Master ICFP

Ultracold Atoms

Exercise class n°6: Spin-orbit coupling in an ultracold Bose gas

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A whole class of physical phenomena, such as the quantum Hall effect, are based on magnetic effects acting on quantum many-body systems. However, orbital magnetism does not naturally appear in ultracold atomic gases, which are made of neutral constituents.

We discuss here a recently developed scheme to emulate spin-orbit coupling in ultracold atoms, and its consequence on Bose-Einstein condensation. Spin-orbit coupling can be viewed as a spin-dependent gauge field acting on an ensemble of spin-1/2 particles, described by the single-particle Hamiltonian

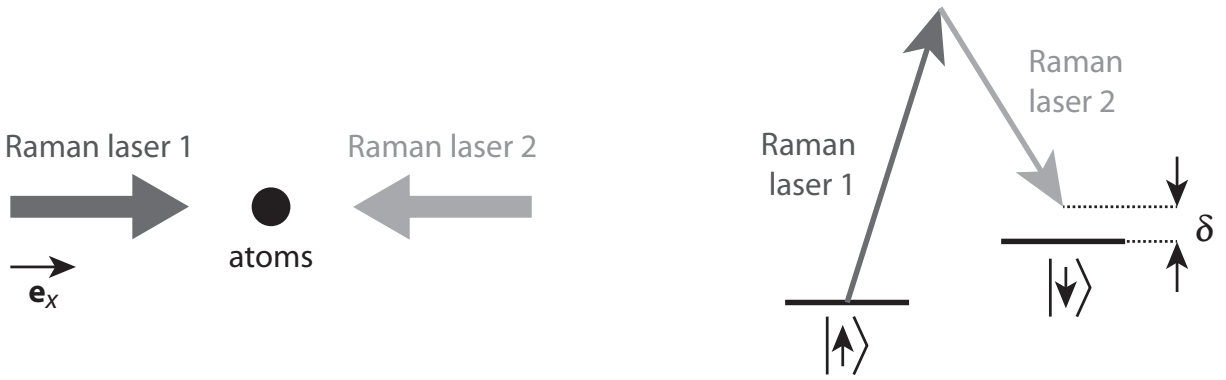
$$\hat{H} = \frac{(\hat{\mathbf{p}} - \lambda\hat{\boldsymbol{\sigma}})^2}{2m},$$

where $\hat{\mathbf{p}}$ and $\hat{\boldsymbol{\sigma}}$ are the momentum and spin operators, respectively. This type of Hamiltonian plays a central role in condensed matter (e.g. spintronics, topological insulators).

1 Spin-orbit coupling implementation

The implementation of spin-orbit coupling is based on laser dressing of atoms prepared into a spin mixture denoted in the following as $|\uparrow\rangle$ and $|\downarrow\rangle$. Due to the Doppler effect, the spin coupling amplitude depends on the atomic motion, leading to a spin-orbit coupling.

The scheme used in recent experiments is represented below. Two Zeeman levels of ^{87}Rb are coupled via a two-photon Raman transition of wavelength λ. The Raman lasers propagate along $\pm\mathbf{e}_x$ and their frequencies differ by the sum of the Zeeman gap and an additional frequency offset δ.



1. Show that the atom dynamics can be described by the Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m}\left(\hat{c}^\dagger_{\uparrow,\mathbf{p}}\hat{c}_{\uparrow,\mathbf{p}} + \hat{c}^\dagger_{\downarrow,\mathbf{p}}\hat{c}_{\downarrow,\mathbf{p}}\right) + \frac{\hbar\Omega}{2}\left(\hat{c}^\dagger_{\uparrow,\mathbf{p}-\hbar\mathbf{k}}\hat{c}_{\downarrow,\mathbf{p}+\hbar\mathbf{k}} + \text{h.c.}\right) + \frac{\hbar\delta}{2}\left(\hat{c}^\dagger_{\uparrow,\mathbf{p}}\hat{c}_{\uparrow,\mathbf{p}} - \hat{c}^\dagger_{\downarrow,\mathbf{p}}\hat{c}_{\downarrow,\mathbf{p}}\right),$$

   where $\mathbf{k} = k\,\mathbf{e}_x$, $k = 2\pi/\lambda$. Does this Hamiltonian conserve momentum? And spin?

2. We perform a gauge transform

$$\hat{a}_{\uparrow,\mathbf{p}} = \hat{c}_{\uparrow,\mathbf{p}-\hbar\mathbf{k}},$$
$$\hat{a}_{\downarrow,\mathbf{p}} = \hat{c}_{\downarrow,\mathbf{p}+\hbar\mathbf{k}}.$$

   Show that in this basis the Hamiltonian conserve momentum.

3. Show that the laser dressing leads to a single-particle Hamiltonian

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \lambda\hat{\sigma}_z)^2}{2m} + \frac{\hbar\Omega}{2}\hat{\sigma}_x + \frac{\hbar\delta}{2}\hat{\sigma}_z, \tag{1}$$

and give the value of the spin-orbit coupling amplitude λ.

2 Ground state degeneracy in the presence of spin-orbit coupling

We are interested in the following in the Bose-Einstein condensation of spin-orbit-coupled Bose gases. The first important aspect to determine is the nature of the single-particle ground state.

4. Calculate the dispersion relation associated with the Hamiltonian (1) and plot it for $(\delta, \Omega) = (0, 0)$, $(\delta, \Omega) = (0, E_r/\hbar)$, $(\delta, \Omega) = (0, 4E_r/\hbar)$, $(\delta, \Omega) = (0, 8E_r/\hbar)$, and $(\delta, \Omega) = (E_r/\hbar, E_r/\hbar)$. We introduced the recoil energy $E_r = \hbar^2 k^2/(2m)$.

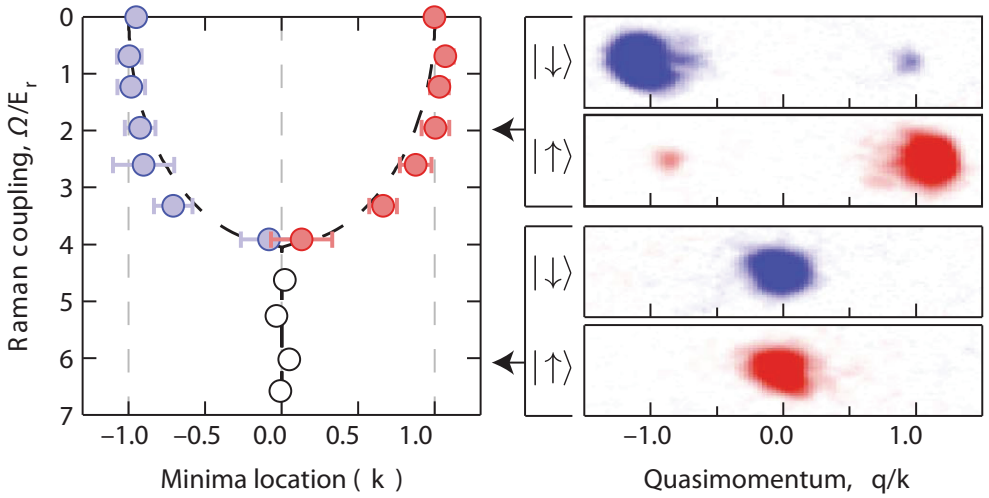
In the following we assume that the Raman lasers are tuned at resonance, i.e. $\delta = 0$.

5. In the regime $\Omega > \Omega_c$, show that the spin-orbit coupling leads to a low-energy quadratic band dispersion with an effective mass m^*.

6. Show that the ground state is doubly degenerate for $\Omega < \Omega_c$, and give the expression of Ω_c.

7. Give an explicit formula for the ground state momenta $\pm k_0$.

8. The following picture represents measurements on spin-orbit coupled BECs of ^{87}Rb, performed in 2011 at NIST (Nature **471**, 83 (2011)). The momentum distribution is measured after time of flight, and the spin states are independently measured after Stern-Gerlach separation. Comment on these measurements.



3 Phase diagram of a spin-orbit coupled BEC

In the regime $\Omega < \Omega_c$ the single-particle ground state is degenerate. This situation goes beyond the usual case of Bose-Einstein condensation consisting of the macroscopic occupation of particles into the unique ground state.

9. Explain why we expect interactions to play a key role for determining the nature of the Bose-Einstein condensate.

10. For simplicity we consider the case $\Omega \ll \Omega_c$. Show that the two single-particle ground states can be written as $|\uparrow, k\rangle$ and $|\downarrow, -k\rangle$.

We assume that the particles form a BEC into the quantum superposition

$$|\psi\rangle = \alpha\,|\uparrow, k\rangle + \beta\,|\downarrow, -k\rangle\,. \tag{2}$$

11. Explain why the interaction energy can be written as

$$E_{\rm int} = \left(\frac{N}{V}\right)^2 \int \mathrm{d}^3 r \left(\frac{g_{\uparrow\uparrow}}{2}\,|\psi_\uparrow(\mathbf{r})|^4 + \frac{g_{\downarrow\downarrow}}{2}\,|\psi_\downarrow(\mathbf{r})|^4 + g_{\uparrow\downarrow}\,|\psi_\uparrow(\mathbf{r})|^2\,|\psi_\downarrow(\mathbf{r})|^2\right),$$

where we introduced spin-dependent coupling constants and the wavefunctions for each spin. For simplicity we assume in the following $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} \equiv g$.

12. Calculate the interaction energy for the ansatz (2) and show that the nature of the ground state is determined by the sign of $g_{\uparrow\downarrow} - g$.

13. For $g_{\uparrow\downarrow} > g$, show that the BEC consists in a spin-polarized plane wave. Which spin component will be observed in the experiment?

14. For $g_{\uparrow\downarrow} < g$, show that the BEC exhibits a spatial modulation of the spin densities along x or y ('stripe phase'). Calculate the typical length scale of this spatial modulation and explain why it has not been observed in experiments yet.

15. We display below measurements of the spin polarization of a spin-orbit coupled BEC of ^{87}Rb, for which $g_{\uparrow\downarrow} < g$ (Nature Physics **10**, 314 (2014)). The histograms gather the measurements of spin polarization along z, taken repeatedly for a given value of Ω. Which part of the measurements correspond to the previous calculations? Comment on the nature of the BEC for large Ω values.

