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Two-loop functional renormalization for elastic manifolds pinned by disorder in *N* dimensions

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We study elastic manifolds in an N-dimensional random potential using a functional renormalization group. We extend to $N>1$ our previous construction of a field theory renormalizable to two loops. For isotropic disorder with $O(N)$ symmetry we obtain the fixed point and roughness exponent to next order in $\epsilon=4-d$, where d is the internal dimension of the manifold. Extrapolation to the directed polymer limit $d=1$ allows some handle on the strong coupling phase of the equivalent N-dimensional Kardar-Parisi-Zhang growth equation, and eventually suggests an upper critical dimension $d_u \approx 2.5$.

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Disordered elastic systems are under extensive study both theoretically and experimentally. They are of interest for a number of physical systems, such as charge density waves (CDW) [1], flux lattices [2,3], wetting on disordered substrates [4], and magnetic interfaces [5], where the interplay between the internal order and the quenched disorder of the substrate produces pinned phases with nontrivial roughness and glassy features [6]. Typically they are described by elastic objects, with internal d-dimensional coordinate x, parametrized by an N-component height, or displacement field $u(x)$. Analytical methods are scarce, and developing a field-theoretical description poses a considerable challenge. One reason is that naive perturbative methods fail, technically due to the breakdown of the dimensional reduction phenomenon [7], and physically because describing the multiple energy minima in a glass seems to contain some nonperturbative features. One subset of these problems, the directed polymer (i.e., $d=1$) in a random potential, maps onto the Kardar-Parisi-Zhang (KPZ) growth problem, well known to exhibit a strong coupling phase, which is out of reach of standard perturbative methods [8]. It is thus important to obtain a field-theoretical description of this phase, since the value and even the existence of its upper critical dimension is still a matter of considerable debate [9,10].

One method that holds promise to tackle this class of problems is the functional renormalization group (FRG). Although it was presented long ago, within a one-loop Wilson scheme [12–14], it is, not so surprisingly, hampered with difficulties, and only recently attempts have been made to push the method further [15–22]. The main problem is that the effective action at zero temperature becomes nonanalytic at a finite scale, the Larkin scale, where metastability appears. Although fixed points are accessible in a $d=4-\epsilon$ expansion, nonanalyticity results in apparent ambiguities in the renormalized perturbation theory at $T=0$ [16,19]. These problems are absent at $T>0$ [23,24] (at least at leading order and for $N=1$) but since temperature is dangerously irrelevant, the finite temperature description is rather complicated [21]. Until now, it has lead to a complete first-principle solution of ambiguities (and calculation of the β function to four loop) only for the toy-model limit $d=0$, $N=1$ [22]. A case where ambiguities have been resolved from first principles at $T=0$ to two-loop order, is the $N=1$ depinning transition [16,18]. Finally, the FRG has also been solved in the large-N limit [20]. Its solution reproduces, apparently with no ambiguity, the main results from the replica-symmetry-breaking saddle point of Ref. [25], and also underlies the importance of specifying the system preparation [20].

In the more difficult case of the statics within the $d=4-\epsilon$ expansion, detailed analysis to two and three loops [16,19,26] for the case of $N=1$ have suggested several methods to construct a renormalizable field theory. These methods give a unique finite β function, with nontrivial anomalous terms. This β function satisfies the potentiality constraint, with anomalous terms distinct from those at depinning, and a fixed point with the same linear cusp nonanalyticity as to one loop, hence confirming the consistency of the picture.

The aim of this paper is to extend these methods to the N-component model. We show how an extended β function can be obtained and point out the specific features of the case $N>1$. For the case of $O(N)$-symmetric disorder we compute the fixed point and roughness exponent ζ to next order in $\epsilon=4-d$, where d is the internal dimension of the manifold. We then study the extrapolations to the directed polymer limit $d=1$, and discuss the various scenarios for the strong-coupling phase of the equivalent N-dimensional KPZ growth equation. In one of them, a value for the upper critical dimension is estimated.

We consider the model for an elastic N-component manifold

$$\mathcal{H} = \int \mathrm{d}^d x \frac{1}{2}(\nabla u)^2 + V(x,u) \tag{1}$$

in a random potential with second cumulant $\overline{V(x,u)V(x',u')}=\delta^d(x-x')R(u-u')$, where $u=u^i$ is an

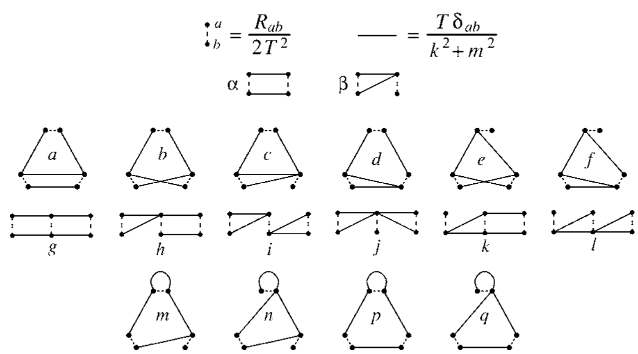

FIG. 1. Graphical rules, one-loop and two-loop diagrammatic corrections to the disorder.

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TABLE I. Numerical results for the exponents ζ_1 and ζ_2 for different values of N.

N	ζ_1	ζ_1^0	ζ_2	ζ_2^0
1	0.2082980	0.2	0.0068573	0
2	0.1765564	0.166667	0.17655636	−0.00555556
2.5	0.1634803	0.153846	−0.000417	−0.00782058
3	0.1519065	0.142857	−0.0029563	−0.00971817
4.5	0.1242642	0.117647	−0.009386	−0.013583
6	0.1043517	0.1	−0.0135901	−0.0155556
8	0.0856120	0.0833333	−0.0162957	−0.016572
10	0.0725621	0.0714286	−0.016942	−0.0166517
12.5	0.0610692	0.0606061	−0.0165154	−0.0161654
15	0.0528216	0.0526316	−0.01564	−0.0154217
17.5	0.046595	0.0465116	−0.0147	−0.014608
20	0.0417	0.0416667	−0.0138	−0.013804

N-component vector. We derive general equations, and later focus on the $O(N)$ isotropic case, noting $R(u)=h(r)$ with $r=|u|$. Introducing replicas, we obtain the replicated action

$$\frac{\mathcal{H}_n}{T} = \int d^d x \frac{1}{2T}\sum_a (\nabla u_a)^2 - \frac{1}{2T^2}\sum_{ab} R(u_a - u_b). \qquad (2)$$

We now carry perturbation theory in the disorder and compute the one-loop and two-loop corrections to the effective action $\Gamma[u]$. We use the usual power counting of the $T=0$ theory, identical to the case $N=1$. Infrared divergences for $d=4-\epsilon$ only occur in the two-replica term, which at zero momentum defines the renormalized disorder; there is no correction to the single-replica term. The graphical rules are depicted in Fig. 1. We use functional diagrams, and mass regularization. The method and notations are identical to [19], to which we refer for details. Here we only stress the differences with the case $N=1$.

The one-loop correction to disorder (see Table I and Fig. 2) reads

$$\delta^1 R(u) = (\tfrac{1}{2}[\partial_{ij}R(u)]^2 - \partial_{ij}R(0)\partial_{ij}R(u))I. \qquad (3)$$

Summation over repeated indices is implicit everywhere, and $I=\int_k G_k^2 = m^{-\epsilon}\tilde{I}$ with $G_k=(k^2+m^2)^{-1}$. We define the dimensionless function $\delta^1(R) := m^{\epsilon}\delta^1 R$ (recognizable by the parenthesis around the argument R). For later use we also denote the bilinear form $\delta^1(R,R) := \delta^1(R)$. This yields the standard one-loop FRG equations, recalled below, and $\partial_{ij}R$ develops a cusp nonanalyticity at $u=0$ beyond the Larkin length scale L_c. For the $O(N)$ model one has $\partial_{ij}R=(h'/r)\delta_{ij}+\hat{u}_i\hat{u}_j[h''-h'/r]=h''(0)\delta_{ij}+\frac{1}{2}h'''(0)r(\delta_{ij}+\hat{u}_i\hat{u}_j)+O(r^2)$ and thus $h'''(0)$ becomes nonzero at L_c ($\hat{u}=u/|u|$).

The two-loop corrections to disorder can be decomposed into a "normal" part, which is the complete result when $R(u)$ is analytic [15], and an "anomalous" part, which arises from nonanalyticity. The normal part reads

$$\begin{aligned}\delta_n^2 R(u) = {} & [\partial_{ij}R(u) - \partial_{ij}R(0)]\partial_{ikl}R(u)\partial_{jkl}R(u)I_A \\ & + (\tfrac{1}{2}\partial_{ijkl}R(u)[\partial_{ik}R(u) - \partial_{ik}R(0)][\partial_{jl}R(u) \\ & - \partial_{jl}R(0)])I^2. \qquad (4)\end{aligned}$$

The first line stems from diagrams b and a of Fig. 1, respectively, and the second and third from g,h,i,j. One has $I_A=\int_{k_1,k_2} G_{k_1}G_{k_2}G_{k_1+k_2}^2 = m^{-2\epsilon}\tilde{I}_A$ and we denote in analogy to $\delta^1(R)$ the dimensionless function $\delta^{(2)}(R) := m^{2\epsilon}\delta^2 R$. The FRG β function is then

$$-m\partial_m R|_{R_0} = \epsilon[R + \delta^1(R) + 2\delta^2(R) - \delta^{1,1}(R)], \qquad (5)$$

where the repeated one-loop counterterm $\delta^{1,1}(R) := 2\delta^1(R,\delta^1(R,R))$ arises when reexpressing the bare disorder R_0 in (2), in terms of the dimensionless renormalized one, defined as $m^{\epsilon}R$, as detailed in [19]. From (3) it reads

$$\delta^{1,1}(R) = \{[\partial_{ij}R - \partial_{ij}R(0)]\partial_{ij}\delta^1(R) - \partial_{ij}\delta^1(R)|_{u=0}\partial_{ij}R\}\tilde{I}^2, \qquad (6)$$

$$\partial_{ij}\delta^1(R) = \partial_{ijkl}R[\partial_{kl}R - \partial_{kl}R(0)] + \partial_{ikl}R\partial_{jkl}R. \qquad (7)$$

The property of renormalizability amounts to cancellation of the $1/\epsilon$ poles between the two last terms in (5) using $\tilde{I}=N_d/\epsilon$ and $\tilde{I}_A - \frac{1}{2}\tilde{I}^2 = N_d^2[1/(4\epsilon)+O(\epsilon^0)]$ [16]. The β function (5) is obtained from (3), (4), and (7) as

$$\begin{aligned}-m\partial_m R(u) = {} & \epsilon R(u) + \tfrac{1}{2}[\partial_{ij}R(u)]^2 - \partial_{ij}R(0)\partial_{ij}R(u) + [\partial_{ij}R(u) \\ & - \partial_{ij}R(0)][\tfrac{1}{2}\partial_{ikl}R(u)\partial_{jkl}R(u) - \alpha_{ij}]. \qquad (8)\end{aligned}$$

The cancellation works perfectly for the normal parts. Anomalous parts, to which we turn now, produce the last term.

We start with the anomalous part of the repeated counterterm,

$$\delta_a^{1,1}(R) = -(\mu_{ij} + \nu_{ij})\partial_{ij}R(u)\tilde{I}^2, \qquad (9)$$

where we denote the limits of small argument $v \to 0$,

$$\mu_{ij} := \partial_{ikl}R(v)\partial_{jkl}R(v)|_{v\to 0}, \qquad (10)$$

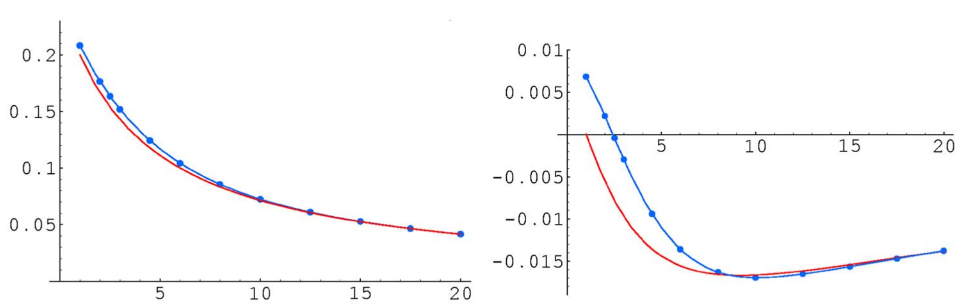

FIG. 2. (Color online) The numerical plots of $\zeta_1(N)$ (left) and $\zeta_2(N)$ (right), in blue with the numerical values from the table as dots. The red curves (no dots) represent the asymptotic expansion.

$$\nu_{ij} := \partial_{ijkl} R(v)[\partial_{kl} R(v) - \partial_{kl} R(0)]|_{v\to 0}, \qquad (11)$$

which, in general, are direction dependent. For an $O(N)$ model, the third derivative tensor,

$$\partial_{ijk} R(v) = A(r)(\delta_{ij}\hat{v}_k + \delta_{ik}\hat{v}_j + \delta_{kj}\hat{v}_i) + B(r)\hat{v}_i\hat{v}_j\hat{v}_k, \quad (12)$$

with $\hat{v}=v/|v|$, $A(r)=(rh''-h')/r^2$, and $B(r)=(r^2h'''-3rh''+3h')/r^2$, has a $\hat{v}$-dependent small-v limit (12) with $A(0)=-B(0)=h'''(0)/2$. This yields

$$\mu_{ij} = h'''(0)^2\left(\frac{1}{2}\delta_{ij} + \frac{N+1}{4}\hat{v}_i\hat{v}_j\right), \qquad (13)$$

and, similarly one finds $\nu_{ij}=((N+1)/4)h'''(0)^2(\delta_{ij}-\hat{v}_i\hat{v}_j)$.

Let us first superficially examine the structure of the two-loop graphs, following the discussion in [19]. As for $N=1$, one can discard $c=d=0$ from parity and similarly set $m+n=0$ and $p+q=0$. One can then write

$$\delta_a^{(2)}(R) = -(\tilde{\mu}_{ij}\tilde{I}_A + \tilde{\nu}_{ij}\tilde{I}^2)\partial_{ij}R(u), \qquad (14)$$

where the first term comes from graphs e (more properly, from the sum of all graphs a to f) and the second from graphs $k+l$ (from the sum of graphs i to l). Global cancellation of the $1/\epsilon$ pole in the β function works provided $\tilde{\mu}_{ij}+2\tilde{\nu}_{ij}=\mu_{ij}+\nu_{ij}$. This then produces $\alpha_{ij}=\tilde{\mu}_{ij}/2$ in the FRG equation above.

We can now use the methods introduced in [19] to analyze the total two-loop contribution to the effective action, including possibly ambiguous graphs. One first computes $\Gamma[u]$ in a region of u where no ambiguity is present, using excluded replica sums, and constraints valid in the zero-temperature theory (the so-called sloop elimination method, Sec. V.B in [19]). One finds that extraction of the two-replica part yields $\tilde{\mu}_{ij}=\mu_{ij}$ and $2\tilde{\nu}_{ij}=\nu_{ij}$, i.e., it works as for $N=1$. This is equivalent to renormalizability diagram by diagram, and thus it satisfies the global renormalizability condition. The background method also yields that result ([19], Sec. V.C). The end result for the β function, $\alpha_{ij}=\mu_{ij}$, although unambiguous for $N=1$, needs further specification for $N>1$, since the limit in (13) is direction dependent.

Another important consideration for the resulting β function is the issue of the "super cusp." For $N=1$ it was found that the β function is such that the cusp nonanalyticity of $R''(u)$ at $u=0$ does not become worse at two loops. That by itself constraints the amplitude of the anomalous term, since any other choice yields a stronger singularity [28]. We now point out that if v and u, in (9), (10), and (14), are colinear, i.e., $\mu_{ij}(v)=\mu_{ij}(u)$ then there is no super cusp. Indeed the result

$$\alpha_{ij}(\hat{u}) = \lim_{r\to 0}\frac{1}{2}\partial_{ikl}R(r\hat{u})\partial_{jkl}R(r\hat{u}) \qquad (15)$$

obviously yields cancellation of the linear term in u in (8) (although it is not the only possibility [27]). Colinearity of $v=u_a-u_c$ and $u=u_a-u_b$ is natural if one computes the effective action in a background configuration breaking the rotational symmetry, which appears to be required for the present theory to hold.

We now specialize to the $O(N)$ model. Starting from (8)

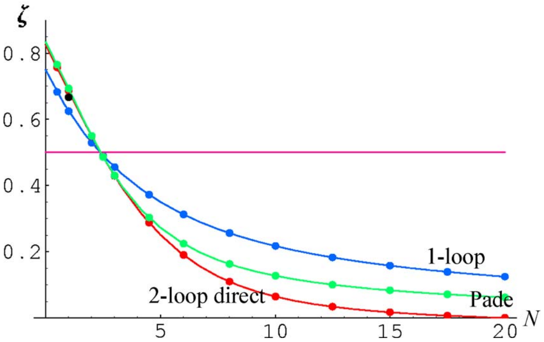

FIG. 3. (Color online) Results for the roughness ζ at one- and two-loop order, as a function of the number of components N. We both show a direct extrapolation and the Pade (1,1): $\zeta_{\text{Pade}}=\epsilon\zeta_1/(1-\epsilon\zeta_2/\zeta_1)$.

and further rescaling $h(r)\to m^{-4\zeta}h(rm^{\zeta})$, using ζ we obtain the following FRG flow equation to two loops:

$$\begin{aligned} -m\partial_m h(r) &= (\epsilon-4\zeta)h(r) + \zeta r h'(r) + \frac{1}{2}h''(r)^2 - h''(0)h''(r) \\ &\quad + \frac{N-1}{2}\frac{h'(r)}{r}\left(\frac{h'(r)}{r} - 2h''(0)\right) + \frac{1}{2}[h''(r) \\ &\quad - h''(0)]h'''(r)^2 + \frac{N-1}{2} \\ &\quad \times \frac{[h'(r)-rh''(r)]^2\{2h'(r)+r[h''(r)-3h''(0)]\}}{r^5} \\ &\quad - h'''(0)^2\left[\frac{N+3}{8}h''(r) + \frac{N-1}{4}\frac{h'(r)}{r}\right], \end{aligned} \qquad (16)$$

where the last line arises from the anomalous term (15). This FRG equation admits for any N a nontrivial attractive fixed point such that $h''(r)$ has a linear cusp at the origin and decays to 0 at infinity faster than a power law, thus corresponding to short range (SR) disorder. Finding the associated $\zeta=\zeta_1\epsilon+\zeta_2\epsilon^2+O(\epsilon^3)$ is an eigenvalue problem, which has to be solved order by order in ϵ following [16,18,19]. Our results for ζ_1 and ζ_2 are given on Fig. 2. Although for the SR disorder no analytical expression can be found for ζ_1 and ζ_2, their large-N behavior can be obtained from an asymptotic analysis of (16). Let us extend the analysis of Balents and Fisher (BF) [13]. Define $h=\hat{h}/N$, $y=r^2/2$, and $\hat{h}(r)=Q(y)$. For $y\gg 1$ the FRG equation can be linearized

$$(\epsilon-2\zeta)Q' + 2\zeta y Q'' - (A+3B)Q'' - 2ByQ''' = 0, \quad (17)$$

with $A=[1-1/N]Q_0'+[(N-1)/(4N^2)]\hat{h}'''(0)^2$ and $B=(1/N)Q_0'+[(N+3)/(8N^2)]\hat{h}'''(0)^2$, $\hat{h}''(0)=Q'(0)=Q_0'$. BF noted that there is an overlapping region $1\gg y\gg N$ where the solution can also be found perturbatively by expansion in $1/N$, yielding for Q a pure exponential. It is indeed an exact solution of (17), with a unique value for ζ_1, the BF result $\zeta_1\approx\zeta_1^0$ with $\zeta_1^0=1/(4+N)$ (i.e., the result from the replica variational method [25]). The corrections (which arise from the neglected nonlinear terms) are shown to be exponentially small; a more accurate estimate being $\zeta_1 \approx \zeta_1^1$ with $\zeta_1^1 = \zeta_1^0 + (N+2)^2/(N+4)^2 2^{-(N+2)/2}/(4e)$. To next order we find similarly the approximation to ζ_2 [29]

$$\zeta_2^0 = -\frac{(N^2-1)(2+N)}{2(4+N)^3(3+N)}, \tag{18}$$

where we have not attempted to estimate further corrections, presumably again exponentially small at large N. We note that ζ_2^0 arises *from the anomalous terms only*. These estimates are listed and plotted on Fig. 2 together with the numerical solution of (16). The quality of the large-N analysis is quite remarkable.

We now discuss the extrapolation of our result to the directed polymer (DP) case $d=1, \epsilon=3$, plotted in Fig. 3. We see that the two-loop corrections are rather big at large N, so extrapolation down to $\epsilon=3$ is difficult. However both one- and two-loop results as well as the Pade (1,1) reproduce well the two known points on the curve: $\zeta=2/3$ for $N=1$ [8] and $\zeta=0$ for $N=\infty$ [20]. This branch in Fig. 3 corresponds to zero temperature and a continuum model. On the other hand we find that for all curves in Fig. 3 the roughness ζ becomes smaller than the thermal $\zeta_{\rm th}=\frac{1}{2}$ at $N=N_{\rm uc}\approx 2.5$. This naturally suggests the scenario that at nonzero temperature $\zeta=1/2$ for $N \geqslant N_{\rm uc}$, i.e., $N_{\rm uc}$ is the upper critical dimension [2]. The same argument gives an upper critical dimension $N_{\rm uc}$ for the KPZ equation of nonlinear surface growth [8,11]. On the other hand, simulations on discretized models of both the directed polymer (at $T=0$) and the KPZ equation [9,10] suggest that $\zeta > 1/2$ in all dimensions, but should be taken with caution [30]. Since the FRG is a systematic expansion in $\epsilon=4-d$, such a scenario seems reconcilable with our above results only through nonperturbative corrections in ϵ, possibly nonanalytic at $\epsilon=2$.

To conclude, we have obtained for the N-component model a FRG description at two-loop order. Various studies, including at large N, are under way to obtain a better understanding of the structure of the theory. For the KPZ growth and the directed polymer we have improved the determination of the possible upper critical dimension. Further numerics, in particular for the directed polymer at $T=0$ would be helpful.

[1] G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988); A. Rosso and T. Giamarchi, Phys. Rev. B **70**, 224204 (2004).
[2] G. Blatter *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).
[3] T. Giamarchi and P. Le Doussal, Phys. Rev. B **52**, 1242 (1995); T. Nattermann and S. Scheidl, Adv. Phys. **49**, 607 (2000).
[4] S. Moulinet, C. Guthmann, and E. Rolley, Eur. Phys. J. A **8**, 437 (2002).
[5] S. Lemerle *et al.*, Phys. Rev. Lett. **80**, 849 (1998).
[6] See reviews in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).
[7] K. Efetov and A. Larkin, Sov. Phys. JETP **45**, 1236 (1977).
[8] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); M. Kardar, Nucl. Phys. B **290**, 582 (1987).
[9] B. M. Forrest and L.-H. Tang, Phys. Rev. Lett. **64**, 1405 (1990); J. M. Kim, M. A. Moore, and A. J. Bray, Phys. Rev. A **44**, 2345 (1991).
[10] E. Marinari, A. Pagnani, and G. Parisi, J. Phys. A **33**, 8181 (2000).
[11] M. Lässig and H. Kinzelbach, Phys. Rev. Lett. **78**, 903 (1997); M. Lässig, *ibid.* **80**, 2366 (1998).
[12] D. S. Fisher, Phys. Rev. Lett. **56**, 1964 (1986).
[13] L. Balents and D. S. Fisher, Phys. Rev. B **48**, 5959 (1993).
[14] T. Nattermann *et al.*, J. Phys. (Paris), Colloq. **2**, 1483 (1992); O. Narayan and D. S. Fisher, Phys. Rev. B **46**, 11 520 (1992).
[15] H. Bucheli *et al.*, Phys. Rev. B **57**, 7642 (1998).
[16] P. Chauve, P. Le Doussal, and K. J. Wiese, Phys. Rev. Lett. **86**, 1785 (2000).
[17] S. Scheidl (unpublished); S. Scheidl and Y. Dincer, e-print cond-mat/0006048; Y. Dincer, Diplomarbeit, Köln, 1999.
[18] P. Le Doussal, K. J. Wiese, and P. Chauve, Phys. Rev. B **66**, 174201 (2002).
[19] P. Le Doussal, K. J. Wiese, and P. Chauve, Phys. Rev. E **69**, 026112 (2004).
[20] P. Le Doussal and K. J. Wiese, Phys. Rev. Lett. **89**, 125702 (2002); Phys. Rev. B **68**, 174202 (2003); Nucl. Phys. B **701**, 409 (2004).
[21] L. Balents and P. Le Doussal, Europhys. Lett. **65**, 685 (2004).
[22] L. Balents and P. Le Doussal, e-print cond-mat/0408048, Adv. Phys. (to be published).
[23] P. Chauve *et al.*, Phys. Rev. B **62**, 6241 (2000).
[24] P. Chauve and P. Le Doussal, Phys. Rev. E **64**, 051102 (2001).
[25] M. Mézard and G. Parisi, J. Phys. I **1**, 809 (1991).
[26] K. J. Wiese and P. Le Doussal (unpublished).
[27] For the $O(N)$ model one must have $\alpha_{ij}=\alpha\delta_{ij}+\beta\hat{u}_i\hat{u}_j$ and the absence of a super cusp reads $\alpha(1+N)+2\beta=[(1+N)/2]h'''(0)^2$.
[28] An alternative scenario is that a nonlinear cusp with a dimension-dependent exponent develops, e.g., $h''(r)\sim r^{1+A\epsilon}$. It may require deviations from the simplest scenario of a uniform density of shocks of codimension one.
[29] We use the one-loop relation $(\epsilon-2\zeta)h''(0)+[(N+3)/(4N)]h'''(0)^2=0$.
[30] To see in the numerics the zero-temperature fixed point of the continuum theory (the descending branch in Fig. 3), a different scaling may be necessary. Also note that the UV cutoff of the RSOS model in [10] is in the middle of the scaling plots for high dimensions.