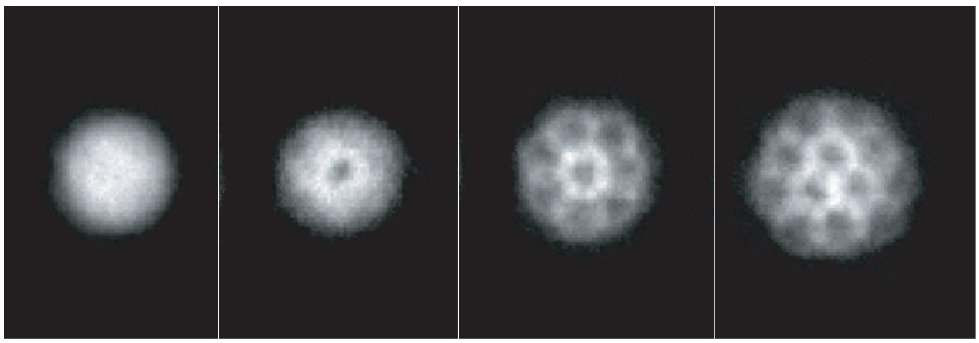
Master ICFP

Ultracold Atoms

Exercise class n°4: Quantized vortices in a Bose-Einstein condensate

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A fundamental signature of superfluidity in an ultracold atomic gas is provided by its response to rotation. As shown below, a superfluid cannot rotate as a solid body, and the rotation is rather concentrated in singularity lines – quantized vortices. The pictures below correspond to Bose-Einstein condensates held in rotating traps, with increasing rotation frequency from left to right (Phys. Rev. Lett. **86**, 4443 (2002)). The small atomic density dips correspond to quantized vortices, whose number increase with the rotation frequency.



1 Notion of superfluid velocity

1. Propose a definition of the superfluid velocity $\mathbf{v}$ based on the density probability current.
2. Give the expression of the superfluid velocity for an atomic field written as $\psi = \sqrt{n}e^{i\varphi}$.
3. Calculate the circulation of this velocity around a closed path.
4. Compare this result this the circulation of velocity for a rigid body rotating at the angular frequency Ω.

2 Density profile close to a vortex line

We describe in this section the density profile of a uniform 3D gas of bosons in a cylindrical geometry, with a vortex line along the cylinder symmetry axis $\rho = 0$. We write the atomic wavefunction as $\psi(\mathbf{r}) = \sqrt{n_0}\,\phi(\rho)\,e^{i\ell\theta}$, where ℓ is the charge of the vortex.

We give:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\Delta f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

5. Calculate the velocity field around the vortex.
6. Calculate the value of the projection of angular momentum along z (per particle).

7. Use the Gross-Pitaevskii equation to find a differential equation for the radial part of the BEC wavefunction.

8. Find the solutions of this equation at short and large distances. Determine the characteristic length defining the crossover between the two regimes and plot the shape of the radial part of the wavefunction.

9. The energy cost associated with a vortex line is dominated by the kinetic energy. Estimate the latter of a gas held in a cylinder of radius R and length L.

10. Compare the energy of a vortex of charge $\ell > 1$ with the energy associated with l distant vortices of charge 1. Do we expect to observe multiple-charged vortices at equilibrium?

3 Nucleation of a single vortex in a trapped, rotating gas

We now discuss the nucleation of a single vortex in a trapped, rotating gas. We consider a BEC of N atoms trapped in an 3D isotropic harmonic trap with an angular frequency ω. We work in the Thomas-Fermi approximation.

11. Estimate the ratio between the healing length ξ and the Thomas Fermi radius R, given by

$$R = a_{\mathrm{ho}} \left(\frac{15Na}{a_{\mathrm{ho}}} \right)^{1/5},$$

where a is the s-wave scattering length and $a_{\mathrm{ho}} = \sqrt{\hbar/m\omega}$ is the extent of the ground-state wavefunction.

12. Calculate numerically the value of the healing length for sodium atoms. We use the following numerical values: $\omega = 2\pi \times 100\,\mathrm{Hz}$, $m = 3.82 \times 10^{-26}\,\mathrm{kg}$, $n_0 = 10^{14}\ \mathrm{cm}^{-3}$ and $a = 2.6$ nm.

13. How does the size of the vortex depend on its position in the cloud?

We consider the nucleation of vortices in a trapped gas set into rotation at the angular frequency Ω. We want to estimate an order of magnitude of the critical rotation frequency required to nucleate one vortex. We approximate the vortex energy by the one calculated for a uniform system.

14. Using the expression of the Hamiltonian $H' = H - \Omega L_z$ in the rotating frame as a function of the Hamiltonian H in the laboratory frame, find an estimate of the critical rotation angular frequency for which one vortex is thermodynamically stable.

4 Abrikosov lattice of vortices

15. In the regime of fast rotation frequency, the gas organizes into a regular lattice of single-charged vortices, with a mean density n_v. What is the relationship between n_v and Ω?

16. Comment on the last picture of rotating BECs.