Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Exercise class n°4: Cavity quantum electrodynamics

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We discuss here the principle of the photon number measurement performed in a high-finesse cavity in the group of S. Haroche. We show that this measurement scheme does not destroy the photon quantum state and can be used to create non-classical states.

1 What is a QND measurement?

1. Let us consider a perfect avalanche photodiode which 'clicks' when it receives a photon number $n \geq 1$. What is the photon state after the detection? Explain why this measurement can be modeled by Kraus operators $\hat{K}_n = |0\rangle \langle n|$. What are the corresponding measurement operators $\hat{\pi}_n$? Show that the $\hat{\pi}_n$'s form a probability operator measure.

2. Show that the Kraus operators $\hat{K}_n = |n\rangle \langle n|$ are associated with the same probability operator measure, but correspond to different measurement outcomes. Are Fock states affected by such a measurement? What about coherent states?

3. Consider the von Neumann position measurement of free particle. Let us assume that one measures the particle position to be $|x\rangle$. What is the quantum state right after the measurement? Is it certain to measure the position x right after the measurement? And after a finite waiting time?

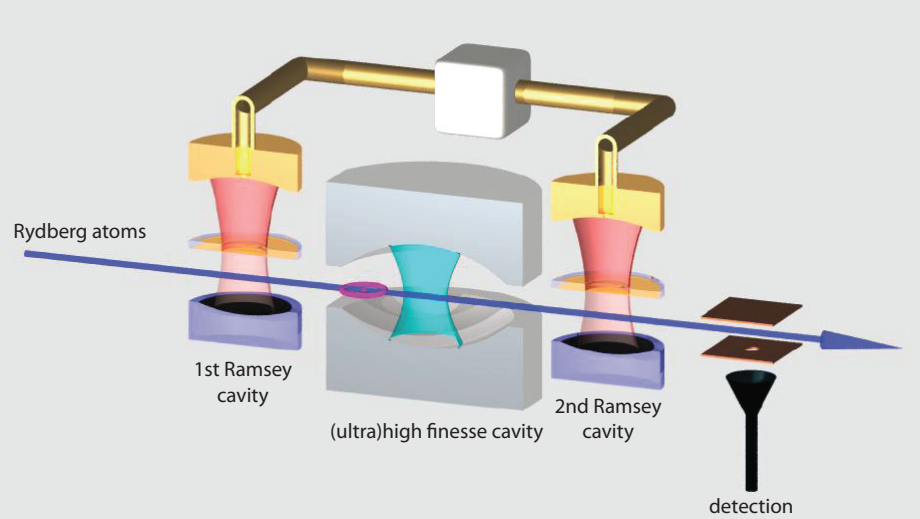
A quantum non-demolition measurement obeys the following properties:

- The measurement is non-destructive, i.e. the measurement itself does not modify the observable value.
- It gives twice the same answer for two measurements whatever the time interval between them.

4. Show that a quantum non-demolition measurement requires the observable to commute with the free evolution Hamiltonian and with all Kraus operators associated with the measurement. Show that the photon number measurement with Kraus operators $\hat{K}_n = |n\rangle \langle n|$ and performed in a high-finesse cavity corresponds to a QND measurement.

2 Principle of the photon number measurement

We discuss in this section the measurement of small photon numbers in a high-finesse cavity, as implemented in the Haroche group. A scheme of the experimental setup is shown below.



The high finesse cavity is placed inside a Ramsey interferometer formed by two lower finesse cavities. Rydberg atoms are sent through the three cavities, and subsequently detected via ionization.

We recall here the principle of a Ramsey interferometer. Two internal states $|g\rangle$ and $|e\rangle$ are manipulated in the Ramsey cavities using resonant microwave pulses. As described in exercise class 1, the interaction between the atom and the electromagnetic field can be described by the Hamiltonian

$$\hat{H} = \frac{\hbar\delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}(\hat{\sigma}_y \cos\phi_r - \hat{\sigma}_x \sin\phi_r),$$

where $\delta = \omega - \omega_0$ is the detuning between the microwave frequency ω and the atomic transition frequency ω_0. ϕ_r is the phase of the Ramsey electric field and Ω is the atom-field Rabi frequency (proportional to the electric field amplitude).

5. The first Ramsey cavity is used to perform the following unitary transform ('$\pi/2$ pulse'):

$$|g\rangle \rightarrow \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle).$$

   What should be the value of the phase ϕ_r in the first cavity? What should be the value of the Ramsey frequency Ω, knowing that the atom interacts with the field for a given time τ?

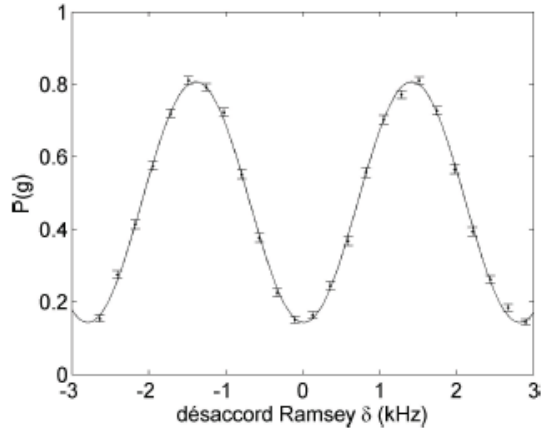
6. What is the quantum state of the atom at the entrance of the second Ramsey cavity if one assumes a free evolution of duration T and a fixed detuning δ?

7. Show that the atom quantum state after a second Ramsey $\pi/2$ pulse of phase ϕ_r is given by

$$e^{-i\phi_r/2} \sin\left(\frac{\phi_r - \delta T}{2}\right)|e\rangle + ie^{i\phi_r/2} \cos\left(\frac{\phi_r - \delta T}{2}\right)|g\rangle\,.$$

8. The ionization detector is able to detect single Rydberg atoms with high fidelity and discriminate between $|g\rangle$ and $|e\rangle$. What is the probability to measure $|e\rangle$?

A typical Ramsey interferometry experiment is shown below, where the probability to measure $|g\rangle$ is plotted as a function of the detuning δ.



In the rest of the exercise sheet we assume that the Ramsey cavities are tuned to resonance, i.e. $\delta = 0$.

The Ramsey interferometer is used to measure the photon number in the high-finesse cavity as described in the following. We assume that the Rydberg atoms are strongly detuned from the cavity resonance frequency ω' and one introduces the detuning $\delta' = \omega' - \omega_0$.

9. Assume that the cavity contains exactly n photons. Show that the atom-photon coupling leads to a slight modification of the eigenenergies of $|g\rangle$ and $|e\rangle$ states of the form

$$\Delta E_{|g\rangle} \simeq \alpha \frac{I(\mathbf{r})}{\delta'} n, \quad \Delta E_{|e\rangle} \simeq -\alpha \frac{I(\mathbf{r})}{\delta'} n,$$

where α is a constant and $I(\mathbf{r})$ is the intensity of the electric field corresponding to one photon.

10. Show that the quantum state of the Rydberg atom after passing through the cavity reads

$$\frac{1}{\sqrt{2}} \left(e^{i\frac{\Phi_0}{2}n} |g\rangle + e^{-i\frac{\Phi_0}{2}n} |e\rangle \right),$$

where you should give the formal expression for Φ_0.

11. Show that the probability to measure the atom in $|e\rangle$ (after the second Ramsey pulse) is given by

$$P_e(n, \phi_r) = \cos^2 \left[\frac{\Phi_0}{2} n - \phi_r \right].$$

12. Assume that one has no information on the quantum state of the cavity photons and that the Rydberg atom is detected in $|e\rangle$. What is the probability $p(n|e)$ that the photon number is n? What is the post-measurement quantum state?

13. Show that, more generally, this measurement can be described by Kraus operators $\hat{K}_g$ and $\hat{K}_e$ that you should write explicitly. What are the corresponding probability operator measures $\hat{\pi}_g$ and $\hat{\pi}_e$?

14. Show that these Kraus operators fulfill the requirements for a quantum non-demolition measurement.

3 Implementation of the photon number measurement

We assume that the cavity is placed in a cryogenic environment in order to maintain the number of thermal photons small. We will assume that the photon number n is necessarily smaller than 8.

We choose the measurement parameters $\Phi_0 = \pi/4$ and $\phi_r = 0$.

15. Plot the conditional probabilities $P_e(n)$.

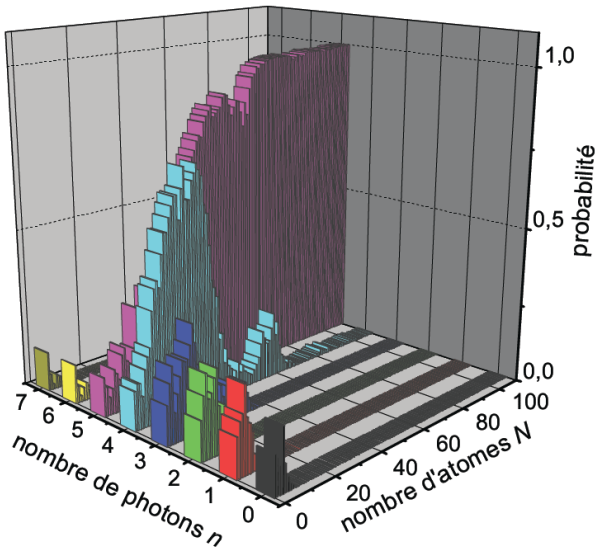
16. More information on the photon number can be obtained by repeating the measurement N times using N Rydberg atoms. The i^{th} measurement outcome is written as $\sigma_i = 0$ for an atom detected in $|g\rangle$ and 1 otherwise. Show that the photon probability distribution is given by

$$p(n) = \frac{1}{Z}\prod_{i=1}^{N}[1 + \cos(n\Phi_0 + \sigma_i\pi)].$$

In practice a more efficient discrimination between photon numbers can be achieved by alternating the Ramsey phase ϕ_r.

17. Guess what is the expected measurement effect for a large number of atoms $N \gg 1$.

18. We show below an experimental evolution of $p(n)$ as a function of the atom number N. Comment on this evolution and on the steady state reached after $\simeq 100$ atoms.



Reference: *Progressive field-state collapse and quantum non-demolition photon counting*, C. Guerlin, J. Bernu, S. Deléglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J.-M. Raimond, S. Haroche, Nature **448**, 889 (2007)

4 Relaxation of the cavity field in a thermal environment

The finite finesse of the cavity leads to a coupling of the cavity field to a thermal environment. Two basic processes can occur: the cavity field either gains or lose one photon from the environment. These processes are described by jump operators $\hat{L}_+$ and $\hat{L}_-$.

19. Show that the evolution of the density matrix is given by the Lindblad equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -i\omega'[\hat{n},\hat{\rho}] + \sum_{\mu=\pm}\left(\hat{L}_\mu\hat{\rho}\hat{L}_\mu^\dagger - \frac{1}{2}(\hat{L}_\mu^\dagger\hat{L}_\mu\hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger\hat{L}_\mu)\right).$$

In the following we assume that the Hamiltonian evolution (the term in ω') can be removed by using a proper interaction picture.

20. Show that a natural choice for the jump operators is given by

$$\hat{L}_+ = \sqrt{\kappa_+}\hat{a}^\dagger, \quad \hat{L}_- = \sqrt{\kappa_-}\hat{a},$$

where $\hat{a}$ is the photon annihilation operator and the κ_μ are (unknown) relaxation rates.

21. A thermal density matrix should not evolve under the Lindblad equation. Show that this allows one to write the relaxation rates as

$$\kappa_- = \kappa(n_{\mathrm{th}}+1), \quad \kappa_+ = \kappa\, n_{\mathrm{th}},$$

where n_{th} is the mean number of thermal photons in the cavity.

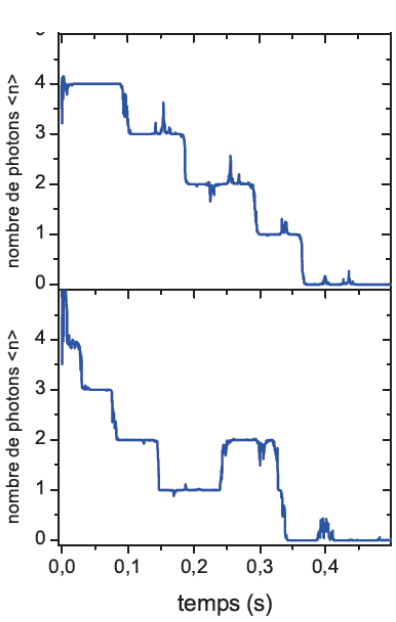
22. Show that the mean number of photons relaxes as

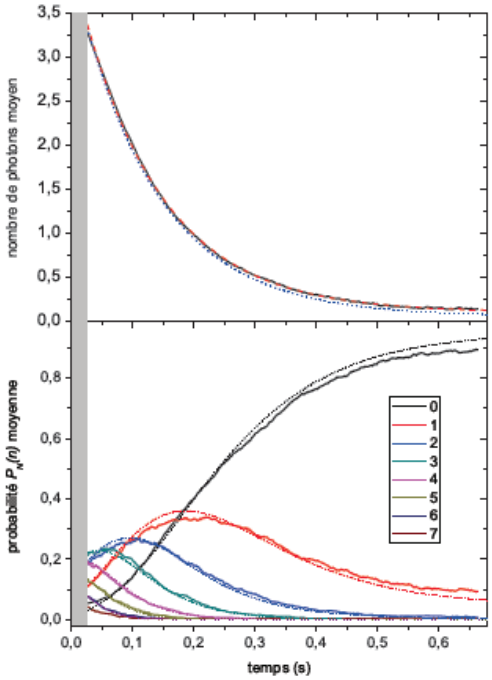
$$\frac{\mathrm{d}\langle\hat{n}\rangle}{\mathrm{d}t} = -\kappa(\langle\hat{n}\rangle - n_{\mathrm{th}}).$$

23. Assume that one prepares a Fock state $|n\rangle$ at $t = 0$. Show that the photon distribution probability evolves at short times as

$$\frac{\mathrm{d}p(n)}{\mathrm{d}t} = -\kappa[n(n_{\mathrm{th}}+1) + (n+1)n_{\mathrm{th}}]p(n).$$

We show below the evolution of the photon number on long time scales (left: two single traces, right: average over 2000 traces).





5 Schödinger cats and decoherence

The QND measurement scheme discussed above can be used to generate small Schrödinger cats and observe their decoherence. For simplicity we consider in this section a zero-temperature environment ($n_{\rm th} = 0$).

The Schrödinger cats realized in experiments are a quantum superposition of two coherent states

$$|\psi_{\rm cat}\rangle \simeq \frac{1}{\sqrt{2}}(|0\rangle + |-\alpha\rangle),$$

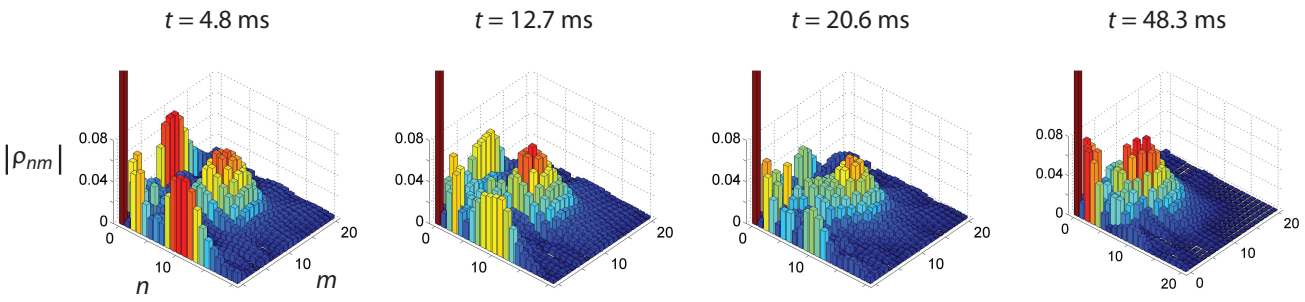
where $|0\rangle$ is the photon vacuum state and $|\alpha\rangle$ is a coherent state of amplitude α (i.e. $|\alpha\rangle \propto \alpha^n/\sqrt{n!}\,|n\rangle$).

24. Before dealing with the Schrödinger cat state, let us consider the evolution of a coherent state $|\psi(t=0)\rangle = |\alpha\rangle$, which can be prepared by injecting photons in the cavity with a classical source. Show that after a relaxation time t the photon state remains a coherent state of amplitude $\alpha(t) = e^{-\kappa t/2}\alpha$.

25. We now consider the relaxation of the Schrödinger cat $|\psi_{\rm cat}\rangle$. Show that the density matrix after an evolution of duration t is given by

$$\hat{\rho}(t) = \frac{1}{2}\left(|0\rangle\langle 0| + |\alpha(t)\rangle\langle\alpha(t)|\right) + \frac{e^{-|\alpha|^2\kappa t/2}}{2}\left(|\alpha(t)\rangle\langle 0| + |0\rangle\langle\alpha(t)|\right).$$

26. Comment on this result in terms of decoherence and show its compatibility with the measured evolution of the density matrix shown below.



Reference: *Reconstruction of non-classical cavity field states with snapshots of their decoherence*, S. Deleglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J.-M. Raimond, S. Haroche, Nature **455**, 510 (2008)