hbar s}{M\rho}\boldsymbol{e}_\phi$

Gross-Pitaevskii equation in a cylindrical container of radius R and length L:

cylindrical coordinates (ρ, ϕ, z)

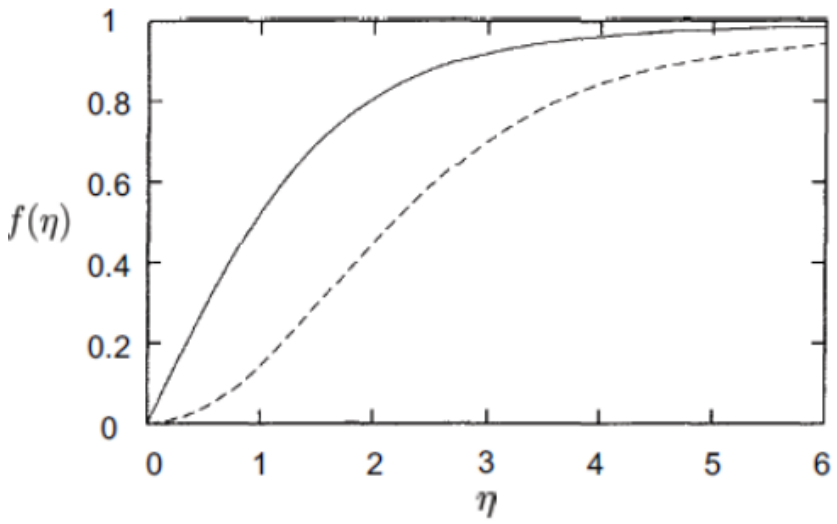
$$\Psi(\boldsymbol{r}) = \sqrt{n_0} e^{is\varphi} f(\eta), \qquad \eta = \frac{\rho}{\zeta}$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right) + \left(1 - \frac{s^2}{\eta^2}\right) f - f^3 = 0$$

$$f(x) \underset{x\to\infty}{\to} 1, \; f(x) \underset{x\to 0}{\sim} x^s,$$

Total angular momentum: $L_z = N\hbar s$

Numerical solution :



solid : $s = 1$, dashed : $s = 2$
vortex core size $\sim$ healing length ζ

Quantized vortices-III

Vortex energy in the lab frame :

$$E(s) - E(s=0) = \int d^3\boldsymbol{r} \left[\frac{M}{2} n_0 \boldsymbol{v}_s^2 + \frac{\hbar^2 n_0}{2M} |\boldsymbol{\nabla} f|^2 + \frac{g n_0^2}{2} \left(f^4 - 1\right) \right]$$

We introduced a cutoff $r^* \geq \zeta$, such that $f(r > r^*) \approx 1$. The contribution of the velocity flow far from the vortex is

$$\begin{aligned} E_{\text{vortex flow}} &= \frac{\hbar^2 s^2}{2M} \times 2\pi n_0 L \int_{r^*}^{R} d\rho \frac{|f(\rho)|^2}{\rho} d\rho \\ &\approx (n_0 \pi R^2 L) \times \frac{\hbar^2 s^2}{M R^2} \ln\left(\frac{R}{r^*}\right) \end{aligned}$$

The other terms correspond to contributions from the region $\rho \lesssim \zeta$ in the vicinity of the vortex core. We rewrite the energy as

$$\frac{E(s) - E(s=0)}{N} = \frac{\hbar^2 s^2}{M R^2} \ln\left(\frac{R}{r_c}\right)$$

with for instance $r_c \approx 1.5\zeta$ for $s = 1$.

- Large $s > 1$ vortices are *energetically unstable* to configurations with many $s = 1$ vortices with the same total angular momentum.

A more sophisticated analysis show that they are also *dynamically unstable*, *i.e.* they have excitation modes with an imaginary frequency (implying exponential growth when seeded from an infinitesimal perturbation).

Rotation of a trapped BEC

Vortex energy in the rotating frame :

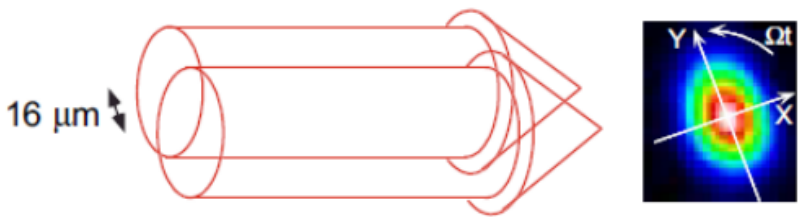
$$H_{\text{rotating}} = H_{\text{lab}} - \boldsymbol{\Omega} \cdot \boldsymbol{L}_z$$

$$\left(\frac{E(s) - E(s=0)}{N} \right)_{\text{rotating}} = \frac{\hbar^2 s^2}{MR^2} \ln\left(\frac{R}{r_c}\right) - s\hbar\Omega$$

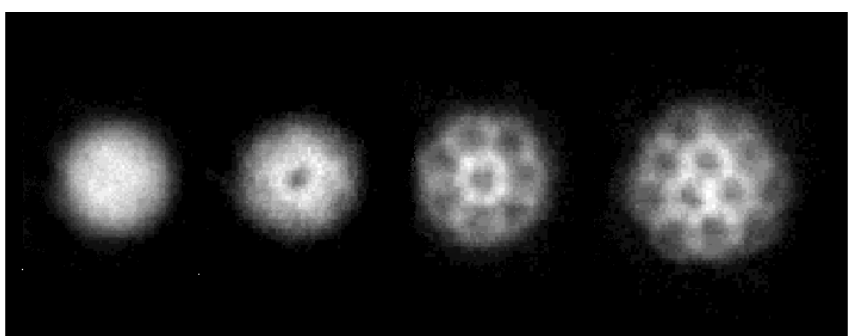
- critical rotation speed $\Omega_* = \frac{\hbar^2 s}{MR^2} \ln\left(\frac{R}{r_c}\right)$ above which the $s = 1$ vortex is the lowest energy state.

Experiments with an "optical spoon" stirring the condensate into rotation :

- spoon = fast rotating anisotropy on top of an otherwise rotationally symmetric potential
- BEC formed in a magnetic trap, then set in rotation
- Turbulent relaxation, coming down to an equilibrium state in the rotating frame after $\sim 1\,\text{s}$
- vortex nucleation above a critical rotation speed $\Omega_c \neq \Omega^*$, determined by dynamics (and not by thermodynamics).

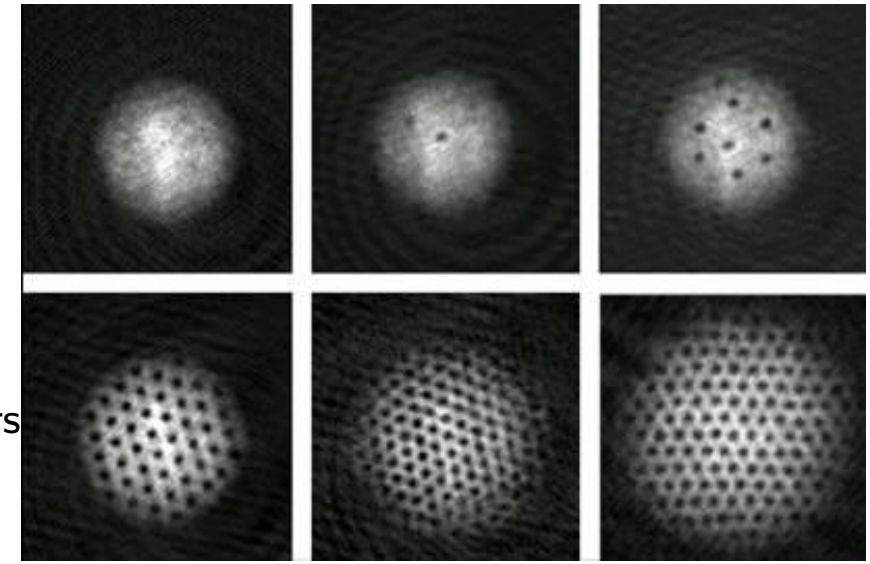


$$\delta U(\vec{r}) = \frac{1}{2} m\omega_\perp^2 \left(\varepsilon_X X^2 + \varepsilon_Y Y^2\right)$$



Abrikosov vortex lattice

- large BEC and high rotation : many vortices (up to 100 observed)
- vortices organize into a regular, triangular array (vortex lattice)
- also present in type-II superconductors



Feynman's argument :

The vorticity field is highly singular,

$$\boldsymbol{\nabla} \times \boldsymbol{v}_s = \sum_i \mathcal{C}_i \delta^2(\boldsymbol{\rho} - \boldsymbol{\rho}_i) \boldsymbol{e}_z,$$

but on a coarse scale $\gg$ vortex spacing, it averages out to the same value as for a rigidly rotating body ($\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \implies \boldsymbol{\nabla} \times \boldsymbol{v} = 2\Omega \boldsymbol{e}_z$). On a "patch" with area S containing N_v vortices, one has

$$\frac{1}{S} \int_S d^2\boldsymbol{\rho} \boldsymbol{\nabla} \times \boldsymbol{v}_s = 2\Omega \boldsymbol{e}_z = \frac{h}{M} \frac{N_v}{S}$$

We end up with a vortex density

$$n = \frac{N_v}{S} = \frac{2M\Omega}{h}$$

Scaling solution of the time-dependent GPE

Castin & Dum , PRL 1996; Kagan, Sur'kov, Shlyapnikov, PRA 1997

- condensate in a harmonic trap with time-dependent frequencies, $U(\{x_i\}) = \sum_i \frac{1}{2} M \omega_i(t)^2 x_i^2$
- Thomas-Fermi regime : quantum pressure negligible

$$\psi(\boldsymbol{r}, t) = \frac{1}{\mathcal{V}(t)} \psi_0 \left(\left\{ \frac{x_i}{b_i} \right\} \right) e^{i\theta}, \qquad \mathcal{V}(t) = \prod_i b_i$$

From the continuity equation :

$$\theta = \sum_i \frac{M}{\hbar} \frac{\dot{b}_i}{b_i} x_i^2 \implies \boldsymbol{v_s} = \sum_i \frac{\dot{b}_i}{b_i} x_i \boldsymbol{e}_i$$

Velocity field linear in position, resulting from mean-field pressure forces driving the expansion.

From Euler equation (neglecting quantum pressure):

$$\ddot{b}_i + \omega_i(t)^2 b_i = \frac{\omega_i(0)^2}{\left(\prod_j b_j \right) b_i}$$

Non-perturbative result, relies only on Thomas-Fermi approximation and harmonicity of the trap.

Scaling solution for time-of-flight expansion

Cigar-shaped trap :

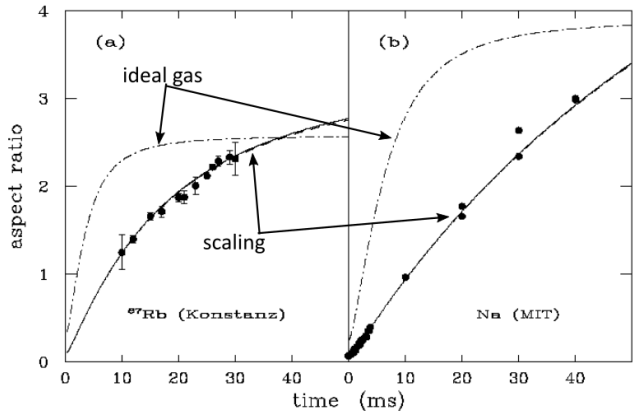
$$U(\boldsymbol{r}) = \frac{1}{2}M\omega_\perp^2(x^2+y^2) + \frac{1}{2}M\omega_z^2 z^2 \implies \begin{cases} \ddot{b}_\perp + \omega_\perp(t)^2 b_\perp = \frac{\omega_\perp(0)^2}{b_\perp^3\, b_z} \\ \ddot{b}_z + \omega_z(t)^2 b_z = \frac{\omega_z(0)^2}{b_\perp^2\, b_z^2} \end{cases}$$

For very anisotropic traps, $\omega_\perp \gg \omega_z$, one finds to lowest order in $\lambda = \omega_z/\omega_\perp$

$$\left.\begin{aligned} \ddot{b}_\perp &\approx \tfrac{\omega_\perp^2}{b_\perp^3} \\ \ddot{b}_z &\approx 0 \end{aligned}\right\} \implies \begin{cases} b_\perp \approx \sqrt{1+(\omega_\perp t)^2} \\ b_z \approx 1 + \frac{\pi\omega_z^2 t}{2\omega_\perp} \quad (t \gg \omega_\perp^{-1}) \end{cases}$$

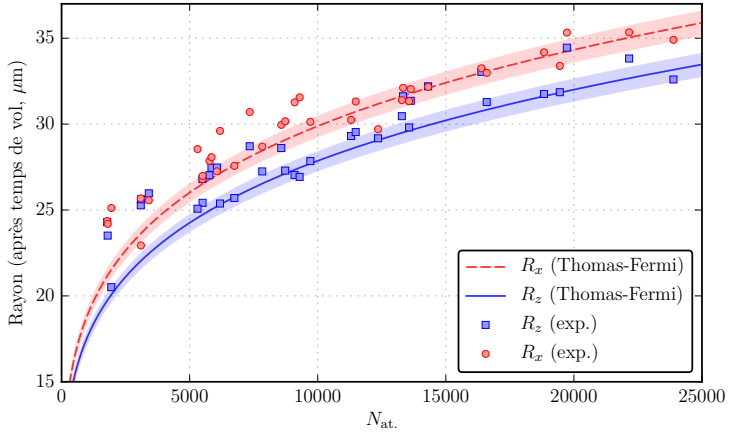
- For long time of flight, $\omega_\perp t \gg 1$, $R_\perp(t) = R_0 b_\perp \approx \sqrt{\frac{2\mu}{M}}t$: the condensate radius expands at the speed of sound.

Scaling solution versus experimental results



Dalfovo *et al.*, RMP 1998

BEC size after time of flight :
$R_{\mathrm{TF}} \propto N^{1/5}$
useful to calibrate absolute atom number in an experiment.



Conclusion : The density profile of an interacting gas (BEC or not) after time of flight expansion is dominated by mean-field repulsion, and does *not* reflect the initial momentum distribution as in the case of non-interacting particles.

1. Refresher on second-quantization

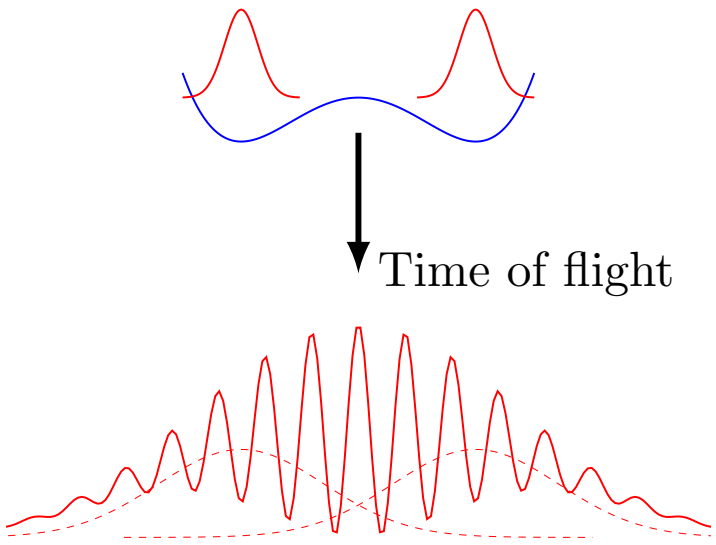
2. The Gross-Pitaevskii equation
   - Static properties of a weakly-interacting BEC

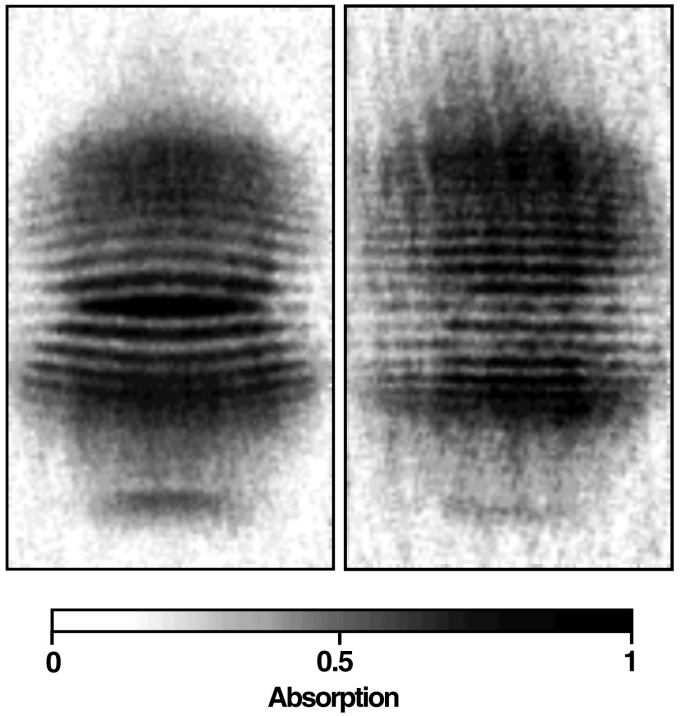
3. Superfluid hydrodynamics and collective modes
   - Superfluid hydrodynamics
   - Collective modes: low-energy excitations
   - Superfluid vortices
   - Scaling solutions in a harmonic trap

4. Coherence
   - Interference fringes and first-order coherence
   - Measurement of the coherence length
   - Penrose-Onsager criterion
   - Spontaneous symmetry breaking
   - Higher order correlation functions : second and third order coherences

Interference between two independent condensates

MIT experiment [Andrews *et al.*, Science 1998]





Two independent condensates brought to overlap (in time of flight) display clear interference fringes.

BEC for an interacting system : off-diagonal long range order

Gross-Pitaevskii approach : We assumed that the many-body wavefunction at $T = 0$ keeps the same structure as for an ideal gas,

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N | \Psi_{\rm GP} \rangle = \Psi_{\rm GP}(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_N) = \phi(\boldsymbol{r}_1) \times \cdots \times \phi(\boldsymbol{r}_N) \qquad \text{(first quantization)}$$

$$|\Psi_{\rm GP}\rangle = \frac{1}{\sqrt{N!}} \left(\hat{a}_\phi^\dagger\right)^N |{\rm vac}\rangle \qquad \text{(second quantization)}$$

with ϕ the condensate wavefunction determined self-consistently and where $\hat{a}_\phi^\dagger$ creates a particle in the orbital ϕ.

Single-particle density matrix :

$$\hat{\rho}^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \int d^3\boldsymbol{r}_2 \cdot \int d^3\boldsymbol{r}_N \Psi_N^*(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \Psi_N(\boldsymbol{r}', \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \qquad \text{(first quantization)}$$

$$= \langle \Psi_N | \hat{\psi}^\dagger(\boldsymbol{r}) \hat{\psi}(\boldsymbol{r}') | \Psi_N \rangle \qquad \text{(second quantization)}$$

$\langle \cdot \rangle$ denotes an average over the equilibrium states: many-body ground state $|\Psi_N\rangle$ at $T = 0$, thermal average at $T \neq 0$

Using $|\Psi_N\rangle = |\Psi_{\rm GP}\rangle$:

$$\rho^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = N\phi^*(\boldsymbol{r})\phi(\boldsymbol{r}') = \psi^*(\boldsymbol{r})\psi(\boldsymbol{r}') = \sqrt{n(\boldsymbol{r})n(\boldsymbol{r}')}$$

- Factorization of the single-particle density matrix,
- Decay at long distances determined by the density envelope $n(\boldsymbol{r})$.

First-order correlation function

Significance of $\rho^{(1)}$: contrast of interference fringes when two matter waves overlap

$$\hat{\psi}(\boldsymbol{r}) \underset{\text{coherent splitting}}{\longrightarrow} \hat{\psi}'(\boldsymbol{r}) = \frac{1}{\sqrt{2}}\left(\hat{\psi}(\boldsymbol{r}) + \hat{\psi}(\boldsymbol{r}+\boldsymbol{d})\right)$$

$$n(\boldsymbol{r}) = \langle \hat{\psi}'(\boldsymbol{r})\hat{\psi}'^{\dagger}(\boldsymbol{r})\rangle = \frac{1}{2}\left(\underbrace{n(\boldsymbol{r}) + n(\boldsymbol{r}+\boldsymbol{d})}_{\text{incoherent sum}} + \underbrace{\rho^{(1)}(\boldsymbol{r}+\boldsymbol{d},\boldsymbol{r}) + \text{c.c.}}_{\text{interference}}\right)$$

Alternative definition (analogy with optics) :

$$g^{(1)}(\boldsymbol{r},\boldsymbol{r}') = \frac{\langle \hat{\psi}^{\dagger}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')\rangle}{\sqrt{\langle \hat{\psi}^{\dagger}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\rangle\langle \hat{\psi}^{\dagger}(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}')\rangle}} = \frac{\rho^{(1)}(\boldsymbol{r},\boldsymbol{r}')}{\sqrt{n(\boldsymbol{r})n(\boldsymbol{r}')}}$$

Coherence length : typical distance over which the $g^{(1)}$ goes to zero

Uniform system :

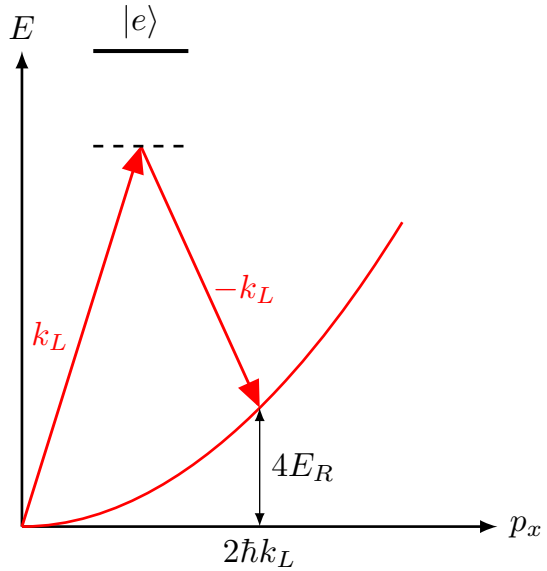
$$g^{(1)}(\boldsymbol{r},\boldsymbol{r}') = g^{(1)}(|\boldsymbol{r}-\boldsymbol{r}'|) = \frac{\rho^{(1)}(|\boldsymbol{r}-\boldsymbol{r}'|)}{n}$$

Non-uniform system :

$$\overline{g}^{(1)}(\boldsymbol{s}) = \frac{1}{N}\int d^3\boldsymbol{R}\,\rho^{(1)}\left(\boldsymbol{R}+\frac{\boldsymbol{s}}{2},\boldsymbol{R}-\frac{\boldsymbol{s}}{2}\right)$$

Coherent atom optics : Bragg diffraction

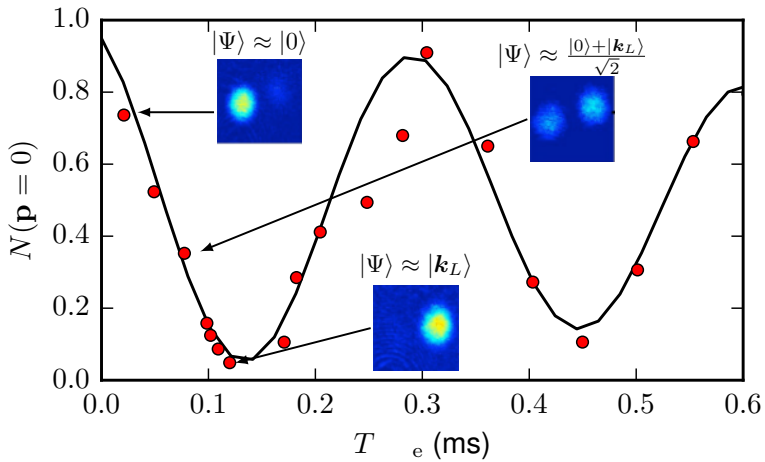
- Standing wave with frequency-shifted beams
- Detuning far from atomic resonances
- Two-photon transition between momentum states (without change in internal state)
- Two photon Rabi frequency : $\frac{\Omega_{\text{Bragg}}}{2} = \frac{\Omega_R^2}{4\hbar\delta_L}$
- Energy conservation :
  $\hbar\Delta_{\text{Bragg}} = \frac{(p_i+2\hbar k_L)^2}{2M} - \frac{p_i^2}{2M} = 4E_R + \frac{2\hbar k_L p_i}{M}$



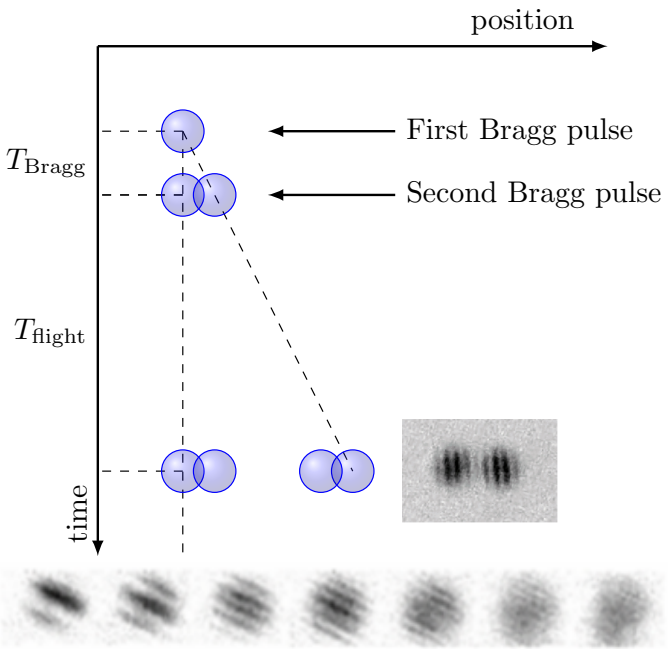
Rabi oscillations between momentum states :

$$|g, \boldsymbol{p} = 0\rangle \xrightarrow[\pi/2 \text{ pulse}]{} \frac{|g, \boldsymbol{p} = 0\rangle + |g, \boldsymbol{p} = 2\hbar k_L\rangle}{\sqrt{2}}$$

A $\pi/2$ pulse acts as a beamsplitter for external degrees of freedom.



Ramsey-Bragg interferometer



$\pi/2$ Bragg pulse as coherent beam splitter

Ramsey scheme to produce interferences of motional degrees of freedom (instead of internal in traditional scheme)

Interference fringes due to the phase profile of an expanding condensate :
$\phi(\boldsymbol{r}+\boldsymbol{d})-\phi(\boldsymbol{r})=\frac{M\,dx}{\hbar t}$

Contrast : $\mathcal{C}=\int g^{(1)}(\boldsymbol{r}+d\boldsymbol{e}_x,\boldsymbol{r})d^3\boldsymbol{r}$

Blue line : Thomas-Fermi envelope

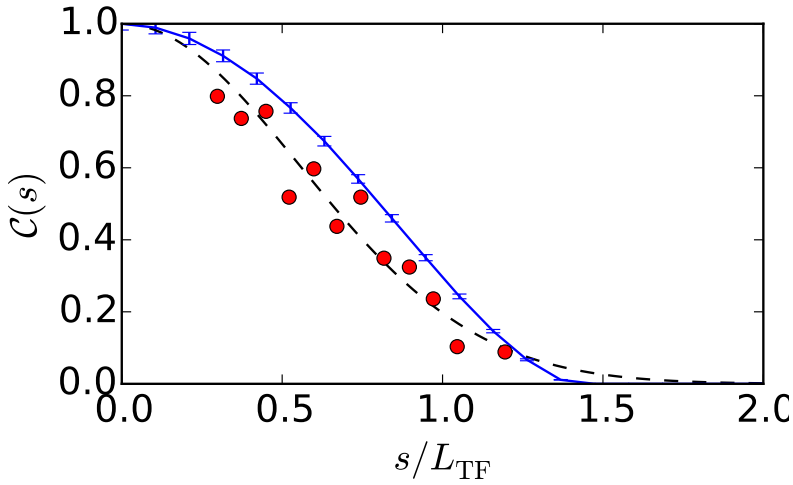
$\mathcal{C}=\int\sqrt{n_{\text{TF}}(\boldsymbol{r}+d\boldsymbol{e}_x)n_{\text{TF}}(\boldsymbol{r})}d^3\boldsymbol{r}$

Dashed line : Gaussian fit $\sigma_c=0.78L_{\text{TF}}$

see also :

E. Hagley *et al.*, PRL 1999

J. Stenger *et al.*, PRL 1999



M. Hugbart *et al.*, EPJD 2005

Penrose-Onsager criterion

Single-particle density matrix :

Symmetric, can be diagonalized (symmetric) with eigenvalues and eigenvectors $(N_0, \phi_0), (N_1, \phi_1), , \cdots$

$$\rho^{(1)}(\boldsymbol{r}, \boldsymbol{r'}) = N_0 \phi_0^*(\boldsymbol{r}) \phi_0(\boldsymbol{r'}) + \sum_{\nu \neq 0} N_\nu \phi_\nu^*(\boldsymbol{r}) \phi_\nu(\boldsymbol{r'})$$

General definition of Bose-Einstein condensation (Penrose-Onsager):

- One (and only one) eigenvalue N_0 is macroscopic,
- all others remain microscopic.

Uniform system (volume $\mathcal{V}$) : plane wave expansion of $\rho^{(1)}$ gives

$$g^{(1)}(|\boldsymbol{r} - \boldsymbol{r'}|) = \frac{n_{\boldsymbol{k}=0}}{n} + \frac{1}{n\mathcal{V}} \int d^3\boldsymbol{k}\, n(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r'})} \underset{|\boldsymbol{r}-\boldsymbol{r'}|\to\infty}{\longrightarrow} \frac{n_{\boldsymbol{k}=0}}{n}$$

At long distances, $\rho^{(1)}$ decays to a non-zero asymptotic value : the condensate fraction.

This property is called off-diagonal long range order (ODLRO), and is considered as the definition of a condensate for an interacting system.

- For an ideal gas at $T = 0$, $N_0 = N$.
- For liquid Helium (also at $T = 0$), neutron scattering measurements indicate $N_0/N \sim$ a few percent.
- What about a weakly interacting gas ? Finite temperatures ?

Spontaneous symmetry breaking

Single-particle density matrix :

$$\rho^{(1)}(\boldsymbol{r},\boldsymbol{r}') = N_0\phi_0^*(\boldsymbol{r})\phi_0(\boldsymbol{r}') + \sum_{\nu\neq 0} N_\nu\phi_\nu^*(\boldsymbol{r})\phi_\nu(\boldsymbol{r}')$$

In terms of field operator :

$$\hat{\psi} = \phi_0\hat{a}_0 + \delta\hat{\psi}$$

Classical field approximation (Bogoliubov): neglect the quantum character of $\hat{a_0}$,

$\hat{a}_0^\dagger\hat{a}_0 \approx \hat{a}_0\hat{a}_0^\dagger \approx N_0$.

$$\hat{a}_0 \to \sqrt{N_0} \qquad \hat{\psi} \to \psi_0 + \delta\hat{\psi}, \qquad \psi_0 = \sqrt{N_0}\phi_0.$$

N_0 denotes the *average* number of atoms in the condensate.

Spontaneous symmetry breaking point of view : $\langle\hat{\psi}\rangle = \psi_0$

The matter wave field acquires a non-zero expectation value, which plays the role of an *order parameter* for the BEC transition. This order parameter is a *classical field*.

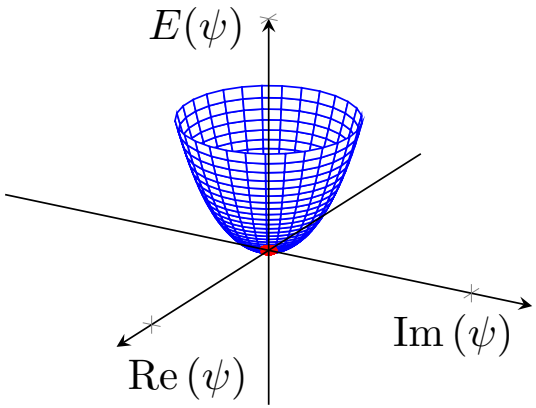
Gauge symmetry (symmetry under *global* gauge transformations $\hat{\psi} \to e^{i\theta}\hat{\psi}$) spontaneously broken.

Landau theory of the BEC transition

Free energy :

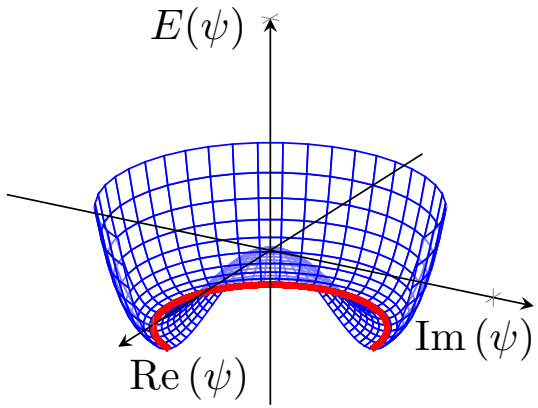
$$E[\psi,\psi^*] = \int d^3\boldsymbol{r}\left(\frac{g}{2}|\psi|^4 - \mu|\Psi|^2\right) + \cdots$$

$T > T_c$



$$\psi_{\mathrm{eq}} = 0$$
$$E[\psi_{\mathrm{eq}}] = 0$$

$T < T_c$



$$\psi_{\mathrm{eq}} = \frac{\mu}{g}$$
$$E[\psi_{\mathrm{eq}}] = -\frac{N_c g n_c}{2}$$

Repulsive interactions stabilize the order parameter :

For an ideal gas, the probability distribution of $N_{\boldsymbol{k}=0}$ is $P(N_{\boldsymbol{k}=0}) \propto e^{-\beta|\mu|N_{\boldsymbol{k}=0}}$.

This leads to very large fluctuations of the population of the condensate $\langle N_{\boldsymbol{k}=0}\rangle \sim N$, $\langle N^2_{\boldsymbol{k}=0}\rangle \sim \langle N_{\boldsymbol{k}=0}\rangle^2$.

This pathology of the grand canonical ensemble disappears when repulsive interactions are included, leading to $\langle N^2_{\boldsymbol{k}=0}\rangle \sim \sqrt{\langle N_{\boldsymbol{k}=0}\rangle} \ll \langle N^2_{\boldsymbol{k}=0}\rangle$.

1. Refresher on second-quantization

2. The Gross-Pitaevskii equation
   - Static properties of a weakly-interacting BEC

3. Superfluid hydrodynamics and collective modes
   - Superfluid hydrodynamics
   - Collective modes: low-energy excitations
   - Superfluid vortices
   - Scaling solutions in a harmonic trap

4. Coherence
   - Interference fringes and first-order coherence
   - Measurement of the coherence length
   - Penrose-Onsager criterion
   - Spontaneous symmetry breaking
   - Higher order correlation functions : second and third order coherences

Second-and third-order coherence

Second-order correlation function :

$$\rho^{(2)}(\boldsymbol{r},\boldsymbol{r}') = \langle \hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}')\rangle = \langle \hat{n}(\boldsymbol{r})\rangle\langle \hat{n}(\boldsymbol{r}')\rangle g^{(2)}(\boldsymbol{r},\boldsymbol{r}')$$

For a pure BEC ($N_0 \approx N$),

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}') = 1$$

For a thermal gas of non-interacting bosons (or more generally, any system with Gaussian statistics),

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}') = 1 + g^{(1)}(\boldsymbol{r},\boldsymbol{r}')$$

Bunching of identical bosons: increased probability to find two bosons at the same point ($g^{(2)}(\boldsymbol{r},\boldsymbol{r}) = 2$).

For an ideal gas of fermions, one finds instead antibunching:

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}') = 1 - g^{(1)}(\boldsymbol{r},\boldsymbol{r}')$$

Decreased probability to find two bosons at the same point ($g^{(2)}(\boldsymbol{r},\boldsymbol{r}) = 0$), consequence of the Pauli exclusion principle.

Second-and third-order coherence

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Measurements of $g^{(2/3)}$: local observables

Two-body interaction energy as a measure of $\rho^2(\boldsymbol{r},\boldsymbol{r})$:

$$E_{\rm int} = \int d^3\boldsymbol{r} d^3\boldsymbol{r}'\, V_{\rm int}(\boldsymbol{r}-\boldsymbol{r}')\langle\hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}')\rangle = \frac{g}{2}\int d^3\boldsymbol{r}\rho^{(2)}(\boldsymbol{r},\boldsymbol{r})$$

- early measurements showed that $g^{(2)}(\boldsymbol{r},\boldsymbol{r}) \approx 1$ (with an uncertainty $\sim 20\%$)

Three-body recombination rate as a measure of $\rho^3(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$:

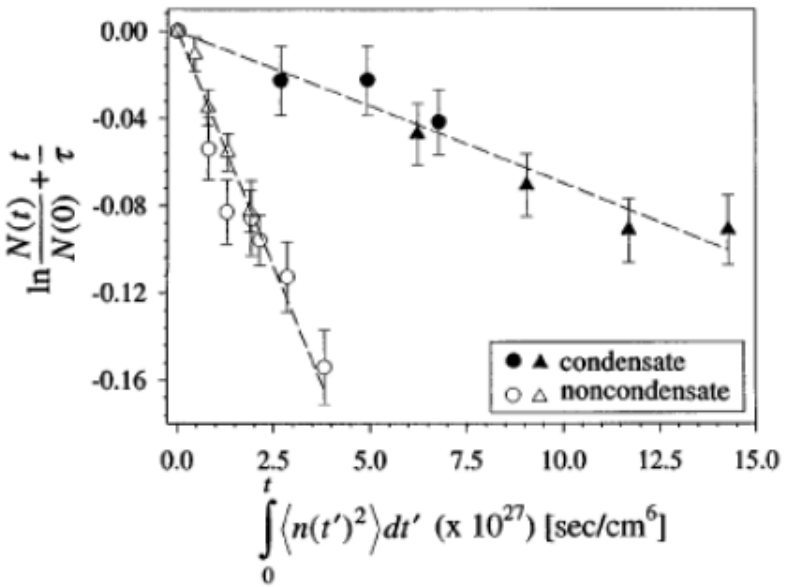
$$\frac{\dot{N}}{N} = -\Gamma_{\rm 3B} = -\gamma_0 \int d^3\boldsymbol{r}\langle\hat{n}^3(\boldsymbol{r})\rangle$$
$$= -\gamma_0 \int d^3\boldsymbol{r}\rho^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$$

BEC:

$$\rho^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r}) = \langle\hat{n}(\boldsymbol{r})\rangle^3$$

thermal gas:

$$\rho^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r}) = 6\langle\hat{n}(\boldsymbol{r})\rangle^3$$



Reduction of three-body recombination as evidence for third-order coherence

[Burt *et al.*, PRL 1997]