Master ICFP

Ultracold Atoms

Exercise class n°2: Phase coherence of an ultracold Bose gas

Sylvain Nascimbene

1 Phase coherence of a 3D Bose gas

The notion of phase coherence describes the extent of spatial ordering of the atomic field. It is described by the first-order correlation function

$$g_1(\mathbf{r}) = \left\langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{0}) \right\rangle .$$

1. Expand the field operators in momentum space and show that the g_1 function can be expressed as
$$g_1(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, n_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}},$$
where n_k is the thermal occupation probability of the momentum component $\mathbf{k}$.

2. Give the expressions of n_k as a function of the temperature T, the chemical potential μ and the macroscopic occupation of the $\mathbf{k} = \mathbf{0}$ state occurring below the condensation threshold $T = T_c$.

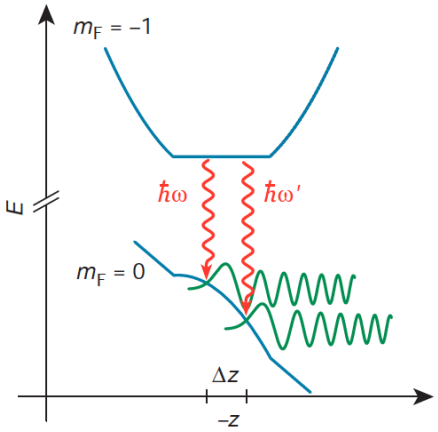
3. In the high-temperature regime $T \gg T_c$, the fugacity $z = e^{\mu/(k_B T)} \ll 1$. Show that the g_1 function is given by
$$g_1(r) = n\, e^{-\pi r^2/\lambda_{\mathrm{dB}}^2}, \quad \lambda_{\mathrm{dB}} = \sqrt{\frac{2\pi\hbar^2}{mk_BT}},$$
where n is the atom density.

4. ($\star$) Right above the condensation temperature, the momentum distribution is dominated by low momenta modes. Show that it can be approximated by
$$n_k = \frac{z}{1 - z + (\lambda_{\mathrm{dB}}k)^2/(4\pi)},$$
and deduce the expression for the g_1 function. The following integral may be useful:
$$\int \frac{e^{i\mathbf{x}\cdot\mathbf{u}}}{1+u^2}\mathrm{d}^3u = 2\pi^2\frac{e^{-x}}{x}.$$
Comment on this result.

5. Assuming for simplicity that the non-condensed atoms contribute according to the question 3, give the expression of the g_1 function below the condensation threshold.

2 Measuring phase coherence via interference experiments

The first-order correlation function was measured experimentally from the interference pattern of two matter waves originating from two spatial regions of a Bose-Einstein condensate.



As described in the figure, the BEC is prepared into a magnetic trap, polarized in the Zeeman level $|F = 1, m_F = -1\rangle$. A radio frequency field transfers the atoms towards the Zeeman state $|F = 1, m_F = 0\rangle$, which is insensitive to magnetic fields and drops due to gravity.

6. Explain the variation of potential energy felt by an atom in the Zeeman levels displayed above.

7. Let us first discuss the outcoupling of an atomic matter wave in the presence of a single radiofrequency. Explain why the outcoupling occurs at a given vertical position that we take here equal to 0.

8. We search for a semi-classical expression of the atomic wavefunction of a particle falling due to gravity. In the framework of WKB semi-classical theory, we locally describe the wavefunction as a plane wave of momentum $k(z)$, such that $E = \hbar^2 k(z)^2/(2m) + V(z)$, where E is the particle energy and $V(z)$ is a potential energy. Explain why this theory leads here to the expression

$$\psi_{\text{out}}(z) \propto \frac{1}{\sqrt{k(z)}} e^{i\phi(z)}, \quad \phi(z) = \int \mathrm{d}z\, k(z), \quad k(z) = \frac{m}{\hbar}\sqrt{2gz}.$$

   Give an explicit expression for $\psi_{\text{out}}(z)$. We assume the z axis to be oriented along gravity.

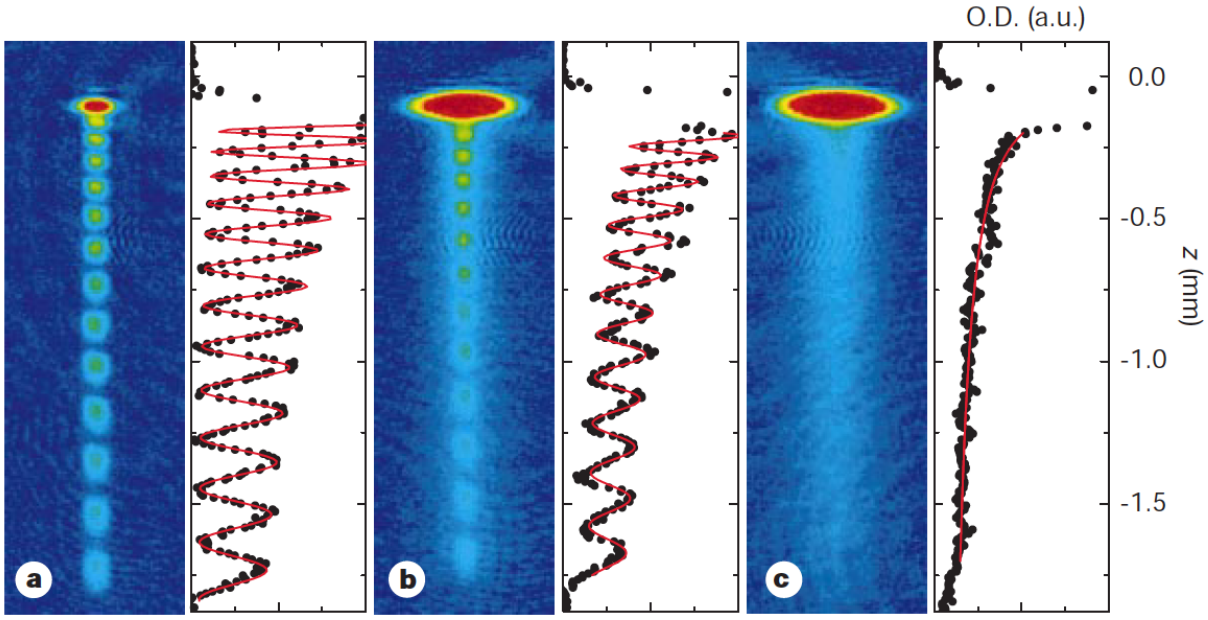
9. Consider now the outcoupling of two matter waves induced by two waves of frequencies ω and ω'. Show that the outcouplings occur at positions z_0 and z_1 offset by $\Delta z = z_0 - z_1 = \hbar(\omega - \omega')/(mg)$.

10. Explain why we expect the atomic density to be given by

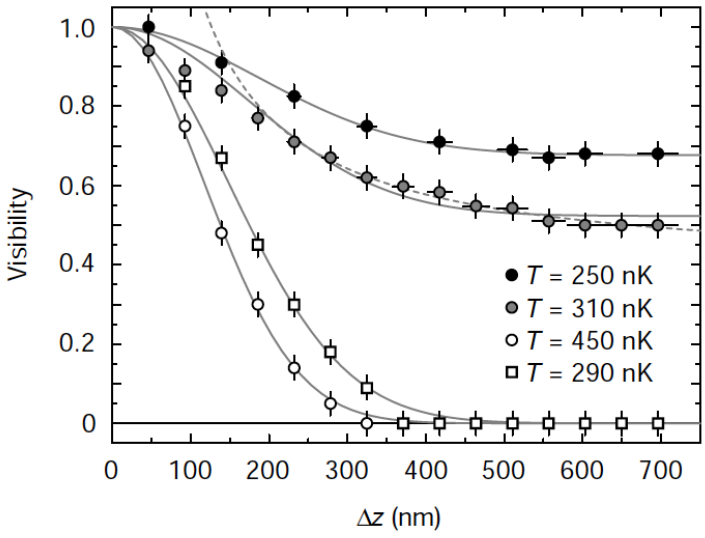
$$n_{\text{out}}(z) = \frac{1}{\sqrt{z}}\left[1 + V\cos(q\sqrt{z} + (\omega - \omega')t)\right], \quad q = \frac{m\sqrt{2g}\Delta z}{\hbar},$$

    where V is the visibility of interferences. Show that $V = 1$ at zero temperature and propose in the general case an expression for the visibility V in terms of a first-order correlation function.

11. Give an interpretation of the experimental results displayed below.



→ increasing temperature →



3 Does the ideal Bose gas condense in two dimensions?

We remind the expression of the density of states of a spinless 2D Bose gas in a square box of size L.

$$\rho(\epsilon) = \frac{mL^2}{2\pi\hbar^2}.$$

1. Show that the number of particles in excited states is given by

$$N_{\rm ex} = -\left(\frac{L}{\lambda_{\rm dB}}\right)^2 \log\left(1 - e^{\beta\mu}\right),$$

   where $\lambda_{\rm dB}$ is the thermal de Broglie wavelength.

2. Does an ideal Bose gas undergo Bose-Einstein condensation in 2D?

In two dimensions the first-order correlation function remains equal to the Fourier transform of the momentum distribution

$$g_1(\mathbf{r}) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\, n_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}$$

3. Show briefly that in the high-temperature regime the g_1 function is given by the expression

$$g_1(r) = n\, e^{-\pi r^2/\lambda_{\rm dB}^2},$$

   where n is the atom density.

4. In the low-temperature regime, show that the low-momentum states are occupied according to

$$n_k = \frac{k_B T}{\epsilon_k + |\mu|}, \quad \epsilon_k = \frac{\hbar^2 k^2}{2m}.$$

   Deduce the expression of the g_1 correlation function at large distances

$$g_1(r) \propto e^{-r/l}/\sqrt{r}, \quad l = \frac{\lambda_{\rm dB}}{\sqrt{4\pi}} e^{n\lambda_{\rm dB}^2/2},$$

   where n is the 2D atomic density. One may use

$$\int \frac{\mathrm{d}u}{\sqrt{1+u^2}} e^{-iau} \underset{a\gg 1}{\simeq} \sqrt{\frac{\pi}{2a}} e^{-a}.$$

5. Plot the coherence length l versus the phase space density $n\lambda_{\rm dB}^2$ and comment on this result.

4 Quasi-condensation of an interacting 2D Bose gas

A statistical physics argument – the so-called Mermin-Wagner theorem – states that Bose-Einstein condensation cannot exist in 2D, even in the presence of interatomic interactions. However, a phase transition occurs at low temperature towards a superfluid state, via the Berezinskii-Kosterlitz-Thouless mechanism. The latter applies to various physical systems in 2D such as liquid helium films, the superconducting transition in arrays of Josephson junctions, and the collision physics of 2D atomic hydrogen. We discuss here the extent of phase coherence in the superfluid phase of a 2D Bose gas.

In the low-temperature regime repulsive interactions favor a homogeneous atom density. Thermal density fluctuations are thus energetically suppressed and the matter wave field can be expressed as

$$\psi(\mathbf{r}) = \sqrt{n}\, e^{i\theta(\mathbf{r})},$$

where $\theta(\mathbf{r})$ describes phase fluctuations and n is the (uniform) superfluid density.

6. Show that the energy corresponding to the field $\psi(\mathbf{r})$ reads

$$E = \frac{\hbar^2}{2m} n \int \mathrm{d}^2 r \; (\nabla\theta(\mathbf{r}))^2.$$

We expand the phase fluctuations in Fourier series as

$$\theta(\mathbf{r}) = \left(\frac{L}{2\pi}\right)^2 \int \mathrm{d}^2 k \; [c_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{r}) + d_{\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{r})].$$

The long-distance behavior of the correlation function g_1 is determined by small-k behavior.

7. Show that the energy can be written as

$$E = nL^2 \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \frac{c_{\mathbf{k}}^2 + d_{\mathbf{k}}^2}{2}.$$

We thus obtain a quadratic energy with respect to each mode $\mathbf{k}$.

8. We assume a regime of classical fields, for which the energy associated with the mode $\mathbf{k}$ is $k_B T/2$. Show that one expects a thermal occupation with

$$\langle c_{\mathbf{k}} \rangle^2 = \langle d_{\mathbf{k}} \rangle^2 = \frac{4\pi}{n\lambda_{\mathrm{dB}}^2} \frac{1}{L^2 k^2}.$$

We admit that each mode $\mathbf{k}$ behaves as a random Gaussian variable, u, which obeys the property

$$\left\langle e^{iu} \right\rangle = e^{-\langle u^2 \rangle/2}.$$

9. Show that

$$g_1(\mathbf{r}) = n \left\langle e^{i(\theta(\mathbf{r})-\theta(\mathbf{0}))} \right\rangle = n \exp\left(-\frac{1}{2\pi n \lambda_{\mathrm{dB}}^2} \int \frac{1-\cos(\mathbf{k}\cdot\mathbf{r})}{k^2} \mathrm{d}^2 k \right)$$

10. The integral is divergent for large k, but we assumed from the beginning that the modes behave classically, which imposes a maximum energy before their quantum nature makes the model fail. We thus heuristically impose that the maximum momentum in the integral is $k = 1/\xi$, where ξ is a correlation length. Explain why we expect, for $r \gg \xi$,

$$\int \frac{1-\cos(\mathbf{k}\cdot\mathbf{r})}{k^2} \mathrm{d}^2 k \simeq 2\pi \log(r/\xi).$$

11. Show that this leads to an algebraic decay of the correlation function

$$g_1(\mathbf{r}) = n \left(\frac{\xi}{r}\right)^{1/n\lambda_{\mathrm{dB}}^2}.$$

This power-law decay of the correlation function is typical of the BKT superfluid transition. It is consistent with experiments on quasi-2D Bose and Fermi gases (Hadzibabic et al, Nature **441**, 1118 (2006) and Murthy et al, Phys. Rev. Lett. **115**, 010401 (2015)).