Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Examination 14 November 2014

3 hours - lecture notes authorized
Answers to questions can be written either in French or in English

1 Simultaneous measurement of position and momentum

We are interested in the simultaneous measurement of the position x and the momentum p of a 1D particle that is in the quantum state $|\psi\rangle$ described by the wavefunction $\psi(x) = \langle x|\psi\rangle$ where $|x\rangle$ is a position eigenstate.

One gives the integrals

$$\int_{-\infty}^{+\infty} dx e^{-x^2/a^2+bx} = \sqrt{\pi}ae^{a^2b^2/4} \quad \int_{-\infty}^{+\infty} dx x^2 e^{-(x-x')^2/a^2} = \sqrt{\pi}a(x'^2 + a^2/2)$$
$$\int_{-\infty}^{+\infty} dx e^{ikx} = 2\pi\delta(k)$$

We call $\mathcal{P}roba(x_m, p_m)$ the probability density for a joint measurement that gives a position between x_m and $x_m + dx_m$ and a momentum between p_m and $p_m + dp_m$.

1) What is the value of $\mathcal{P}roba(x_m, p_m)$ if one performs an ideal Von Neumann measurement of x, then immediately after, of p?

2) We now use the description of such a measurement within the frame of non-ideal measurements. We call $\hat{\pi}(x_m, p_m)$ the Positive Operator Valued Measure corresponding to the measurement of x, then immediately after, of p. Write $\mathcal{P}roba(x_m, p_m)$ in terms of the POVM. What is the normalization relation fulfilled by the operators $\hat{\pi}(x_m, p_m)$?

3) We consider the quantum state $|x_m, p_m\rangle$ of the particle described by the wave functions in the x and p representations:

$$\psi_{x_m,p_m}(x) = (2\pi\sigma^2)^{-1/4} \exp\left[-\frac{(x-x_m)^2}{4\sigma^2} + i\frac{p_m x}{\hbar}\right] \tag{1}$$

$$\psi_{x_m,p_m}(p) = (2\pi\sigma'^2)^{-1/4} \exp\left[-\frac{(p-p_m)^2}{4\sigma'^2} - i\frac{x_m p}{\hbar}\right] \tag{2}$$

with $\sigma' = \hbar/(2\sigma)$

4) It can be shown that the Gaussian state $|x_m = 0, p_m = 0\rangle$ is a "minimum state", i.e. saturates the Heisenberg inequality: $\Delta x = \sigma; \Delta p = \sigma'$. Show without calculating the integrals that this is also true for all states $|x_m, p_m\rangle$.

5) Show that

$$\int_{-\infty}^{\infty} dx_m \int_{-\infty}^{\infty} dp_m |x_m, p_m\rangle\langle x_m, p_m| = 2\pi\hbar\hat{\mathrm{I}}\,. \tag{3}$$

6) Tell why

$$\hat{\pi}(x_m, p_m) = \frac{1}{2\pi\hbar}|x_m, p_m\rangle\langle x_m, p_m|\,. \tag{4}$$

is a reasonable choice for the probability operators $\hat{\pi}(x_m, p_m)$. To what limit value of σ does the Von Neumann measurement of x correspond ?

7) Give the probability distributions for measuring the values x_m and p_m on state $|\psi\rangle$, then for measuring on the same state and with the same set-up the position x_m only, then the momentum p_m only. Calculate the variances $\Delta' x_m$ and $\Delta' p_m$.

8) What is the value of σ giving the minimum value to the product $\Delta' x_m \Delta' p$? What conclusion do you draw about the accuracy of such a simultaneous measurement?

9) Is it possible to deduce from the expression of the POVM the expression of the state of the particle after the measurement when it has given the values x_m and p_m of these quantities ? Can one have $|\Psi^{out|x_m,p_m}\rangle = |x_m, p_m\rangle$?

10) Imagine an experimental set-up allowing to measure both the position and the velocity of an atom that is likely to scatter in all directions the resonant light to which it is submitted.

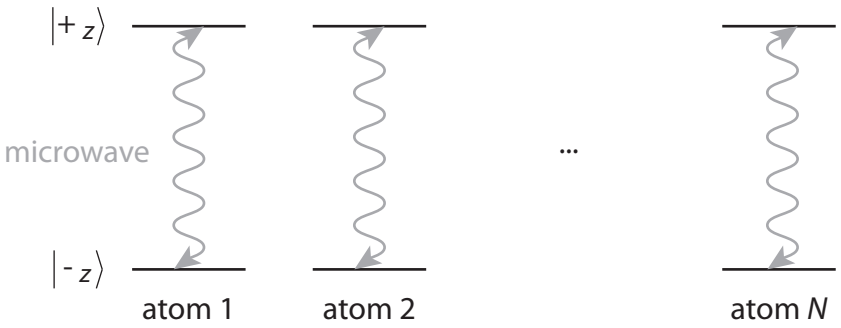
2 Entangled states of more than 40 atoms

This exercise deals with the generation of interesting quantum states of a large number of atoms using a collective measurement technique. It is adapted from a recent article of the group of J. Reichel located at ENS (Science 344, 180 (2014)).

2.1 Symmetric states of N qubits

We are interested in a particular family of quantum states of N qubits - the so-called Dicke states. They can be realized and studied in an ensemble of N identical atoms whose internal degree of freedom can be restricted to a two-level system denoted as $\{|-_z\rangle, |+_z\rangle\}$. We assume that the atoms occupy different positions and are thus distinguishable.

A coherent manipulation of the qubits can be performed by sending a microwave field resonant on the $\{|-_z\rangle \leftrightarrow |+_z\rangle\}$ transition. To a very good approximation the microwave field can be considered uniform on the scale of the atomic ensemble. All qubits are thus driven identically by the microwave field.



1. Consider a single atom irradiated by a microwave of Rabi frequency Ω. The evolution of the atomic state is described by the Hamiltonian

$$\hat{H} = \frac{\hbar\Omega}{2}\hat{\sigma}_y,$$

where $\hat{\sigma}_y$ is the Pauli matrix along y. The atom is prepared in the state $|\psi\rangle = |-_z\rangle$ at $t = 0$. Give the expression of the atomic state $|\psi(t)\rangle$ after an evolution of duration t. We

recall the formulas

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \exp(-i\xi\hat{\sigma}_y) = \cos\xi\,\mathbb{1} - i\sin\xi\,\hat{\sigma}_y.$$

2. Let us now consider an ensemble of N atoms labeled by index i and prepared at $t = 0$ in the state $|0_N\rangle = |-_z, -_z, \ldots, -_z\rangle$. We shine a microwave field of Rabi frequency Ω. Show that for short evolution times the expression of the state can be expanded as

$$\psi(t) = (1 - \alpha^2 t^2/2)\,|0_N\rangle + \alpha t\,|1_N\rangle + \beta t^2\,|2_N\rangle + O(t^3), \tag{5}$$

and give the expression of the coefficients α, β and of the states $|1_N\rangle$, $|2_N\rangle$. In particular one should write the state $|1_N\rangle$ as

$$|1_N\rangle = \frac{1}{\sqrt{N}} \sum_{1\le i\le N} |e_i\rangle\,,$$

where $|e_i\rangle$ is the quantum state with all atoms in $|-_z\rangle$ except the atom i in $|+_z\rangle$. The state $|1_N\rangle$ is a N-particle entangled state called the W state.

3. Show that the W state contains exactly one atom in the state $|+_z\rangle$.

4. As the microwave acts symmetrically on all atoms, a natural symmetry of the problem is described by the operator $\hat{\mathcal{S}}$ defined as

$$\begin{aligned} \hat{\mathcal{S}}\,|\psi\rangle &= |\psi\rangle \quad \text{if } |\psi\rangle \text{ is invariant with respect to the exchange of atoms } i \text{ and } j, \text{ for all pairs } (i,j) \\ &= 0 \quad \text{in the orthogonal subspace.} \end{aligned}$$

Show that the states $|0_N\rangle$, $|1_N\rangle$ and $|2_N\rangle$ belong to the symmetric subspace.

5. One introduces another state with exactly one atom in $|+_z\rangle$, defined as the statistical mixture

$$\hat{\rho}_1 = \frac{1}{N} \sum_{1\le i\le N} |e_i\rangle\langle e_i|\,.$$

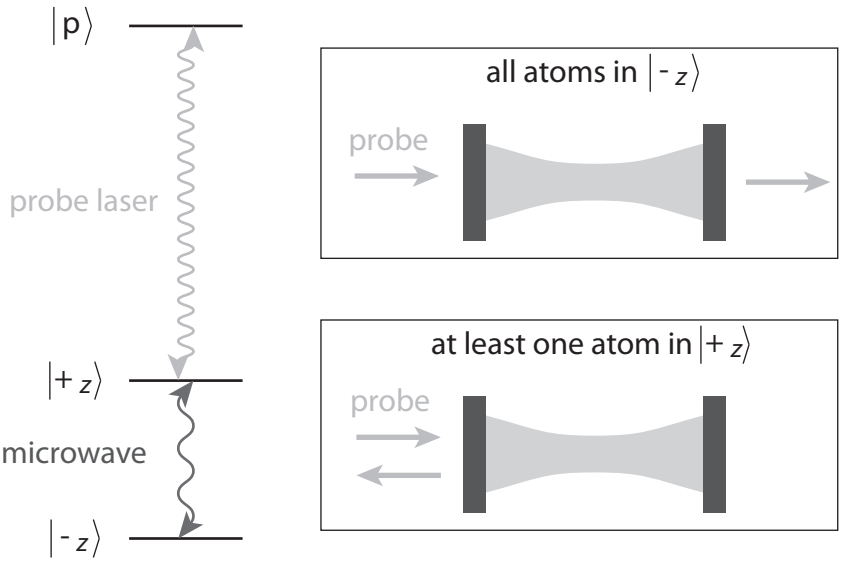
Does $\hat{\rho}_1$ represent a symmetric state? Give the relationship between the density matrix $\hat{\mathcal{S}}\hat{\rho}_1\hat{\mathcal{S}}$ and the state $|1_N\rangle$.

6. Is the state $\hat{\rho}_1$ entangled? How to characterize it ?

2.2 Experimental realization of the W state

The atomic system is probed using a collective QND measurement represented below. The atoms are placed inside a high finesse cavity. The cavity length is tuned in resonance with the wavelength of the $|+_z\rangle \leftrightarrow |p\rangle$ optical transition, where $|p\rangle$ is third internal state that is used only for probing the atoms.

When all the atoms are in the $|-_z\rangle$, the light is unaffected by the presence of the atoms and light is transmitted through the cavity. On the other hand, as soon as one atom occupies the state $|+_z\rangle$, the light becomes strongly coupled to the $|+_z\rangle$ atoms. We admit that this leads to a suppression of the light transmission through the cavity.



In first approximation, measuring the light transmission through the cavity realizes a non-destructive measurement of the state $|0_N\rangle$. One can use this technique to conditionally prepare and characterize of the W state.

One initializes an ensemble of atoms in the state $|0_N\rangle$ and sends a weak excitation in order to prepare the state $|\psi(t)\rangle$ written in equation (**??**). One then sends the probe light to measure the cavity transmission.

7. What is the probability to observe a non-zero cavity transmission, to second order in t?

8. What is the quantum state after measuring a non-zero cavity transmission?

9. What is the quantum state after measuring a zero cavity transmission?

10. Propose a procedure leading to the preparation of the W state $|1_N\rangle$.

In practice the measurement suffers from several imperfections. The most important one consists in a residual probability $p \simeq 0.1$ that the atom in the state $|+_z\rangle$ may scatter one photon from the probe laser.

We recall that the atoms can be distinguished as they occupy different positions $\mathbf{r}_i$. By observing from which position the photon is scattered one can thus reveal which atom is in the state $|+_z\rangle$.

11. Assume that one observes a photon scattered from the position $\mathbf{r}_i$. Give the expression of the post-measurement operator associated with this measurement technique. What is the post-measurement state?

12. In the experiment the only information we have is the probability p that the atom in the state $|+_z\rangle$ scatters one photon. Show that the experimental procedure described here actually leads to the final state

$$\hat{\rho} = (1-p)\,|1_N\rangle\langle 1_N| + p\,\hat{\rho}_1.$$

Write the quantum map describing the evolution from $|1_N\rangle$ to the mixed state $\hat{\rho}$ in terms of Kraus operators.

2.3 Decoherence of the W state

We study in this part the decoherence of the W state due to the thermal random dephasing of the $|+_z\rangle$ states.

The atoms are held in a harmonic trap whose amplitude is slightly dependent on the atom internal states. Consider an atom i prepared in the state $|-_z\rangle$ in an eigenstate of the harmonic oscillator of energy E_i. If the microwave drives it to the excited state $|-_z\rangle$, the energy associated with its motion will then be slightly modified to a value $(1+\eta)E_i$, with $\eta \ll 1$.

13. Assume that one prepares the W state at $t = 0$. Show that it evolves according to

$$|\psi(t)\rangle = \frac{1}{\sqrt{N}} c_i(t) |e_i\rangle, \quad \text{where} \quad c_i(t) = e^{-i\eta E_i t/\hbar}.$$

14. Because the atoms are not at zero temperature, they are not all in the same harmonic oscillator energy state E_i. Show that the thermal average of the overlap with the W state can be written as

$$|\langle 1_N|\psi(t)\rangle|^2 = \frac{1}{N} + \frac{N(N-1)}{N^2} C(t), \quad \text{where} \quad C(t) = \left\langle e^{-i\eta E t/\hbar} \right\rangle^2.$$

The thermal averaging can be calculated using the formula

$$\left\langle e^{-i\eta E t/\hbar} \right\rangle = \int p(E) e^{-i\eta E t/\hbar} dE,$$

where $p(E) = \beta^3 E^2 e^{-\beta E}/2$ is the density of states in a harmonic trap and $\beta = 1/(k_B T)$.

15. Give the explicit expression of the function $C(t)$ and show that it can be written as a function of a dimensionless parameter t/τ_D. Give the expression of the characteristic decoherence time τ_D. How does it vary with the parameter η and the temperature T?

16. Show that the thermally averaged quantum state can be described by the density matrix

$$\hat{\rho}(t) = C(t) |1_N\rangle \langle 1_N| + (1 - C(t))\hat{\rho}_1.$$

17. One gives the value of the parameter $\eta = 2 \times 10^{-5}$. Can you estimate the temperature of the atomic ensemble from the decoherence data below?

