Formation interuniversitaire de physique

Evolution and Measurement of Quantum States

Exercise class n°5: Entangled photons and quantum communication

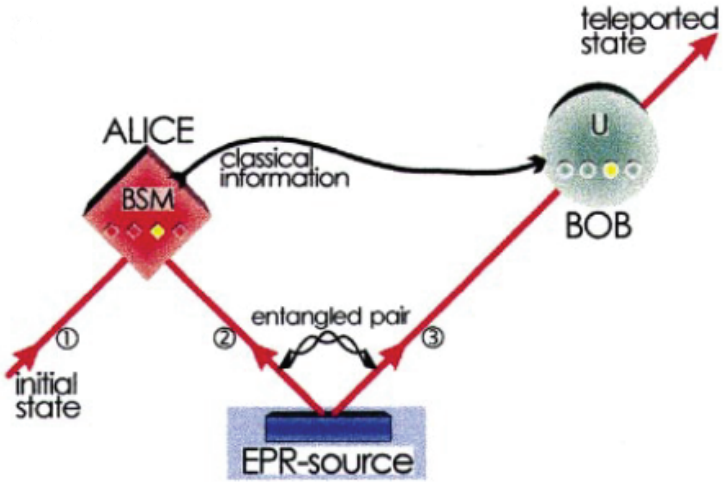
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We recall the expression of the four Bell states:

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(|{+}{+}\rangle \pm |{-}{-}\rangle\right), \quad \left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle \pm |{-}{+}\rangle\right).$$

1 Quantum teleportation

The notion of entanglement is at the basis of quantum information processing that have no equivalent within classical physics. A celebrated example consists in the notion of quantum teleportation, of which the principle is pictured below.



Quantum teleportation consists in the ability to replicate an unknown qubit at a distant location by conveying 2 bits of classical information. The initial qubit $|\psi\rangle = \alpha\,|+\rangle_1 + \beta\,|-\rangle_1$ is encoded on a particle 1 initially held by Alice. Let us assume that Alice and Bob share an entangled qubit pair in the state $|\Psi^-\rangle_{23} = (|{+}{-}\rangle_{23} - |{-}{+}\rangle_{23})/\sqrt{2}$ encoded in particles 2 and 3.

1.1 Detection of the antisymmetric Bell state $|\Psi^-\rangle$

The Bell state $|\Psi^-\rangle_{12}$ can easily be detected using a 1:1 beam splitter in which one sends the 1 and 2 particles at both entrance arms A and B. The exit arms are labelled C and D.

1. Let us first model the action of the beam splitter for a single particle. Show that it can be described by a 2×2 unitary operator $\hat{U} = (u_{i,j})$ acting as

$$|A\rangle \to u_{1,1}\,|C\rangle + u_{1,2}\,|D\rangle, \quad |B\rangle \to u_{2,1}\,|C\rangle + u_{2,2}\,|D\rangle.$$

2. Show that a 1:1 beam splitter cube is characterized by $|u_{i,j}|^2 = 1/2$. Show that by choosing appropriate phases for the C and D states the beam splitter matrix $\hat{U}$ is given by

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

3. Let us consider a quantum state of two indistinguishable photons, with one photon incoming on each entrance arm of a beam splitter. Give the outcome of the beam splitter for incoming states described as one of the four Bell states.

4. Show that the antisymmetric Bell state $|\Psi^-\rangle_{12}$ can be distinguished by recording coincidences on the detectors C and D.

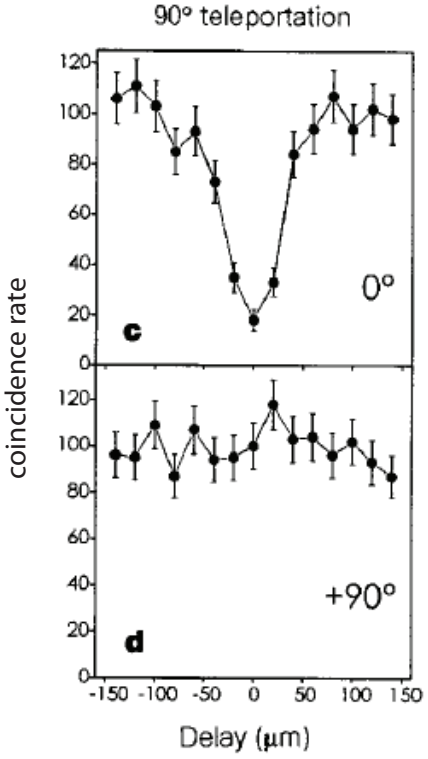
1.2 Teleportation scheme

5. What is the probability to measure particles 1 and 2 in one of the Bell states?

6. Assume that one measures particles 1 and 2 in the antisymmetric Bell state $|\Psi^-\rangle_{12}$. What is the quantum state of particle 3 after the measurement?

7. In order to demonstrate the qubit teleportation, one characterizes the quantum state of the particle after teleportation for several choices of polarization for the photon 1. Assume for example that the photon is polarized along y. What should be the probability to measure a teleportation event with a photon polarized along x?

8. We introduce a small delay τ in the arrival time of the photon 2. What is the criterion on τ for coincidence clicks of the detectors 1 and 2 correspond to an entangled two-photon state? Show that for large delays the photon 3 looks like unpolarized even for coincident events. Explain the experimental observations reproduced below.



2 Purification of entanglement

In a quantum communication channel decoherence leads to an unavoidable decrease of the quality of entangled states with the channel length. Entanglement purification is a technique allowing one to extract states with high entanglement and purity from a larger set of less entangled states.

We describe here the first experimental realization of entanglement purification. Suppose that we are interested in using photon pairs in the Bell state $|\Phi^+\rangle$ in a quantum communication protocol. For each photon pair one photon a is kept by Alice and the other one b by Bob. Due to environmental noise the actual state of photon pairs is rather described by a mixed state

$$\hat{\rho}_{ab} = F\,|\Phi_+\rangle_{ab}\,\langle\Phi_+| + (1-F)\,|\Psi_-\rangle_{ab}\,\langle\Psi_-|\,,$$

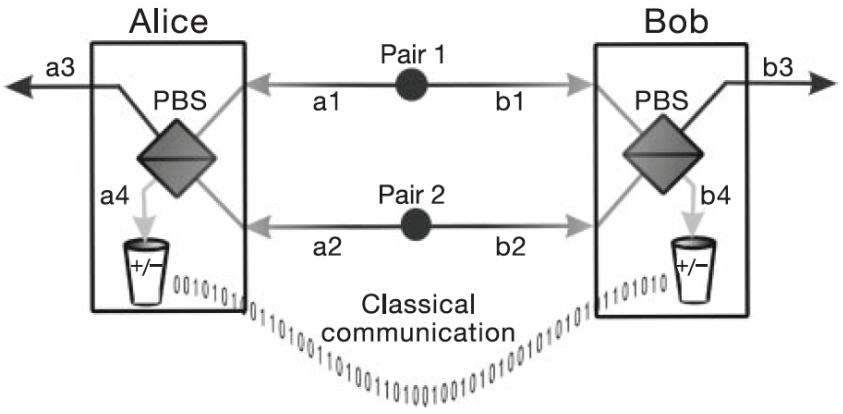
where $|\Psi_-\rangle$ is the unwanted state and F is the fidelity in the wanted state.

The aim of the purification protocol is to use two entangled pairs in the state $\hat{\rho}_{ab}$ to prepare a single photon pair in a state

$$\hat{\rho}'_{ab} = F'\,|\Phi_+\rangle_{ab}\,\langle\Phi_+| + (1-F')\,|\Psi_-\rangle_{ab}\,\langle\Psi_-|\,,$$

with $F' > F$.

The setup is described in the figure below.



Alice mixes the two photons she owns into the two arms of a polarizing beam splitter, and Bob does the same one his side. They only keep the cases where there is exactly one photon in the output modes a_4 and b_4. Furthermore they perform a polarization measurement of the a_4 and b_4 photons along the direction x. Depending on the results, Alice may perform a specific operation on the a_3 photon. We show in the following that this procedure leads to a photon pair in the modes a_3 and b_3 with a higher fidelity F' than the original photon pairs.

1. Show that the density matrix for the two photon pairs can be written as a mixture of the four pure states $|\Phi_+\rangle_{a_1b_1}\,|\Phi_+\rangle_{a_2b_2}$, $|\Phi_+\rangle_{a_1b_1}\,|\Psi_-\rangle_{a_2b_2}$, $|\Psi_-\rangle_{a_1b_1}\,|\Phi_+\rangle_{a_2b_2}$, and $|\Psi_-\rangle_{a_1b_1}\,|\Psi_-\rangle_{a_2b_2}$. Explain why among those four states the second and third ones never lead to the detection of exactly one photon in a_4 and b_4.

2. Consider now the state $|\Phi_+\rangle_{a_1b_1}\,|\Phi_+\rangle_{a_2b_2}$. What is the probability to deal with this state? Assuming that this state is realized, what is the probability to measure exactly one photon in each arm? What is the quantum state after such a measurement?

3. Assume that the photons a_4 and b_4 are measured with equal polarizations along x. Show that the state of the a_3 and b_3 pair is then $|\Phi_+\rangle_{a_3b_3}$.

4. Assume that the photons a_4 and b_4 are measured with different polarizations along x. Show that the state of the a_3 and b_3 pair is then $|\Phi_-\rangle_{a_3b_3}$. When this measurement result is recorded, Alice applies a phase slip operation on the qubit a_3 in order to transform $|\Phi_-\rangle$ into $|\Phi_+\rangle$. Give the definition of the phase slip operation in order to obtain the desired action.

5. Consider now the state $|\Psi_-\rangle_{a_1b_1}|\Psi_-\rangle_{a_2b_2}$. What is the probability to deal with this state? Assuming that this state is realized, what is the probability to measure exactly one photon in each arm? In that case, show that performing the x-polarization measurement + phase slip procedure leads to a photon pair $|\Psi_+\rangle_{a_3b_3}$.

6. Show that this procedure leads to the preparation of photon pairs in the a_3 and b_3 arms in the state

$$\hat{\rho}'_{ab} = F' |\Phi_+\rangle_{ab} \langle\Phi_+| + (1-F') |\Psi_+\rangle_{ab} \langle\Psi_+|, \quad \text{with} \quad F' = \frac{F^2}{F^2+(1-F)^2}.$$

   What is the condition for a successful purification $F' > F$?

References:

Experimental quantum teleportation, D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature **390**, 575 (1997)

Experimental entanglement purification of arbitrary unknown states, J.-W. Pan, S. Gasparoni, R. Ursin, G.Weihs, and A. Zeilinger, Nature **423**, 417 (2003)