Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Exercise class n°1: Observing and manipulating individual qubits in ion traps

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Single trapped ions are model quantum systems which are very well isolated from the environment. Once brought almost to rest using laser cooling, one can coherently manipulate their internal electronic degrees of freedom using laser fields. The very high degree of control one can achieve make them ideal for experimental studies of small quantum systems. We discuss here how to encode a single qubit with one ion, and how to make state-of-the-art measurements of its quantum state.

1 Preliminary: Bloch sphere for mixed states of a qubit

Before dealing with quantum measurements in ion traps we review some basic properties of density matrices on the simplest system of a single qubit.

1. Let us consider a single qubit in a mixed state. Show that its density matrix can be written in terms of the Pauli matrices as

$$\hat{\rho} = \frac{1}{2}\left(\hat{\mathbb{1}} + r_1\hat{\sigma}_1 + r_2\hat{\sigma}_2 + r_3\hat{\sigma}_3\right).$$

   The vector $\mathbf{r}$ is called the Bloch vector associated with the density matrix.

2. Show that the components of the Bloch vectors are given by

$$r_i = \mathrm{Tr}\left[\hat{\rho}\,\hat{\sigma}_i\right].$$

3. What are the possible values for the norm $||\mathbf{r}||$ of the Bloch vector? Show that $||\mathbf{r}|| = 1$ corresponds to a pure quantum state.

4. Calculate the value of the purity of a density matrix of Bloch vector $\mathbf{r}$.

5. Determine the direction of the Bloch vector $\mathbf{r}$ for a general pure state

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\,, \quad \theta \in [0,\pi]\,,\ \phi \in [0,2\pi)\,. \tag{1}$$

6. We consider two density matrices $\hat{\rho}_1$ and $\hat{\rho}_2$. Show that their overlap is given by

$$O_{12} = \mathrm{Tr}\left(\hat{\rho}_1\hat{\rho}_2\right) = \frac{1}{2}\left(1 + \mathbf{r}_1\cdot\mathbf{r}_2\right).$$

2 Preparation of a trapped ion as a single qubit

Single ions can be stored in vacuum using suited combinations of electrostatic potentials and radio-frequency fields, essentially leading to a conservative harmonic trapping with a cylindrical symmetry

$$\hat{V} = \frac{1}{2}m\omega_r^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2,$$

where m is the ion mass. In the so-called linear traps the axial trapping frequency ω_z is much smaller that the radial one ω_r. In the following we assume for simplicity that the only motional degree of freedom is along the z direction.

Internal electronic degrees of freedom can be manipulated using quasi-resonant laser fields. We first consider the coupling between two electronic states $|g\rangle$ and $|e\rangle$ using a laser field $\mathcal{E}e^{i(kz-\omega t)}\boldsymbol{\epsilon}$.

1. Show that the Hamiltonian can be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_z^2\hat{z}^2 + \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \frac{1}{2}\hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\left(e^{i(kz-\omega t)} + e^{-i(kz-\omega t)}\right)$$

   where $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$ and $\hbar\omega_0$ is the energy separation between $|g\rangle$ and $|e\rangle$. What is the value of the Rabi frequency Ω?

2. By introducing the annihilation operator $\hat{a}$ associated to the harmonic motion along z, show that the Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega_z\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0\hat{\sigma}_z + \frac{1}{2}\hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\left(e^{i\eta(\hat{a}^\dagger+\hat{a})}e^{-i\omega t} + e^{-i\eta(\hat{a}^\dagger+\hat{a})}e^{i\omega t}\right),$$

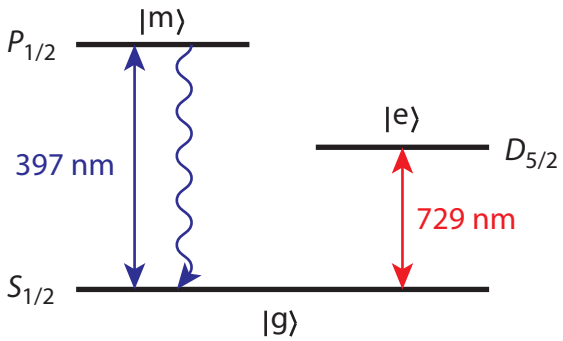
   where η is a dimensionless number called the Lamb-Dicke parameter.

3. Close to the resonance $\omega \simeq \omega_0$, it is natural to make use of the interaction picture. Recall the principle of the interaction picture, and show that it leads to a simplified Hamiltonian

$$\hat{H} = \hbar\omega_z\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{1}{2}\hbar\Omega\left(e^{i\eta(\hat{a}^\dagger+\hat{a})}\hat{\sigma}_+e^{-i\delta t} + e^{-i\eta(\hat{a}^\dagger+\hat{a})}\hat{\sigma}_-e^{i\delta t}\right), \tag{2}$$

   after removing fast oscillating terms and introducing the detuning $\delta = \omega - \omega_0$.

4. We consider the case of a $^{40}Ca^+$ ion held in a trap of axial frequency $\omega_z = 2\pi \times 1\,\text{MHz}$. The relevant electronic levels are displayed below (the level $|m\rangle$ will be used in the next section).



Explain why the ground state has a $S_{1/2}$ electronic configuration. The electronic level $|e\rangle$ is a metastable state of lifetime $T \simeq 1\,\mathrm{s}$. Can you explain why it does not decay faster to the ground state? Calculate the value of the Lamb-Dicke parameter η for the 729 nm 'clock' transition.

5. The Lamb-Dicke regime corresponds to small values of $\eta \ll 1$. The main physical content of the Hamiltonian (2) is captured by expanding it to first order in η. Show that in this picture only a few transitions $|g, n\rangle \leftrightarrow |e, m\rangle$ are allowed and give the value of their Rabi frequency Ω_{nm}.

6. Explain why one expects only three resonance frequencies, denoted as carrier frequency and red/blue sidebands.

7. Explain how the excitation of one of the sidebands can be used to optically pump the ion into its motional ground state. This mechanism called sideband cooling is extensively used in ion trap experiments for the state preparation.

3 Detection of internal states: the electron shelving technique

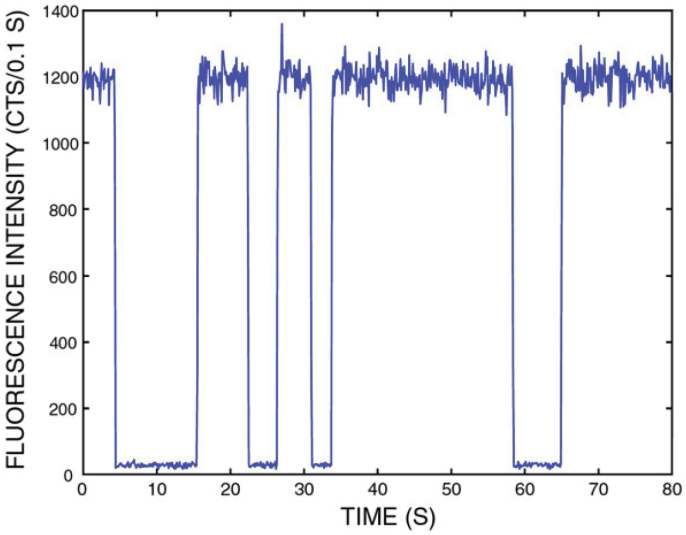
In this section we assume that the ion remains in its motional ground state. The levels $|g\rangle$ and $|e\rangle$ thus form an effective spin 1/2 that can be manipulated using laser light at the carrier frequency.

3.1 Principle of the measurement and estimation of the error probability

The measurement of σ_z makes use of a third level $|m\rangle$ coupled with the ground state $|g\rangle$ by a broad optical transition at $\lambda = 397\,\mathrm{nm}$ (natural linewidth $\Gamma = 2\pi \times 22\,\mathrm{MHz}$).

1. We shine a laser at $\lambda = 397\,\mathrm{nm}$. Explain how the observable σ_z can be measured via the observation of the ion fluorescence.

2. We show below a typical measurement of the fluorescence over time detected by a photomultiplier. Interpret the observation of plateaus of fluorescence at two possible values.



In practice stray light can never be completely avoided and one observes photon counts on the photomultiplier even when the ion is in the $|e\rangle$ state. This limits the maximum efficiency of the detection. Let us denote Γ the photon count rate observed for the level $|g\rangle$ and $\epsilon\,\Gamma$ the rate for the state $|e\rangle$ (of course $\epsilon \ll 1$).

3. Assuming that the photon counts are Poissonian random variables, calculate for both levels the mean values and variances of the photon counts measured for a duration τ.

4. The state discrimination is performed by defining a threshold value S such that if the number of counts is larger than S the ion is assumed to be in the $|g\rangle$ level. Show that the likelihood of errors is minimized for $S = \sqrt{\epsilon}\,\Gamma\tau$ and give the minimal error probability.

5. Show that for long measurement times the minimal error probability is given by

$$P_{\text{error}} = \frac{e^{-\Gamma\tau/2}}{\sqrt{2\pi\Gamma\tau}}.$$

6. For long measurement times τ one has to take into account the finite lifetime T of the excited level $|e\rangle$. In practice we will choose $\tau \ll T$. Show that for an atom in $|e\rangle$ one can approximate the photon count probability distribution by

$$p(n) = \left(1 - \frac{\tau}{T}\right) p_{T=\infty}(n) + \frac{1}{\Gamma T}\Theta(\Gamma\tau - n),$$

where Θ is the Heaviside function.

7. Assuming that the optimal threshold value S remains unchanged, show that the error probability becomes

$$P_{\text{error}} = \frac{e^{-\Gamma\tau/2}}{\sqrt{2\pi\Gamma\tau}} + \frac{\tau}{2T}.$$

8. What is the optimal value of τ ? Show that the minimum error probability is

$$P_{\text{error}} \simeq \frac{\log(\Gamma T)}{\Gamma T}$$

and evaluate it for typical experimental parameters $\epsilon = 1/10$, $\Gamma = 10^4\,\text{s}^{-1}$ and $T = 1\,\text{s}$.

3.2 Describing measurements with errors

9. Let use assume for simplicity that when measuring $\hat{\sigma}_z$ one records the wrong state with probability p (independent of the state). What are the probabilities p_g and p_e for the two measurement outcomes?

10. Show that these probabilities can be written as a Born rule result

$$p_g = \text{Tr}\,(\hat{\rho}\,\hat{\pi}_g), \quad p_e = \text{Tr}\,(\hat{\rho}\,\hat{\pi}_e),$$

where $\hat{\pi}_g$ and $\hat{\pi}_e$ are two probability operators. Are they projectors? Show that they form a probability operator measure.

3.3 Quantum state tomography

11. We would like to measure the full density matrix $\hat{\rho} = \frac{1}{2}\left(\hat{\mathbb{1}} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}}\right)$ using repeated measurements on many copies of the ion prepared in identical conditions. Give a protocol for measuring the Bloch vector $\mathbf{r}$ using the electron shelving technique (we neglect here P_{error}).

Reference: *Observation of quantum jumps in a single atom*, J. C. Bergquist, R. G. Hulet, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. 57, 1699 (1986)

4 Application: the quantum Zeno effect

The quantum Zeno effect is the inhibition of quantum state evolutions due to frequent measurements. Ion traps constitute an ideal system for its observation on the simplest quantum system of a single qubit.

1. We shine the clock laser at the carrier frequency, inducing $|g\rangle \to |e\rangle$ transitions with a Rabi frequency Ω. Show that the Bloch vector evolves as

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\mathbf{r},$$

   where $\boldsymbol{\Omega} = \Omega\,\mathbf{u}_x$.

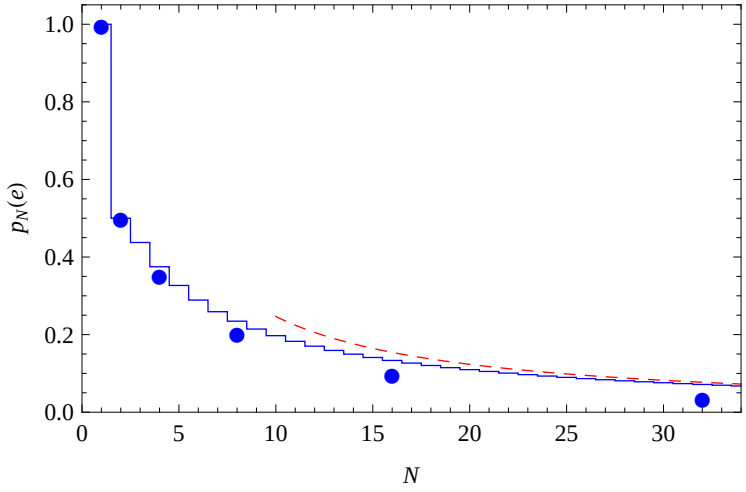
2. Let us consider an ion prepared in the state $|g\rangle$. What is the quantum state after an evolution under the clock laser of duration $T = \pi/\Omega$?

3. We compare the former evolution with an evolution of same total duration T but with additional measurements of $\hat{\sigma}_z$ performed at the times $t_n = nT/N$, where N and $1 \leq n \leq N$ are integers. What is the expression of the density matrix right before the first measurement? And right after it?

4. What is the exact expression of the density matrix after the last measurement? Show that the probability to find the ion in the state $|e\rangle$ is given by

$$p_N(e) = \frac{1}{2}\left[1 - \cos^N(\pi/N)\right] \underset{N\gg 1}{\simeq} \frac{\pi^2}{4N}.$$

5. The measurements performed by the group of D. Wineland are represented below, together with the theoretical value of $p_N(e)$ (the asymptotic expresson is in dashed line). Can you explain possible reasons for the discrepancy of the measurements compared with theory for large values of N ?



Reference: *Quantum zeno effect*, W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A 41, 2295 (1990)