Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Exam 2015

3 hours - lecture notes authorized.
Answers to questions can be written either in French or in English.
Please write your answers for the two exercices on **separate** sheets.

1 Conditional preparation of a single photon qubit $\lambda|0\rangle + \mu|1\rangle$

1.1 Introduction

We recall that a single mode quantum field of frequency ω is mathematically equivalent to a harmonic oscillator, with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ fulfilling $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{N} = \hat{a}^\dagger\hat{a}$ being the photon number operator, and $|n\rangle$ denoting the n-photon state, eigenstate of $\hat{N}$ with eigenvalue n.

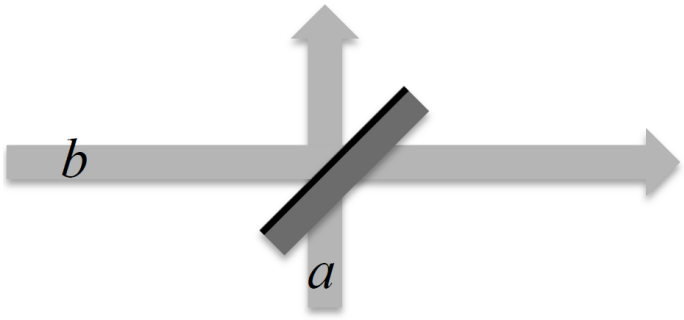
Experimentally, one is able to straightforwardly produce the vacuum state $|0\rangle$, the single photon state $|1\rangle$ and the so-called coherent state $|\alpha\rangle$, where α is any complex number, that is generated by a single mode laser. The coherent state can be expanded over the n-photon state basis as:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{1}$$

in which the c_n coefficients are non zero for all the photon numbers n.

The aim of the exercise is to show how to produce from these resources and in a conditional way any quantum state in the two-dimensional (or qubit) space spanned by the photon states $|0\rangle$ and $|1\rangle$ only, hence the name of "quantum scissors" given to this device, able to "cut" in the state the contribution of all n values larger than 1.

1.2 Beamsplitter



A beamsplitter is a glass plate that couples by reflection and transmission two field modes of same frequency and different directions, labeled a and b, and described respectively by annihilation operators $\hat{a}$ and $\hat{b}$ (see figure above). A two-mode input field quantum state $|\psi_{ab}^{in}\rangle$ is transformed by a beamsplitter of 50% reflection and 50% transmission into an output state $|\psi_{ab}^{out}\rangle$ such that:

$$|\psi_{ab}^{out}\rangle = \hat{U}_{ab}|\psi_{ab}^{in}\rangle \quad , \quad \hat{U}_{ab} = e^{\phi \hat{M}_{ab}} \quad ; \quad \hat{M}_{ab} = \hat{a}^\dagger \hat{b} - \hat{a}\hat{b}^\dagger \tag{2}$$

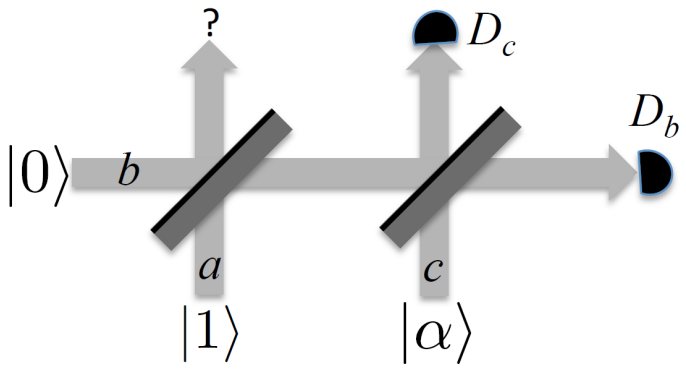
with $\phi = \pi/4$.

1. Show that $|\psi_{ab}^{out}\rangle = (|a:0,b:1\rangle - |a:1,b:0\rangle)/\sqrt{2}$ when $|\psi_{ab}^{in}\rangle$ is equal to $|a:0,b:1\rangle$, where $|a:m,b:m'\rangle$ describes the state with m photons in mode a and m' photons in mode b. Calculate also $|\psi_{ab}^{out}\rangle$ when $|\psi_{ab}^{in}\rangle$ is respectively equal to $|a:0,b:0\rangle$ and $|a:1,b:0\rangle$. Are these expressions plausible ? Hint: calculate $\hat{M}_{ab}^{2n}|a:0,b:1\rangle$ and $\hat{M}_{ab}^{2n+1}|a:0,b:1\rangle$.

2. Calculate also $\hat{U}_{ab}^\dagger|a:0,b:1\rangle$.

1.3 Ideal quantum scissors

One considers now the two-beamsplitter set-up pictured below, which couples three modes labeled a, b and c, transforming a three-mode input state $|\psi_{abc}^{in}\rangle$ into an output state $|\psi_{abc}^{out}\rangle = \hat{U}_{abc}|\psi_{abc}^{in}\rangle$.



3. Give the expression of $\hat{U}_{abc}$ in terms of $\hat{U}_{ab}$ and $\hat{U}_{bc}$.

4. The input field state consists of a single photon in mode a, the vacuum in mode b, and a coherent state in mode c: $|\psi_{abc}^{in}\rangle = |a:1,b:0,c:\alpha\rangle$. Determine the conditional state of the system in mode a $|\psi_{a,cond}^{out}\rangle$ after an ideal photon number measurement by photon counters D_b and D_c has been performed on output modes b and c which has given the result: one photon in b and no photon in c.

5. What is the probability of such an event?

6. Write the post-measurement operator associated with such a partially destructive measurement.

7. In a coherent state, $c_1/c_0 = \alpha$. For which value of α do we obtain the conditional state $(|0\rangle + |1\rangle)/\sqrt{2}$ in mode a ?

8. One can show that $\hat{U}_{ab}|a:1,b:1\rangle = (|a:2,b:0\rangle - |a:0,b:2\rangle)\sqrt{2}$. Is this result intuitive? One has also $\hat{U}^{\dagger}_{ab}|a:1,b:1\rangle = (|a:0,b:2\rangle - |a:2,b:0\rangle)\sqrt{2}$.

9. One now takes as an input state the state $|\psi'^{in}_{abc}\rangle = |a:1,b:1,c:\alpha\rangle$. Determine the conditional state of the system in mode a $|\psi'^{out}_{a,cond}\rangle$ after an ideal photon number measurement by photon counters D_b and D_c has been performed on output modes b and c which has given the result: one photon in b and no photon in c.

1.4 Non ideal quantum scissors

Actually photon counters D_b and D_c are not perfect. In this section, we go back to the situation described in section 1.3.b. In the following questions, one will use the expressions of the probability operators given in the lectures.

10. Will the result be the same if the detector D_b is not photon number resolving, i.e. gives the same click when the input state is not only $|1\rangle$, but also $|n\rangle$ whatever $n > 1$?

11. Give the state produced conditionally and its probability of occurrence when the detector D_b has a limited quantum efficiency η; Explain physically the result.

12. Is the state produced conditionally the same when the detector D_c has unit quantum efficiency, and dark counts with probability p? Same question when the detector D_b has unit quantum efficiency and dark counts with probability p?

2 Schmidt decomposition

We consider a quantum system made of two qubits labelled a and b. The total Hilbert space $\mathcal{H}_{ab}$ is given by the tensor product of the two Hilbert spaces associated with each qubit, as $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$.

We have seen in the lectures that an arbitrary state $|\psi\rangle$ can be written as a Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{2} \sqrt{\lambda_i}\, |u_i\rangle_a\, |v_i\rangle_b\,, \tag{3}$$

where $\{|u_i\rangle_a\}_{i=1,2}$ is an orthogonal basis for the qubit a (and similarly for b). The coefficients λ_i are positive real numbers.

1. Consider the quantum states

$$|\phi_1\rangle = \sqrt{\frac{2}{3}}\,|00\rangle - \sqrt{\frac{1}{3}}\,|11\rangle\,,$$
$$|\phi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2},$$
$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}.$$

   Calculate the reduced density matrices $\rho_a = \mathrm{Tr}_b\,|\psi\rangle\langle\psi|$ and $\rho_b = \mathrm{Tr}_a\,|\psi\rangle\langle\psi|$ for those states.

2. Find the Schmidt decomposition associated with the states $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$.

We now consider a system made of three qubits a, b and c. A Schmidt decomposition for a quantum state $|\psi\rangle$ would be defined as

$$|\psi\rangle = \sum_{i=1}^{2} \sqrt{\lambda_i}\,|u_i\rangle_a\,|v_i\rangle_b\,|w_i\rangle_c\,, \tag{4}$$

where $\lambda_i \geq 0$ and the $\{|u_i\rangle_a\}_{i=1,2}$, $\{|v_i\rangle_b\}_{i=1,2}$, $\{|w_i\rangle_c\}_{i=1,2}$ are orthogonal basis for the qubits a, b and c, respectively.

Let us a consider the so-called GHZ (Greenberger-Horne-Zeilinger) state

$$|\psi_{\mathrm{GHZ}}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

3. Write down the two-particle density matrix ρ_{ab} corresponding to the qubits a and b. Does it correspond to an entangled state?

4. Quantum states of several qubits are called *maximally entangled* when the single-qubit density matrices (here ρ_a, ρ_b, ρ_c) are proportional to the identity. Show that the GHZ state is maximally entangled.

5. Can one write the GHZ state as a Schmidt decomposition (4)?

We now consider another famous state, the so-called W (Werner) state

$$|\psi_{\mathrm{W}}\rangle = \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}.$$

6. Write down the two-particle density matrix ρ_{ab} corresponding to the qubits a and b. Does it correspond to an entangled state of two qubits?

7. Based on the previous questions, one of the two states $|\psi_{\mathrm{GHZ}}\rangle$ and $|\psi_{\mathrm{W}}\rangle$ exhibits an entanglement between particles more robust with respect to qubit losses. Which one?

8. Show that the W state cannot be written as a Schmidt decomposition (4). It may be useful to consider the reduced density matrices ρ_a, ρ_b, ρ_c.