Master ICFP

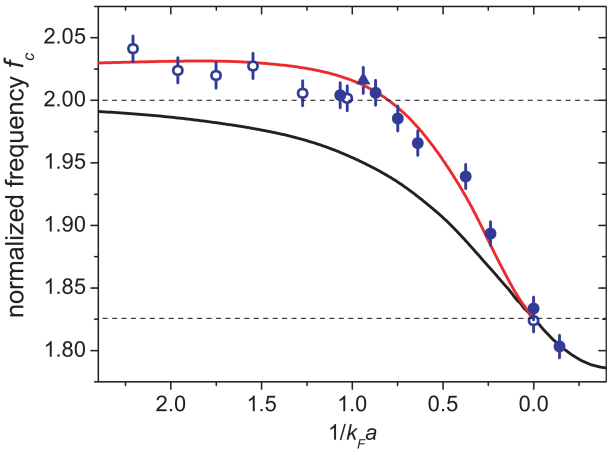
Ultracold Atoms

Exercise class n°5: Breathing mode of a strongly-interacting Fermi gas

Sylvain Nascimbene

We consider the small oscillations of a strongly-interacting Fermi gas around equilibrium. We will model the breathing mode corresponding to the transverse oscillation of an elongated gas.

We show below a measurement of the mode frequency, along the BEC-BCS crossover (from PRL 98:040401 (2007)). A precise measurement of the mode frequency can be used to discriminate between the result of Monte-Carlo simulations (upper curve) and mean-field BCS theory (lower one).



In the following we consider a zero temperature Fermi gas trapped in an cylindrical harmonic potential $V(\mathbf{r}) = \frac{1}{2}M\omega_r^2 r^2 + \frac{1}{2}M\omega_z^2 z^2$.

Useful formulas : Operators in cylindrical coordinates

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad \boldsymbol{\nabla} f = (\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z})$$

Preliminary

1. Which values of $1/k_F a$ correspond to the unitary regime and the molecular BEC regime?
2. In the BEC regime, write down the expression for the density profile at rest $n_0(\mathbf{r})$, as a function of the global chemical potential μ_0, the coupling constant g and the trapping potential $V(\mathbf{r})$.
3. Same question in the unitary regime. One should introduce the Bertsch parameter $\xi = \mu/E_F$.
4. For both the BEC and unitary regimes the local chemical potential can be written as a polytropic law of density $n_0(\mathbf{r})$, i.e. $\mu(n_0) = \alpha n_0(\mathbf{r})^\gamma$. Give the values of γ corresponding to the two regimes.

Hydrodynamics equations

We assume in the following that the dynamics of the interacting Fermi gas is described by hydrodynamics equations:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{1}$$

$$M\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\frac{1}{2}M\mathbf{v}^2 + \mu(n) + V(\mathbf{r})) = 0 \tag{2}$$

where $\mathbf{v}$ is the velocity and $\mu(n)$ is the local chemical potential for a density $n(r)$.

5. Write down the second equation at equilibrium. Comment on the result.

6. Explain why in the second equation we do not take into account the term in $\boldsymbol{\nabla} \wedge \mathbf{v}$.

7. Show that for a cloud at rest

$$\left.\frac{\partial \mu}{\partial n}\right|_{n_0} \boldsymbol{\nabla} n_0 = -\boldsymbol{\nabla} V(\mathbf{r}). \tag{3}$$

8. We assume that the polytropic law for the chemical potential is still valid for a density profile $n(r)$ even for a cloud not at rest. Find the expression of $n_0 \left.\frac{\partial \mu}{\partial n}\right|_{n_0}$ as a function of α, γ et n_0 and then as a function of γ, μ_0 et $V(\mathbf{r})$.

We study small variations of the density around the equilibrium value and write $n(\mathbf{r}, t) = n_0(\mathbf{r}) + \delta n(\mathbf{r}, \mathbf{t})$ where n_0 is time-independent. We also write the chemical potential as $\mu(n) = \mu(n_0) + \left.\frac{\partial \mu}{\partial n}\right|_{n_0} \delta n(\mathbf{r}, t)$.

9. Linearize the hydrodynamics equations and show that

$$M\frac{\partial^2 \delta n}{\partial t^2} = \nabla \cdot \left[n_0 \boldsymbol{\nabla} \left(\left.\frac{\partial \mu}{\partial n}\right|_{n_0} \delta n \right) \right]. \tag{4}$$

10. We write δn as $\mathrm{Re}[n_0^{1-\gamma}(\mathbf{r}) f(\mathbf{r}) e^{-i\omega t}]$. Show that

$$-M\omega^2 f(\mathbf{r}) = -\boldsymbol{\nabla} V(\mathbf{r}) \cdot \boldsymbol{\nabla} f(\mathbf{r}) + n_0 \left.\frac{\partial \mu}{\partial n}\right|_{n_0} \Delta f(\mathbf{r}). \tag{5}$$

The breathing mode corresponds to oscillations of the transverse size of the cloud. We can write the solution as $\delta n = n_0^{1-\gamma}(\mathbf{r})(a + br^2 + cz^2)$. The trap is very anisotropic so $\omega_r \gg \omega_z$ and we assume $\omega_z \approx 0$.

11. Find a system of three equations between a, b et c.

12. Non-trivial solutions correspond to the cancellation of the matrix determinant. Find a second order equation in ω^2 whose solutions are the frequencies of the excited modes. Give the expression of the solutions as a function of γ.

13. Calculate the frequencies for a unitary Fermi gas and for a molecular BEC and compare with the results in the figure. What is the origin of the discrepancy in the molecular BEC regime?

Answers

1. Unitarity $1/k_F a = 0$. Molecular BEC for $1/k_F a > 0$.

2. $\mu_0 = g n_0(r) + V(r)$. This is the local density approximation which is valid for a potential varying slowly enough so that we can define a local equilibrium for every value of r and assume the gas is described by the physics of the uniform gas with a modified chemical potential.

3. $n_0(\mathbf{r}) = \frac{1}{3\pi^2}(\frac{2m}{\xi\hbar^2}(\mu_0 - V(r))^{3/2})$

4. BEC : $\mu(n_0) = \mu_0 - V(\mathbf{r}) = g n_0(r)$ so $\gamma = 1$.
   Fermi $\mu(n_0) = \frac{\xi\hbar^2}{2m}(3\pi^2 n_0(\mathbf{r}))^{2/3}$ so $\gamma = 2/3$.

5. $v = 0$. $\mu(n_0(r)) + V(r)$ =constant. Same result as in question 2.

6. We are at $T = 0$. If $T < T_c$, we have a superfluid and then an irrotational flow.

7. $\mu(n_0) = \mu_0 - V(r)$ so $\boldsymbol{\nabla}\mu(n_0) = -\boldsymbol{\nabla}V(r)$ so $\left.\frac{\partial\mu}{\partial n}\right|_{n_0} \boldsymbol{\nabla} n_0 = -\boldsymbol{\nabla}V(\mathbf{r})$.

8. $\mu(n) = \alpha n^\gamma$ gives $\frac{\partial\mu}{\partial n} = \alpha\gamma n^{\gamma-1}$. We take this value in $n = n_0$ and find $\left.\frac{\partial\mu}{\partial n}\right|_{n_0} = \alpha\gamma n_0^{\gamma-1}$. We also have for a cloud at rest $\mu_0 - V(r) = \alpha n_0^\gamma$. By replacing in the previous equation we find $n_0 \left.\frac{\partial\mu}{\partial n}\right|_{n_0} = \gamma(\mu_0 - V(r))$.

9. For a BEC, $\gamma = 1$ and $\left.\frac{\partial\mu}{\partial n}\right|_{n_0} = g$.

10. Linearization of equations (1) and (2) gives

$$\frac{\partial^2\delta n}{\partial t^2} + \boldsymbol{\nabla}\cdot(n_0\frac{\partial\mathbf{v}}{\partial t}) = 0 \tag{6}$$

$$M\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\left.\frac{\partial\mu}{\partial n}\right|_{n_0}\delta n) = 0. \tag{7}$$

    Combining these two equations we get (4).

11.

$$-M\omega^2\delta n = \nabla\cdot(n_0\boldsymbol{\nabla}(\left.\frac{\partial\mu}{\partial n}\right|_{n_0}\delta n)) \tag{8}$$

$$-M\omega^2 n_0^{1-\gamma} f(r) = \nabla\cdot(n_0\boldsymbol{\nabla}(\alpha\gamma n_0^{\gamma-1} n_0^{1-\gamma} f(r))) \tag{9}$$

$$-M\omega^2 n_0^{1-\gamma} f(r) = \alpha\gamma\nabla\cdot(n_0\boldsymbol{\nabla}(f(r))) \tag{10}$$

$$-M\omega^2 f(r) = \alpha\gamma n_0^{\gamma-1}(\boldsymbol{\nabla}n_0\boldsymbol{\nabla}f(r) + n_0\Delta f(r))) \tag{11}$$

$$-M\omega^2 f(r) = \left.\frac{\partial\mu}{\partial n}\right|_{n_0}(\boldsymbol{\nabla}n_0\boldsymbol{\nabla}f(r) + n_0\Delta f(r)) \tag{12}$$

$$-M\omega^2 f(r) = -\boldsymbol{\nabla}V(r)\boldsymbol{\nabla}f(r) + n_0\left.\frac{\partial\mu}{\partial n}\right|_{n_0}\Delta f(r) \tag{13}$$

12. $\boldsymbol{\nabla}V = (M\omega_r^2 r, 0, 0)$ and $\Delta f = 4b + 2c$ and $n_0 \left.\frac{\partial\mu}{\partial n}\right|_{n_0} = \gamma(\mu_0 - \frac{1}{2}M\omega_r^2 r^2)$.

$$\begin{pmatrix} \omega^2 & 4\gamma\mu_0/M & 2\gamma\mu_0/M \\ 0 & \omega^2 - 2\omega_r^2(1+\gamma) & -\gamma\omega_r^2 \\ 0 & 0 & \omega^2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

13. $(\omega^2)^2(\omega^2 - 2\omega_r^2(1+\gamma)) = 0$.
    $\omega = 0$ or $\omega = \omega_r\sqrt{2(1+\gamma)}$.

14. BEC : $\omega = 2\omega_r$, Unitarity $\omega = \sqrt{10/3}\,\omega_r \approx 1.83\,\omega_r$.
    The BEC is not well described by the weak interaction regime.