Self Organized Percolation and Critical Sales Fluctuations

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Abstract

There is a discrepancy between the standard view of equilibrium through price adjustment in economics and the observation of large fluctuations in stock markets. We study here a simple model where agents decisions not only depend upon their individual preferences but also upon information obtained from their neighbours on a social network. The model shows that information diffusion coupled to the adjustment process drives the system to criticality with large fluctuations rather than converging smoothly to equilibrium.

1 Introduction

In the perspective of an economy at equilibrium, random deviations from equilibrium should appear as some Gaussian uncorrelated noise. This is often the model used by economists when discussing risk and uncertainty. In fact markets such as stock markets display lot of fluctuations, and time series analysts recognize the existence of "fat tails" in returns distributions, namely the fact the relative probability of very large fluctuations is larger than a Gaussian distribution would predict (see for instance Bouchaud and Potters (2000), Lux and Ausloos (2000)). Apparently the "market" has some memory of previous fluctuations. Such memory effects could always be attributed to external causes, but the purpose of the present paper is to show that simple endogenous effects such as information propagation could account for not IID (independent identically distributed) fluctuations in the market. An alternative explanation, based on learning agents, has been proposed by Steyer and Zimmermann (2000). We will compare both approaches in the conclusion section.

2 The model

2.1 The basic percolation model

We will use a "social percolation" model introduced by Solomon et al. (2000). A product is characterised by a scalar quality q, taken on interval [0,1]. A customer i is characterised by a scalar preference p_i taken on interval [0,1]. The idea is that customers have different expectations (or preferences) about products and that they will not accept a product of quality q lesser than their preference p_i. We will always use in the present paper preferences randomly distributed from a uniform distribution among customers.

Purchase of a product is limited by the availability of private information from neighbours supposedly trusted by the purchasing agent. We have in mind products for which customers trust their acquaintances rather than products with well defined specifications. Another possibility is the existence of positive externalities, namely the fact that the utility of posessing an equipment is increased when your neighbour already has the same equipment through exchange of help and/or information. Classical examples of positive externalities are communication and computer equipments. In the basic model a customer then purchases products under two conditions:

- he get some private information from one of his neighbours;
- the quality q of the product is higher than his preference p_i.

The symmetrical neighbourhood relationship defines a social network of connected agents. The construction of the social network and the repartition of agents properties across it can be the big unknown of the problem. Because of the lack of relevant information about the specifics of the network and of agents repartition on it, one generally considers certain classes of networks, for instance periodic lattices (with periodic boundary conditions) or random networks, and random distributions of preferences p_i on the network. When one considers a product which agents usually need only once, e.g. going to a specific movie, or buying some equipment, the density of potential adopters, i.e. those agents such $p_i > q$ is the critical factor which determines the market share of the product. If this density is larger than a critical threshold p_c, the percolation threshold (Stauffer and Aharony 1994) related to the connectivity of the network, the influence of even a small fraction of initial purchasers propagates across the network and a large majority of agents purchase the product. One says that percolation of influence occurs. On the opposite, for smaller densities of potential adopters, influence does not propagates and market shares remain close to the fraction of initial adopters. Because this dynamical behaviour is so contrasted, one accepts its conclusion derived for many instances of periodic and random networks and apply them to social networks which structure is largely unknown. The percolation property is "generic", which means that it is predictable with a large degree of accuracy for any network on each side of the percolation threshold, but not at the percolation threshold itself. These conclusions are valid for any kind of network, though the percolation threshold p_c changes according to the dimensionality of regular lattices or the connectivity of random nets.

2.2 The iterated percolation model

The basic model that we just described supposes a good which is purchased only once by buyers. Different versions can be imagined when a good is repetitively presented to customers. In Solomon et al. (2000) for instance, a movie maker is proposing a series of movies at intervals larger than the time necessary for the information to percolate across the network. Here instead we suppose that the time necessary for the information to propagate across the social net is larger than the time between successive purchases by the same buyer. In other words, after purchasing a good, a buyer is not active during some kind of a refractory period m in analogy with the phenomena observed in nerve tissues or chemical reactions. The rationale for this hypothesis in economics is the existence of some consumption time for a commodity or some decay time for an investment between successive purchases. In the iterated percolation model three conditions are then necessary for purchase by an agent:

- The agent has not taken any decision (to purchase or not) for a period at least equal to the refractory period m;

- he get some private information from one of his neighbours;
- the quality q of the product is higher than his preference p_i.

We further add some adjustment dynamics to the above decision process. We suppose that agents which purchased before the refractory period increase their preference coefficient for the next period by choosing p_i through a random sampling between q and 1. Symmetrically, we suppose that agents who did not purchase before the refractory period decrease their preference coefficient for the next period by choosing p_i through a random sampling between 0 and q. This adjustment dynamics is not very different from the tatonnement process proposed by economists such as Lesourne (1991) and Diamond (1989), but its coupling with information propagation results in a different dynamic behaviour.

In the case of iterative purchase, waiting until one of your neighbours purchases still make sense for the interpretation based on positive externalities. For the interpretation based on information propagation, waiting for private information from a neighbour after the first purchase might seem unnecessary, since the agent should already know about q. The idea here is that q might change through some adjustment process of the producer, hence the necessity for a prospective buyer to wait for up-to-date private information before any decision. Since only the sign of the quantity $q - p_i$ matters for the decision process, readjusting randomly p_i should be equivalent to readjusting randomly both q and p_i. We ran most simulations with fixed q and tested the hypothesis on equivalence by a few runs with adaptative q.

3 Simulation results

3.1 General conditions

The above defined model was run on square lattices with four neighbours and periodic boundary conditions. We ran many numerical simulations. We used the Leath algorithm to follow the time evolution of purchases. In the description of the results, one time step corresponds to the propagation of information and of the purchasing process from all purchasers at the previous time step to all the immediate neighbours. The following quantities were monitored:

- Time evolution of purchases at each time step.

- Patterns of purchase in the network at given time steps (in fact we also observed on line the time evolution of these patterns during the simulations).
- Average fraction of agents below the q threshold, i.e. average fraction of agents willing to purchase when updated.
- Average fraction of actual purchase at any given time step. (These average fractions were taken after an initial growth process of 100 time steps).
- The purchase process eventually comes to an end if no purchase is made during a time step. We have measured the average final time whenever the purchase process actually stops, and the frequency of such occurrences. Averages were most often made over 5 000 runs. (The averages for the fractions of purchase and of agents willing to purchase are only taken for the set of non-stopping runs).
- For the set of non-stopping runs we averaged the power spectrum of purchases.

The simulations were made on square lattices of size L equal to 40, 60, 80 and 160. We used periodic boundary conditions. The refractory period was varied from 4 to L. For m larger than L the purchase process stops after one sweep of the network by the purchase front (see further) by lack of potential purchasers. One single run normally goes on for 4096 time steps: this is more than necessary to take fast Fourier transform since coherence is lost after roughly $2L$ time steps.

3.2 Typical results for a representative set of parameters

Let us analyse a first experimental condition with $L = 40, m = 10$, runtime of 4096 steps and averages taken on 500 runs. The first figure displays a typical time plot of sales. One can notice large fluctuations of amplitude with short (around 70 steps) time scale correlations, very different from white noise.

A typical pattern displaying the states of the agents is shown in figure 2. White cells correspond to "awake" agents that are ready to make a choice when triggered by their active neighbours. Their previous choices occurred

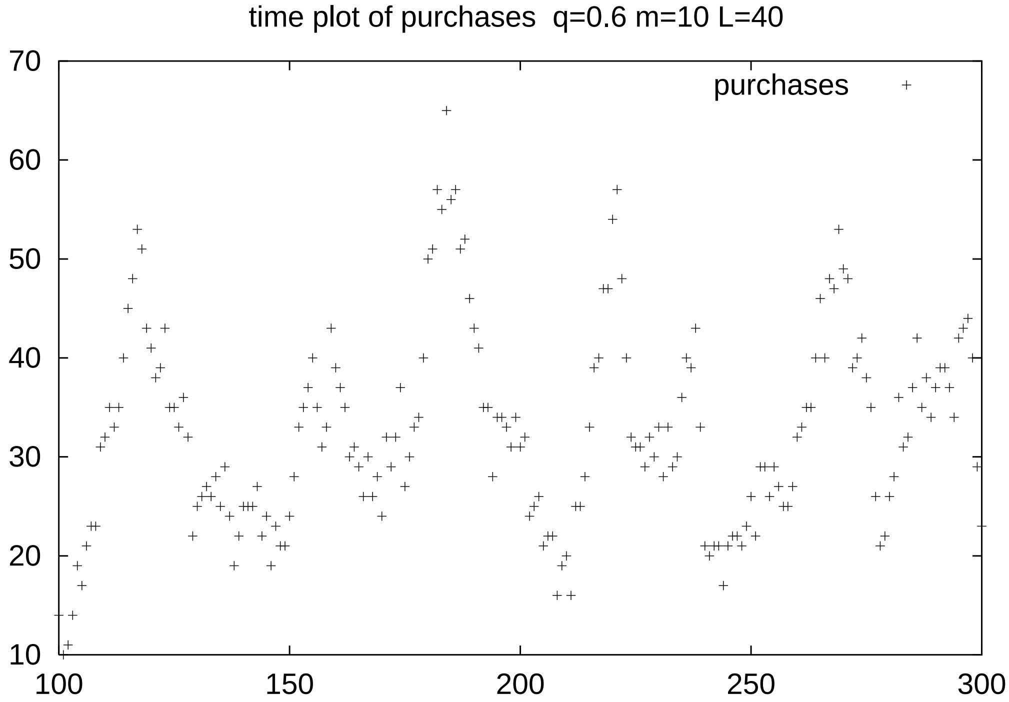

Figure 1: Time plot of purchases after 100 time steps for a 40×40 square lattice, with a refractory period of 10 and quality 0.6 slightly above the percolation threshold.

more than m time steps ago. Other cells made their choice more recently: black cells correspond to agents who purchased and grey cells to agents who refused to purchase. On average (500 runs and 4096 time steps) 2 percent of the agent purchase at any time step. They are located on the black positions at the edges of the dark clusters, forming a disconnected purchasing front. On line monitoring of the patterns dynamics shows the displacement of the purchasing fronts across the lattice. The typical time scale, around 70, observed in figure 1 for the low frequency of the oscillation is similar to the time it takes to a front to move across the lattice and it scales with L.

We observed an average fraction of 0.593 potential purchasers, very close to the percolation threshold, which explains why quite often (19 percent of observed runs) the run stop at an average time of 358, i.e. after a few sweeps across the lattice.

Finally, figure 3 displays the power spectrum of purchases averaged on those 81 percent of the 5000 runs which carried across 4096 time steps without stop. We observe a power law with a -2 exponent between periods 4 and 40. The flat spectrum at larger periods corresponds to the loss of coherence for times larger than the sweep time of fronts across the lattice.

3.3 Influence of simulation parameters

The quality parameter q

When q is less than the percolation threshold purchases stop very early

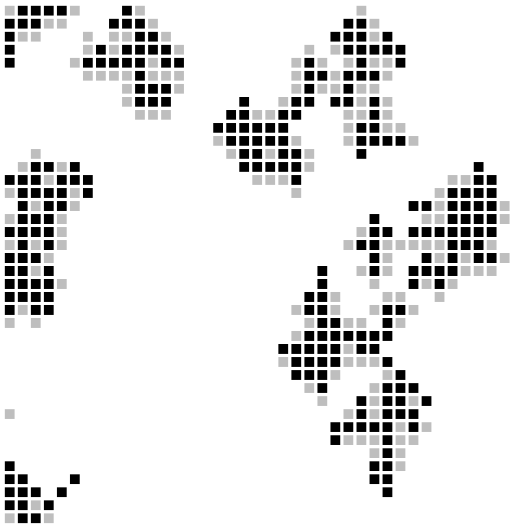

Figure 2: Pattern of purchases on a 40 × 40 square lattice. White cells correspond to "awake" agents that are ready to make a choice when triggered by their active neighbours. Black cells correspond to agents during their refractory period who purchased and grey cells to agents who refused to purchase.

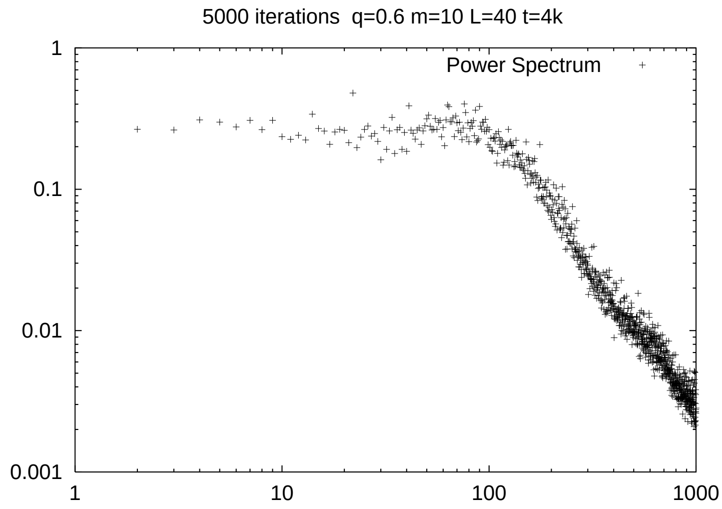

Figure 3: Log-log plot of the power spectrum of purchases computed on a time span of 4096 time steps and averaged over 3300 runs. 1000 on the x axis correspond to 4 periods, 100 to 40 periods. The high frequency spectrum obeys a power law with a -2 exponent.

since there is no percolation across the lattice. When q is higher than the percolation threshold, the dynamics soon reshuffle the distribution of individual preferences p_i such that the fraction of agents with preferences below q get closer to the percolation threshold. For instance when $q = 0.7$, the fraction of purchasers becomes 63 percent. But when q increases the proportion of stopping runs increases dramatically because of the too fast readjustment of p_i: after the first sweep, the distribution of p_i are biased towards high values and percolation is not possible any more. At $q = 0.9$ most runs (95 percent) stop after 51 time steps on average, i.e. soon after the first sweep of the lattice. Non-stopping runs display a fraction of 71 percent purchasers on average.

The refractory period m

At low values of m, the sweeping process of the purchase front across the lattice is less dominant and a lot of re-purchasing can occur locally. Purchase rate is doubled at $m = 4$ with respect to $m = 10$ and nearly all runs persist for long time, except 9 percent which stop very early after 7 time steps on average. The power spectrum does not display a well characterized power law (see figure 4).

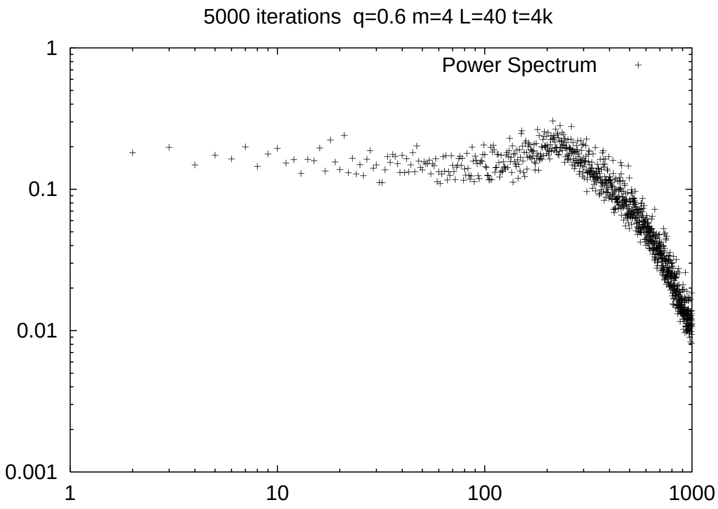

Figure 4: Log-log plot of the power spectrum of purchases for a refractory period of 4 on a 40×40 lattice.

The lattice size L

For the intermediate range of m, $L/4 < m < L/2$ the low frequency cut-off for the power law scales in inverse proportion to L as seen in figure 5. For smaller values of m, $m = 4$ i.e., the power spectrum is insensitive to L. This is a check of the importance of local re-purchase.

3.4 Model variants

We made preliminary studies of several variants of the model.

As mentionned in the introduction, we also tested the equivalence between the model with fixed q and a model with adaptive q to test the equivalence of dynamic properties of both model. We supposed that after each time step, a producer changes the quality of the proposed product by a "tatonnement" process: when sales increased (resp. decreased), she decreases q in proportion (resp. increases) through equation:

$$q(t+1) = q(t) + [sales(t-1) - sales(t)]\delta q, \tag{1}$$

where δq is the proportionality factor. We tried $\delta q = 0.0001$ and $\delta q = 0.00001$ corresponding to ajdustments of roughly 0.1 and 0.01 per lattice sweep for q.

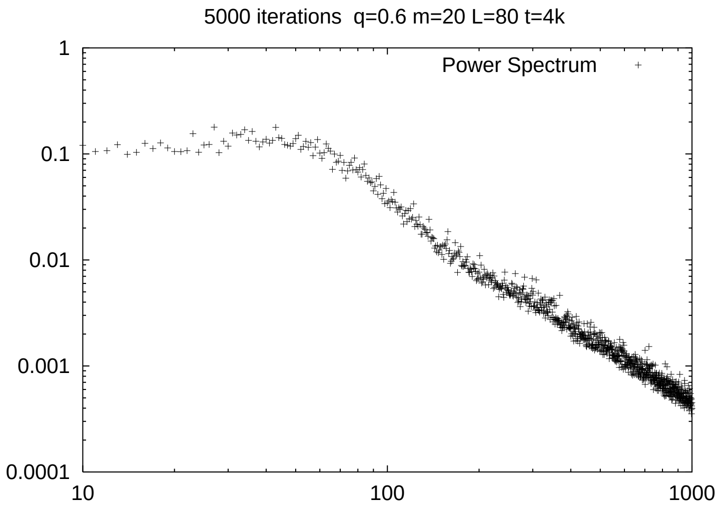

Figure 5: Log-log plot of the power spectrum of purchases for a refractory period of 20 on a 80×80 lattice.

The obtained Fourier spectra display the same scaling behaviour as for fixed q.

In the small world variant, a small fraction (typically a few per thousand) of the connections were randomly assigned across the whole lattice without taking into account any notion of neighbourhood. The results were qualitatively analogous to those obtained when all connections are periodic.

We also tried to change the readjustment process of those agents which refused to purchase. Rather to randomly redistribute them in the segment $[0, q]$, we redistributed them in the segment $[0, p_i]$. The same dynamics were observed.

Finally another change was a slower adjustment of preferences: purchasers preferences being redistributed in $[p_i, p_i + \alpha(1 - p_i)]$ and non purchasers' preferences in $[p_i, (1 - \alpha)p_i]$ with $0 < \alpha < 1$. Once more we did not observe big changes.

4 Conclusions

Apart from the technicalities, the obvious conclusion is that the information contagion process is enough to create the long term correlations which are displayed in the sales fluctuation power spectrum. In other words, when coupled to an imitation based decision process, the tatonnement process does not lead to equilibrium and small fluctuations but to criticality and a power law fluctuation spectrum. The refractory period and the readjustment process sustain exchanges long enough to allow us to measure the power spectrum: correlations only exist as long as activity is maintained in the system.

The alternative mechanism proposed by Steyer and Zimmermann (2000) to explain long term correlations is also based on information diffusion on a social network but implies a Hebbian learning mechanism: connections among neighbours are of varying intensity, and are re-inforced when agents take similar decisions. The decision process of individual agents is based on polling their neighbours rather than simply being triggered by any active neighbour. Steyer and Zimmermann report in the presence of Hebbian learning an even more correlated behaviour with a $1/f$ power spectrum.

Our conjecture based on the two sets of simulations and on reasonable arguments is that long term correlations and non-exponential distribution of fluctuation sizes should be expected whenever decision processes are based on information contagion. Lux and Ausloos (2000) express the same idea by referring more generally to interactions among economic agents as the generating mechanism. Reciprocally, when long term correlations are observed, an information contagion process is a possible explanation for the effect.

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