ICFP M2 – Selected Topics in Statistical Field Theory
TD n°1 & 2 – $O(N)$ model

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Consider the following Landau-Ginzburg Hamiltonian, $\beta\mathcal{H}[\boldsymbol{\phi}] = \int \mathrm{d}^d\boldsymbol{x}\ \mathcal{L}[\boldsymbol{\phi}(\boldsymbol{x})]$, with

$$\mathcal{L}[\boldsymbol{\phi}(\boldsymbol{x})] = \left[\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\phi})^2 + \frac{t}{2}\boldsymbol{\phi}^2 + \lambda\,(\boldsymbol{\phi}^2)^2 - \boldsymbol{h}\cdot\boldsymbol{\phi}\right],$$

where $\boldsymbol{\phi}(\boldsymbol{x})$ is a N-component real vector field, the "mass^2" t and the interaction λ are phenomenological parameters with $\lambda > 0$. Unless otherwise stated, we consider the situation with a vanishing external field, $\boldsymbol{h} = 0$, where the model is $O(N)$-symmetric. The partition function reads

$$\mathcal{Z} = \int \mathcal{D}[\boldsymbol{\phi}]\,\mathrm{e}^{-\beta\mathcal{H}[\boldsymbol{\phi}]}\,.$$

If need be, the theory is regularized in the UV by a microscopic cutoff length scale $a = \Lambda^{-1}$.

A – Mean-field theory

(1) Justify that, in the saddle-point approximation, the free energy per unit volume is given by $\beta\mathcal{F} = \min\{\mathcal{L}(\bar\phi);\ \bar\phi \in \mathbb{R}\}$. Sketch $\mathcal{L}(\bar\phi)$ for $t < 0$ and $t > 0$ at $h = 0$. Discuss the resulting phase transition.

(2) Compute the spontaneous magnetization $\bar\phi(h = 0)$, deduce the critical exponent β.

(3) Compute the longitudinal susceptibility on both sides of the transition, $\chi_\pm \sim |t|^{-\gamma_\pm}$, and identify the critical exponents $\gamma_\pm$.

(4) Sketch the magnetization curves, *i.e.* $\bar\phi$ as a function of h, for $t > 0$, $t = 0$, and $t < 0$.

(5) Compute the zero-field free energy per unit volume $\mathcal{F}$ for $t > 0$ and $t < 0$. Deduce the heat capacity $C(h = 0) = -T\frac{\partial^2\mathcal{F}}{\partial T^2}$.

B – Fluctuations & Failure of mean-field theory

(1) Include longitudinal and transverse fluctuations by setting

$$\boldsymbol{\phi}(\boldsymbol{x}) = \big(\bar\phi + \phi_l(\boldsymbol{x})\big)\,\boldsymbol{e}_1 + \sum_{\alpha=2}^{N}\phi_{t,\alpha}(\boldsymbol{x})\boldsymbol{e}_\alpha\,,$$

and expand $\beta\mathcal{H}$ to second order in the $\boldsymbol{\phi}_\alpha$'s.

(2) Introducing the Fourier modes *via* $\phi_\alpha(\boldsymbol{x}) = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \phi_\alpha(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, compute the two-point correlation function $\langle \phi_\alpha(\boldsymbol{k})\phi_\beta(\boldsymbol{k}')\rangle$ and identify the longitudinal and transverse correlation lengths, ξ_l and ξ_t, respectively. Give the decay of the real-space two-point correlation function at the phase transition.

(3) Recompute the free energy per unit volume, $\mathcal{F}$, and show that the singular contribution to the heat capacity reads

$$C_{\text{sing}}(h=0) \propto -\frac{\partial^2 \beta\mathcal{F}}{\partial t^2} = \begin{cases} 0 + \frac{N}{2}\int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{(k^2+t)^2} & \text{for } t>0 \\ \frac{1}{8u} + 2\int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{(k^2-2t)^2} & \text{for } t<0 \end{cases} \tag{1}$$

Discuss how the corrections due to the Gaussian fluctuations modify the discontinuity of the heat capacity, depending on the dimension d. Conclude on the reliability of the mean-field approach at the transition.

C – Failure of perturbation theory

Let us separate $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$ with the Gaussian Hamiltonian

$$\beta\mathcal{H}_0 = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{2}\left(k^2+t\right)|\boldsymbol{\phi}(\boldsymbol{k})|^2$$

and the interaction term

$$\begin{aligned}\mathcal{U} &= \lambda \int \mathrm{d}^d\boldsymbol{x}\,\left(\boldsymbol{\phi}^2(\boldsymbol{x})\right)^2 \\ &= \lambda \int \frac{\mathrm{d}^d\boldsymbol{k}_1\,\mathrm{d}^d\boldsymbol{k}_2\,\mathrm{d}^d\boldsymbol{k}_3}{(2\pi)^{3d}}\,\phi_\alpha(\boldsymbol{k}_1)\phi_\alpha(\boldsymbol{k}_2)\phi_\beta(\boldsymbol{k}_3)\phi_\beta(-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\end{aligned}$$

(1) Compute $\langle \phi_\alpha(\boldsymbol{k})\phi_\beta(\boldsymbol{k}')\rangle$ to first order in λ.

(2) Deduce the magnetic susceptibility, $\chi = \lim_{k\to 0} S(k)$ with $S(k) = \langle |\phi_1(\boldsymbol{k})|^2\rangle$. How is the critical temperature shifted?

(3) How does the susceptibility diverge at the shifted critical point? Conclude on the applicability of perturbation theory for describing the divergence of the susceptibility close to the critical point in dimensions $d<4$.

D – Perturbative renormalization group

(1) Coarse graining. Write down the partition function $\mathcal{Z}$ by separating the fluctuations into two components as, $\boldsymbol{\phi}(\boldsymbol{k}) = \bar{\boldsymbol{\phi}}(\boldsymbol{k})$ for $0<k<\Lambda/b$ and $\boldsymbol{\phi}(\boldsymbol{k}) = \tilde{\boldsymbol{\phi}}(\boldsymbol{k})$ for $\Lambda/b<k<\Lambda$. Integrate over the short-wavelength fluctuations by first expanding to first order in λ. Write down the coarse-grained Hamiltonian $\beta\overline{\mathcal{H}}[\bar{\boldsymbol{\phi}}]$ in terms of renormalized parameters $\bar{t}$ and $\bar{\lambda}$ to be expressed as a function of the original t and λ.

(2) Rescale by setting $\boldsymbol{k} = b^{-1}\boldsymbol{k}'$.

(3) Renormalize $\bar{\boldsymbol{\phi}} = z\boldsymbol{\phi}'$ with $z = b^{1+d/2}$ and derive the flow equations for the parameters t and λ by setting $b = \mathrm{e}^l$ and considering an infinitesimal δl.

(4) Linearize in the vicinity of the Gaussian fixed point, identify the scaling dimensions Δ_t and Δ_λ associated to the parameters t and λ, respectively, and discuss the stability of the fixed point.

(5) Let us turn on the (relevant) symmetry breaking field $\boldsymbol{h}$. Perform the steps D (1) to D (3) on the associated term in the Landau-Ginzburg Hamiltonian. Deduce Δ_h, the scaling dimension of h.

E – Scaling interlude

The partition function $\mathcal{Z}$ is dimensionless, and by construction it is invariant under the steps D (1) to D (3).

(1) Show that the free energy per unit volume scales as

$$\mathcal{F}(t,h) = t^{d/\Delta_t} f(h/t^{\Delta_h/\Delta_t}) \,, \tag{2}$$

and deduce the Josephson's identity $2-\alpha = d/\Delta_t$ where α is the critical exponent of the (singular part of the) heat capacity defined as $C_{\text{sing}}(h=0) \sim |t|^{-\alpha}$.

(2) Find the scaling of the correlation length $\xi(t,h)$ and express the critical exponent ν, defined as $\xi(h=0) \sim |t|^{-\nu}$, in terms of the scaling dimensions.

(3) Repeat for the spontaneous magnetization $\bar{\phi}(h=0) = -\frac{1}{V}\frac{\partial \ln \mathcal{Z}(t,h)}{\partial h}|_{h=0}$ and the susceptibility $\chi(h=0) = \frac{\partial \bar{\phi}}{\partial h}|_{h=0}$. Express the critical exponents β and γ in terms of the scaling dimensions.

F – Challenge

Perform steps D (1) to D (3) by now expanding up to order λ^2. Identify the new, non-Gaussian, perturbative fixed point, and analyze its stability depending on the dimension. Simplify the expressions by only retaining the first-order terms in $\epsilon = 4-d$ (the so-called ϵ expansion around $d=4$). Compute the critical exponents α, β and γ to order ϵ. Discuss.