ICFP M2 – Selected Topics in Statistical Field Theory
TD n$^{\mathrm{o}}$ 4 – Elements of stochastic calculus

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A – Preamble: damped oscillator

Let us consider a particle of mass m subject to a deterministic force f and in contact with a thermal bath at temperature T. The dynamics of its position (here in 1d) is described by the Langevin equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = f(x) \underbrace{-\eta_0 \frac{\mathrm{d}x}{\mathrm{d}t} + \xi(t)}_{f_{\mathrm{bath}}}\,, \tag{1}$$

where $\eta_0 \geq 0$ is a friction coefficient and $\xi(t)$ is a Gaussian white noise: $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\eta_0 T \delta(t-t')$ where $\langle \ldots \rangle$ indicates the average over the noise realizations.

(1) Discuss briefly the physics behind each term of f_{bath}. Give $P[\xi]$, the probability density for a given realization of $\xi(t)$. Compute $\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle$.

(2) Let us now focus on the case of an harmonic oscillator, *i.e.* $f(x) = -kx$ with $k > 0$. In the absence of a surrounding bath ($\eta_0 = 0$), what is the typical time scale τ_{syst} of the system?

(3) In the presence of a zero-temperature bath (*i.e.* $\eta_0 > 0$ and $T = 0$), use dimensional analysis to identify τ_{relax}, the typical time scale associated with the relaxation due to the bath's friction.

(4) Overdamped limit: when $\tau_{\mathrm{relax}} \ll \tau_{\mathrm{syst}}$, argue that the dynamics may be simplified to the so-called overdamped Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + \xi(t)\,, \tag{2}$$

where the time t has been rescaled such that $\langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')$.

B – Multiplicative noise

Let us now consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)\xi(t)\,, \tag{3}$$

where the function $g(x)$ is a strictly positive regular function for all x. Given that it multiplies a function of the dynamical variable $x(t)$, the noise is now called "multiplicative" [as opposed to "additive" in Eq. (2)].

(1) As it stands, the expression in (3) is not a well-defined equation. Discuss the issue brought by the multiplicative factor $g(x)$.

I. Itô Stochastic Differential Equation (Itô SDE)

The Itô prescription consists in giving a meaning to (3) with the following discretization:

$$\mathrm{d}x_n = f(x_n)\mathrm{d}t + g(x_n)\mathrm{d}W_n\,, \tag{4}$$

where time was discretized in finite intervals $\mathrm{d}t$, $\mathrm{d}x_n \equiv x_{n+1}-x_n$, and $\mathrm{d}W_n \equiv W_{n+1}-W_n$ where W_n is a Wiener process with $\langle \mathrm{d}W_n\rangle = 0$ and $\langle \mathrm{d}W_m \mathrm{d}W_n\rangle = 2T\delta_{mn}\mathrm{d}t$.

(1) Show that $\langle g(x)\xi(t)\rangle = 0$, and deduce the equation describing the dynamics of the average position $\langle x(t)\rangle$.

(2) Itô formula. Using the discrete Itô SDE (4), show that for any function $A(x)$,

$$\frac{\mathrm{d}A(x)}{\mathrm{d}t} \equiv \lim_{\mathrm{d}t\to 0} \frac{A(x_{n+1}) - A(x_n)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} A'(x) + Tg^2 A''(x)\,. \tag{5}$$

(3) Fokker-Planck equation. The average $\langle \frac{\mathrm{d}A(x)}{\mathrm{d}t}\rangle$ can be expressed *via* the above Itô formula, and also in terms of the probability distribution $P(x,t)$ to find the particle in x at time t: $\langle \frac{\mathrm{d}A(x)}{\mathrm{d}t}\rangle = \int \mathrm{d}x\; A(x)\partial_t P(x,t)$. Deduce the Fokker-Planck equation associated with the Itô SDE,

$$\partial_t P(x,t) = -\partial_x \left[f(x)P(x,t)\right] + T\partial_x^2 \left[g^2(x)P(x,t)\right]\,. \tag{6}$$

II. Stratonovich Stochastic Differential Equation (Strato SDE)

The Stratonovich prescription consists in giving a meaning to (3) with the following discretization:

$$\mathrm{d}x_n = f(x_n)\mathrm{d}t + g\left(\frac{x_{n+1}+x_n}{2}\right)\mathrm{d}W_n\,. \tag{7}$$

(1) Compute $\langle g(x)\xi(t)\rangle$, and deduce the equation describing the dynamics of the average position $\langle x(t)\rangle$.

(2) Using the discrete Strato SDE (7), show that any function $A(x)$ obeys the rules of conventional calculus, *i.e.*

$$\frac{\mathrm{d}A(x)}{\mathrm{d}t} \equiv \lim_{\mathrm{d}t\to 0} \frac{A(x_{n+1}) - A(x_n)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} A'(x)\,. \tag{8}$$

(3) Strato to Itô. By developing $g\left(\frac{x_{n+1}+x_n}{2}\right)$ around x_n, bring the Strato SDE (7) to an equivalent Itô SDE.

(4) Fokker-Planck equation. Use the above result and Eq. (6) to derive the Fokker-Planck equation associated with the Strato SDE (7).

III. Non-linear change of variable

Let us consider the Itô SDE in (4), and its associated Fokker-Planck equation (6). Instead of working with the variable x, we are free to perform a change of variable $x \to \tilde{y} = y(x)$ where $y'(x) > 0$ for all x.

(1) Performing the change of variable in Eq. (6), identify the corresponding Fokker-Planck equation on $\tilde{P}(\tilde{y},t)$.

(2) Deduce the corresponding SDE governing the dynamics of the new variable $\tilde{y}$.

(3) Repeat when x obeys the Strato SDE (7).

IV. Itô Vs Strato: which prescription to choose?

In practice, the variance of the bath-induced noise is never infinite: $\langle \xi(t)\xi(t')\rangle = 2T\eta(t-t')$ where $\eta(t) \geq 0$ is a regular function that depends on the details of the bath and its coupling to the system. Typically, $\eta(t \gg \tau_{\text{bath}}) \to 0$ where τ_{bath} is a typical time-scale of the bath.

(1) What are the two physical limits that were performed to get to the pre-equation (3)? Which order of limits yields the Itô prescription? The Stratonovich prescription?

C – Challenge: from multiplicative to additive noise

Find the non-linear change of variable $x \to \tilde{u} = u(x)$ which maps the multiplicative-noise Itô SDE

$$\frac{\mathrm{d}x}{\mathrm{d}t} \overset{\text{Itô}}{=} f(x) + g(x)\xi(t)\,, \tag{9}$$

to an additive-noise SDE

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} \overset{\text{Itô}}{=} \tilde{f}(u) + \tilde{\xi}(t)\,, \tag{10}$$

where the new $\tilde{f}$ and $\tilde{\xi}$ will be expressed in terms of the former f, g, and ξ.