Quantization and Fock spaces

Examen de théorie des champs, M1 2009-2010

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The exam is made of three parts, A, B and C.

Part A is a series of questions that were explicitly covered during the lectures.

Parts B and C need some results given in section A.I, and which can be used and the sequel even without proof.

Part B is a warm-up for part C, which can however be dealt with almost independently.

Questions marked with a $^\flat$ are independent of the rest of the problem and should preferably be covered only after the other ones have been solved (or at least tried).

Apart from the very last question, units are chosen so that $\hbar = c = 1$.

Introduction

In Quantum Field Theory, operator products at coinciding points are singular, divergent. Normal ordering (also called Wick ordering) is a recipe to quantize classical products while avoiding these divergences, at least as far as normal ordered products have finite matrix elements between Fock states of finite excitation number.

The purpose of the problem is to analyze more deeply in a simple situation the pros and contras of normal ordering. The general setting is the following : one adds to the Hamiltonian of a free field acting on usual Fock space a perturbation quadratic in the fields and one studies whether or not the perturbed system can be studied in the original Fock space. The result depends on the dimension of space-time.

Even if the study that follows does not go that far, a concrete case in which the proposed computations are relevant is the computation of the response (for instance the $\hat{S}$ matrix) of a free field perturbed by a time dependent (but constant in space) external source coupled not to the field itself, but to its square. Electron-positron pair creation by a classical electromagnetic field is a typical example.

Partie A : Basic questions.

A.I Wick's theorem

Let $\hat{c}$ and $\hat{d}$ be two operators such that $[\hat{c}, \hat{d}] = h$ where $[h, \hat{c}] = [h, \hat{d}] = 0$. The letters λ and ρ stand for commuting parameters (say complex numbers for instance).

A.1 : Prove that

$$e^{\lambda\hat{c}}\hat{d}e^{-\lambda\hat{c}} = \hat{d} + \lambda h.$$

Show without further computation that

$$e^{-\rho\hat{d}}\hat{c}e^{\rho\hat{d}} = \hat{c} + \rho h.$$

Conclude that $\hat{c}\hat{d}^n = \hat{d}^n\hat{c} + nh\hat{d}^{n-1}$, $n = 2, 3, \cdots$. ✎

A.2 : ♭ Use the previous question to show that

$$e^{-\rho\hat{d}}e^{\lambda\hat{c}}e^{\rho\hat{d}} = e^{\lambda\rho h}e^{\lambda\hat{c}}.$$

It is suggested to take the derivative with respect to λ. ✎

A.3 : ♭ Consider the function $F(t) \equiv e^{-t\rho\hat{d}}e^{t(\rho\hat{d}+\lambda\hat{c})}e^{-t\lambda\hat{c}}$. Check that

$$\begin{aligned} F'(t) &= e^{-t\rho\hat{d}}(-\rho\hat{d})e^{t(\rho\hat{d}+\lambda\hat{c})}e^{-t\lambda\hat{c}} \\ &+ e^{-t\rho\hat{d}}e^{t(\rho\hat{d}+\lambda\hat{c})}(\rho\hat{d} + \lambda\hat{c})e^{-t\lambda\hat{c}} \\ &+ e^{-t\rho\hat{d}}e^{t(\rho\hat{d}+\lambda\hat{c})}(-\lambda\hat{c})e^{-t\lambda\hat{c}} \end{aligned}$$

Use $[\lambda\hat{c} + \rho\hat{d}, \hat{d}] = \lambda h$ to infer from the preceding questions that $F'(t) = t\lambda\rho h F(t)$. ✎

A.4 : ♭ Conclude that

$$e^{\rho\hat{d}+\lambda\hat{c}} = e^{\rho\lambda h/2}e^{\rho\hat{d}}e^{\lambda\hat{c}}.$$

✎

A.II Fock space of an oscillator

Let $\mathcal{F}$ be the Fock space associated to the pair of operators $\hat{a}, \hat{a}^\dagger$ such that $[\hat{a}, \hat{a}^\dagger] = 1$. We remind you that this space comes with

an orthonormal basis $\{|\,n\,\rangle; n = 0, 1, \cdots\}$ such that $\hat{a}|\,0\,\rangle = 0$, $\hat{a}|\,n\,\rangle = \sqrt{n}|\,n-1\,\rangle$ for $n = 1, 2, \cdots$ and $\hat{a}^\dagger|\,n\,\rangle = \sqrt{n+1}|\,n+1\,\rangle$ for $a = 0, 1, \cdots$.

The symbols a^* and a stand for commuting variables and the normal order of a function f of a^* et a with a series expansion $f = \sum_{m,n=0,1,\cdots} f_{m,n}(a^*)^m a^n$ is defined to be

$$:f: \equiv \sum_{m,n=0,1,\cdots} f_{m,n}(\hat{a}^\dagger)^m \hat{a}^n.$$

We define

$$q \equiv \frac{1}{\sqrt{2\omega}}(a^* + a) \quad \hat{q} \equiv :q: \quad p \equiv \sqrt{\frac{\omega}{2}}i(a^* - a) \quad \hat{p} \equiv :p:,$$

where ω is a constant > 0.

A.5 : Compute $[\hat{q}, \hat{p}]$. ✎

Let $\hat{H}_0 \equiv :\frac{1}{2}(p^2 + \omega^2 q^2):$.

A.6 : Write $\hat{H}_0$ as a function of $\hat{a}$ et $\hat{a}^\dagger$. Give an orthonormal basis of eigenstates of $\hat{H}_0$ and the corresponding energy levels. Compare $\hat{H}_0$ and $\frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2)$. ✎

A.III Fock space for the real free field

We work in generic dimension d [1].

A function $f(x_1, \cdots, x_d)$ defined in $\mathbb{R}^d$ is said to be periodic of period L if f is periodic of period L independently along the d components $x_1, \cdots, x_i, \cdots, x_d$, i.e. if f is invariant under the action of translations by all vectors in $I^* \equiv (L\mathbb{Z})^d$.We remind you that $I \equiv (\frac{2\pi}{L}\mathbb{Z})^d$ is the set of vectors that contribute to the Fourier series expansion of functions of period L in $\mathbb{R}^d$, and that for $\vec{k} \in I$,

$$\int_{[0,L]^d} d^d\vec{x}\, e^{i\vec{k}\vec{x}} = L^d \delta_{\vec{k},\vec{0}}.$$

Let $\mathcal{F}_{QFT}$ be the Fock space associated to pairs of operators $\hat{a}_{\vec{k}}$, $\hat{a}^\dagger_{\vec{k}}$, $\vec{k} \in I$ such that

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{l}}] = \delta_{\vec{k},\vec{l}} \qquad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{l}}] = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{l}}] = 0$$

[1] Apart from this slight generalization, the definitions and notations that follow are those used during the lectures and as such should be familiar. For the record, $d \simeq 3$ in the real world.

for $\vec{k}, \vec{l} \in I$. We remind you that this space has an orthonormal basis $\{|\overline{n}\rangle\}$ where $\overline{n} \equiv (n_{\vec{k}})_{\vec{k}\in I} \in \mathbb{N}^I$ runs over the maps from I to $\mathbb{N}$ such that $\sum_{\vec{k}\in I} n_{\vec{k}} < +\infty$. The state $|\overline{0}\rangle$ is the vacuum, annihilated by all the $\hat{a}_{\vec{k}}$'s, $\vec{k} \in I$ and

$$|\overline{n}\rangle \equiv \prod_{\vec{k}\in I} \frac{(\hat{a}^\dagger_{\vec{k}})^{n_{\vec{k}}}}{n_{\vec{k}}!}|\overline{0}\rangle.$$

The symbols $a^*_{\vec{k}}$ and $a_{\vec{k}}$ for $\vec{k} \in I$ denote commuting variables and we define the normal order of a function of the $a^*_{\vec{k}}$'s and $a_{\vec{k}}$'s with a series expansion $f = \sum_{\overline{m},\overline{n}} f_{\overline{m},\overline{n}} \prod_{\vec{k}\in I} (a^*_{\vec{k}})^{m_{\vec{k}}} (a_{\vec{k}})^{n_{\vec{k}}}$ by

$$:f: \equiv \sum_{\overline{m},\overline{n}} f_{\overline{m},\overline{n}} \prod_{\vec{k}\in I} (\hat{a}^\dagger_{\vec{k}})^{m_{\vec{k}}} (\hat{a}_{\vec{k}})^{n_{\vec{k}}}.$$

Let $m > 0$. For $\vec{k} \in I$ set $\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m^2}$. Define

$$\Phi(\vec{x}) \equiv \frac{1}{L^{d/2}} \sum_{\vec{k}\in I} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (a^*_{\vec{k}} e^{-i\vec{k}\vec{x}} + a_{\vec{k}} e^{i\vec{k}\vec{x}}) \qquad \hat{\Phi}(\vec{x}) \equiv :\Phi(\vec{x}):,$$

$$\Pi(\vec{x}) \equiv \frac{1}{L^{d/2}} \sum_{\vec{k}\in I} \sqrt{\frac{\omega_{\vec{k}}}{2}} i(a^*_{\vec{k}} e^{-i\vec{k}\vec{x}} - a_{\vec{k}} e^{i\vec{k}\vec{x}}) \qquad \hat{\Pi}(\vec{x}) \equiv :\Pi(\vec{x}):.$$

These functions, together with the associated operators, are periodic of period L on $\mathbb{R}^d$.

A.7 : What is the value of $[\hat{\Phi}(\vec{x}), \hat{\Pi}(\vec{y})]$? (Don't do the computation)✎

Let

$$\hat{H}_0 \equiv : \int_{[0,L]^d} d^d\vec{x}\, \frac{1}{2}(\Pi^2 + (\vec{\nabla}\Phi)^2 + m^2\Phi^2) :.$$

A.8 : Write $\hat{H}_0$ in terms of the operators $\hat{a}_{\vec{k}}$ an $\hat{a}^\dagger_{\vec{k}}$. Give an orthonormal basis of eigenstates of $\hat{H}_0$ and the corresponding energy levels.

Without computations, explain the difference between $\hat{H}_0$ and $\int_{[0,L]^d} d^d\vec{x}\, \frac{1}{2}(\hat{\Pi}^2 + (\vec{\nabla}\hat{\Phi})^2 + m^2\hat{\Phi}^2)$ and why the first one if preferable✎

Partie B : Harmonic oscillator.

We use again the notations of section A.II.

Let $\hat{H} = :\frac{1}{2}(p^2 + \Omega^2 q^2):$, where Ω is a constant > 0.

B.1 : Write $\hat{H}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Explain, without computation if possible, why $\langle 0|\hat{H}|0\rangle = 0$. ✎

Define operators $\hat{b}$ et $\hat{b}^\dagger$ acting on $\mathcal{F}$ via

$$\hat{b} \equiv \frac{(\Omega+\omega)\hat{a} + (\Omega-\omega)\hat{a}^\dagger}{2\sqrt{\omega\Omega}} \qquad \hat{b}^\dagger \equiv \frac{(\Omega+\omega)\hat{a}^\dagger + (\Omega-\omega)\hat{a}}{2\sqrt{\omega\Omega}}.$$

B.2 : Check (using that $\hat{a}$ and $\hat{a}^\dagger$ are adjoints of each other) that $\hat{b}$ and $\hat{b}^\dagger$ are adjoints of each other and compute $[\hat{b}, \hat{b}^\dagger]$. ✎

B.3 : Show that

$$\hat{H} = \Omega\hat{b}^\dagger\hat{b} - E_0(\Omega,\omega)$$

and give the formula for $E_0(\Omega,\omega)$. Deduce the formal spectrum of $\hat{H}$. By "formal" we mean "the spectrum in the usual Fock space of $\hat{b}$ and $\hat{b}^\dagger$" (whose relationship with $\mathcal{F}$ is not totally obvious). ✎

B.4 : We observe that $E_0(\Omega,\omega)$ is always positive, and we want to understand that. Starting from questions B.1 and B.3, show that

$$E_0(\Omega,\omega) = \Omega\|\hat{b}|0\rangle\|^2.$$

✎

To go beyond the formal spectrum, we look for a basis of eigenspaces of $\hat{H}$ on $\mathcal{F}$. We remind you that, for $|u| < 1$,

$$(1-u)^{-1/2} = \sum_{n=0}^{\infty} u^n \frac{(2n)!}{4^n n!^2} = \sum_{n=0}^{\infty} u^n \frac{(2n-1)!!}{2^n n!}.$$

B.5 : Show that there is in $\mathcal{F}$ a unique normalized state, denote it by $|0\rangle_\Omega$, annihilated by $\hat{b}$ and whose component along $|0\rangle$ is real and positive. What is the energy (for $\hat{H}$) of this state ?

We suggest you to look for this state as a sum $\sum_{n=0}^{\infty} c_n|n\rangle$, give a recursion relation for the c_n's, solve it, and then compute the normalization.

Check that $|0\rangle_\Omega$ can be written compactly as:

$$|0\rangle_\Omega = \sqrt[4]{\frac{4\omega\Omega}{(\omega+\Omega)^2}}\, e^{\frac{1}{2}(\hat{a}^\dagger)^2\frac{\omega-\Omega}{\omega+\Omega}}|0\rangle.$$

✎

Define

$$|n\rangle_\Omega \equiv \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle_\Omega, \quad n = 0, 1, \cdots .$$

B.6 : Explain without further computations why the states $\{|n\rangle_\Omega\}_{n=0,1,\cdots}$ are an orthonormal basis of $\mathcal{F}$. ✎

The computations from question B.5 show in particular that the expansion of $|0\rangle_\Omega$ on the states $|n\rangle$ involves only even n's. To interpret that, we define linear operator $\hat{U}$ on $\mathcal{F}$ by $\hat{U}|n\rangle = (-1)^n|n\rangle$

B.7 : Check that $\hat{U} = \hat{U}^\dagger = \hat{U}^{-1}$, and then that, as operators on $\mathcal{F}$,

$$\hat{U}\hat{a}\hat{U}^{-1} = -\hat{a} \quad \hat{U}\hat{a}^\dagger\hat{U}^{-1} = -\hat{a}^\dagger.$$

Infer that

$$\hat{U}\hat{b}\hat{U}^{-1} = -\hat{b} \quad \hat{U}\hat{b}^\dagger\hat{U}^{-1} = -\hat{b}^\dagger.$$

and then that $\hat{U}$ commutes with $\hat{H}_0$ et $\hat{H}$. What can you conclude for the eigenstates of $\hat{H}$ and in particular for $|0\rangle_\Omega$. ✎

Partie C : Quantum Field Theory.

We use again the notations of section A.III.

Let

$$\hat{H} \equiv : \int_{[0,L]^d} d^d\vec{x}\, \frac{1}{2}(\Pi^2+(\vec{\nabla}\Phi)^2+M^2\Phi^2) : = \hat{H}_0 + : \int_{[0,L]^d} d^d\vec{x}\, \frac{M^2-m^2}{2}\Phi^2 :.$$

where M is a constant > 0. For $\vec{k} \in I$ set $\Omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + M^2}$.

C.1 : Write $\hat{H}$ in terms of the $\hat{a}_{\vec{k}}$'s and $\hat{a}^\dagger_{\vec{k}}$'s. Express the result as a sum over $\vec{k} \in I$ of Hamiltonians $\hat{H}^{(\vec{k})}$ such that $\hat{H}^{(\vec{k})} = \hat{H}^{(-\vec{k})}$ depends only on $\hat{a}_{\vec{k}}$, $\hat{a}^\dagger_{\vec{k}}$, $\hat{a}_{-\vec{k}}$ and $\hat{a}^\dagger_{-\vec{k}}$. Justify that $\left[\hat{H}^{(\vec{k})}, \hat{H}^{(\vec{l})}\right] = 0$ for $\vec{k}, \vec{l} \in I$. ✎

Define operators $\hat{b}_{\vec{k}}$ and $\hat{b}^\dagger_{\vec{k}}$ acting on $\mathcal{F}_{QFT}$ by

$$\hat{b}_{\vec{k}} \equiv \frac{(\Omega_{\vec{k}} + \omega_{\vec{k}})\hat{a}_{\vec{k}} + (\Omega_{\vec{k}} - \omega_{\vec{k}})\hat{a}^\dagger_{-\vec{k}}}{2\sqrt{\omega_{\vec{k}}\Omega_{\vec{k}}}} \qquad \hat{b}^\dagger_{\vec{k}} \equiv \frac{(\Omega_{\vec{k}} + \omega_{\vec{k}})\hat{a}^\dagger_{\vec{k}} + (\Omega_{\vec{k}} - \omega_{\vec{k}})\hat{a}_{-\vec{k}}}{2\sqrt{\omega_{\vec{k}}\Omega_{\vec{k}}}}.$$

C.2 : Show that the transformation is such that $\hat{b}_{\vec{k}}$ et $\hat{b}^\dagger_{\vec{k}}$ are adjoints of each other and satisfy

$$[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{l}}] = \delta_{\vec{k},\vec{l}} \quad \vec{k}, \vec{l} \in I.$$

✎

You shall take for granted in the sequel that $[\hat{b}_{\vec{k}}, \hat{b}_{\vec{l}}] = [\hat{b}^\dagger_{\vec{k}}, \hat{b}^\dagger_{\vec{l}}] = 0$ $\vec{k}, \vec{l} \in I$.

C.3 : Set $s_{\vec{k}} \equiv \frac{\omega_{\vec{k}} - \Omega_{\vec{k}}}{\omega_{\vec{k}} + \Omega_{\vec{k}}}$. Show that (for $\vec{k} \neq \vec{0}$ fixed) the state

$$e^{s_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}}|\,\overline{0}\,\rangle$$

is normalizable with squared norm

$$\|e^{s_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}}|\,\overline{0}\,\rangle\|^2 = \frac{(\omega_{\vec{k}} + \Omega_{\vec{k}})^2}{4\omega_{\vec{k}}\Omega_{\vec{k}}},$$

and is annihilated by $\hat{b}_{\vec{k}}$ and $\hat{b}_{-\vec{k}}$.

It is suggested to expand the exponential in powers of $s_{\vec{k}}$ and to observe that $\hat{b}_{\vec{k}}$ is proportional to $\hat{a}_{\vec{k}} - s_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}$. ✎

C.4 : Show that $\Omega_{\vec{k}} \left(\hat{b}^\dagger_{\vec{k}} \hat{b}_{\vec{k}} + \hat{b}^\dagger_{-\vec{k}} \hat{b}_{-\vec{k}} \right) = \hat{H}^{(\vec{k})} + \hat{H}^{(-\vec{k})} + 2E_0(\Omega_{\vec{k}}, \omega_{\vec{k}})$ where E_0 is given by the same formula as in question B.3.

Infer, by use of the previous question, the spectrum of $\hat{H}^{(\vec{k})} + \hat{H}^{(-\vec{k})}$ acting on $\mathcal{F}_{QFT}$. ✎

In the rest of the problem, one can use without justification [2] the following fact. If $(u_{\vec{k}})_{\vec{k} \in I}$ is a family of reals indexed by I such that $u_{\vec{k}} \sim C|\vec{k}|^{-\delta}$ for $|\vec{k}| \to \infty$ the the sum $\sum_{\vec{k} \in I} u_{\vec{k}}$ is convergent if and only if $\delta > d$.The notation $|\vec{k}|$ stands for the Euclidean norm of $\vec{k}$.

C.5 : Show that the infinite product $\prod_{\vec{k} \in I} \frac{(\omega_{\vec{k}} + \Omega_{\vec{k}})^2}{4\omega_{\vec{k}} \Omega_{\vec{k}}}$ converges if and only if $d < 4$. You can reduce the question to the consideration of a sum by taking the logarithm. ✎

C.6 : Conclude using question C.3 that for $d < 4$ the state

$$e^{\frac{1}{2} \sum_{\vec{k} \in I} \hat{a}^\dagger_{\vec{k}} \hat{a}^\dagger_{-\vec{k}} \frac{\omega_{\vec{k}} - \Omega_{\vec{k}}}{\omega_{\vec{k}} + \Omega_{\vec{k}}}} |\overline{0}\rangle$$

is normalizable and annihilated by all $\hat{b}_{\vec{k}}$'s. Give its norm. The corresponding normalized state will be denoted by $|\overline{0}\rangle_M$.

What happens for $d \geqslant 4$? Are the usual Fock representation of the $\hat{b}_{\vec{k}}$'s and $\hat{b}^\dagger_{\vec{k}}$'s, and their representation in $\mathcal{F}_{QFT}$ equivalent? ✎

C.7 : Show that $\sum_{\vec{k} \in I} \frac{(\Omega_{\vec{k}} - \omega_{\vec{k}})^2}{4\omega_{\vec{k}}}$ converges if and only if $d < 3$. ✎

C.8 : Give, for $d < 3$, the value of $\hat{H}$ acting on $|\overline{0}\rangle_M$, and infer that $\hat{H}$ is diagonalizable on $\mathcal{F}_{QFT}$. Give its spectrum without further computations.

Show that $\hat{H}$ is not bounded below if $d \geqslant 3$. ✎

C.9 : For $d < 3$, write the energy density (i.e. the energy per unit volume) of the ground state of $\hat{H}$ acting on $\mathcal{F}_{QFT}$ in the limit $L \to \infty$ ✎

C.10 : Compute this integral explicitly for $d = 2$. Restore the powers of c and $\hbar$. ✎

[2] But preferably with some understanding of why it is true!