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The Passive Polymer Problem

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In this article, we introduce a generalization of the diffusive motion of point-particles in a turbulent convective flow with given correlations to a polymer or membrane. In analogy to the *passive scalar problem* we call this the *passive polymer* or *membrane* problem. We shall focus on the expansion about the marginal limit of velocity–velocity correlations which are uncorrelated in time and grow with the distance x as $|x|^{\varepsilon}$, and ε small. This relation gets modified in the case of polymers and membranes (the marginal advecting flow has correlations which are shorter ringed.) The construction is done in three steps: First, we reconsider the treatment of the passive scalar problem using the most convenient treatment via field theory and renormalization group. We explicitly show why IR-divergences and thus the system-size appear in physical observables, which is rather unusual in the context of ordinary field-theories, like the ϕ^4-model. We also discuss, why the renormalization group can nevertheless be used to sum these divergences and leads to anomalous scaling of $2n$-point correlation functions as e.g., $S^{2n}(x) := \langle [\Theta(x, t) - \Theta(0, t)]^{2n} \rangle$. In a second step, we reformulate the problem in terms of a Langevin equation. This is interesting in its own, since it allows for a distinction between single-particle and multi-particle contributions, which is not obvious in the Focker–Planck treatment. It also gives an efficient algorithm to determine S^{2n} numerically, by measuring the diffusion of particles in a random velocity field. In a third and final step, we generalize the Langevin treatment of a particle to polymers and membranes, or more generally to an elastic object of inner dimension D with $0 \leqslant D \leqslant 2$. These objects can intersect each other. We also analyze what happens when self-intersections are no longer allowed.

KEY WORDS: Passive scalar; turbulence; passive advection; polymer; polymerized membrane; renormalization group; multiscaling.

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1. INTRODUCTION AND OUTLINE

For now more than 5 decades, turbulence has resisted a satisfying theoretical treatment. The principle question asked since Kolmogorov's pioneering work[1] in 1941 is whether there are corrections to the simple scaling behavior predicted in ref. 1 for higher correlation functions.[2, 3] The most natural tool to answer this question is the renormalization group. However, all attempts to go beyond Kolmogorov's analysis have essentially failed so far. To better pin down the problem, simpler toy models have been proposed. The probably most prominent such model is the *passive scalar* model, introduced by Obukhov[4] and Kraichnan.[5] This model describes the diffusion of a point-particle in a turbulent flow with given correlations. For simplicity these correlations are taken to be Gaussian. Nevertheless, the model is far from beeing simple, and shows multi-scaling, i.e., higher correlation functions of the particle density scale independently of the second moment, characterized by new critical exponents. More explicitly, particles, or equivalently heat is injected in a finite range of size $L \sim 1/M$, whereas the turbulent flow grows up to a bound of $l \sim 1/m$, which finally shall be taken to infinity. This is possible, if the total number of particles, or the total heat, injected into the system is conserved. In that case, L and not l sets the largest scale in the problem, as visualized in Fig. 1.1.

In this article, we introduce the generalization from point particles to higher dimensional elastic objects, as e.g., polymers and membranes. In analogy to the *passive scalar* problem we call this the *passive polymer* or *passive membrane* problem.

We start by considering the passive polymer problem. Much has been learned during the last years about higher correlation functions due to a common effort of mathematicians and physicists.[6–27] Whereas the first to calculate the 4-point function by considering the 0-modes of the steady state are,[7, 9] the calculatory most convenient scheme, based on the perturbative renormalization group, was introduced in ref. 16. Contrary to the sometimes heard claim, the renormalization group is able to handle large eddy motion. The expansion is performed about the marginal limit of velocity–velocity correlations which are uncorrelated in time and grow with the distance x as $|x|^{\varepsilon}$, and ε small, a relation which gets modified for polymers and membranes (the marginal advecting flow has correlations which are shorter ranged.)

The generalization to polymers and membranes is then performed in three steps: First, we reconsider the treatment of the passive scalar problem using the most convenient treatment via field theory and renormalization group. We explicitly show why IR-divergences and thus the system-size appear in physical observables, which is rather unusual in the context of

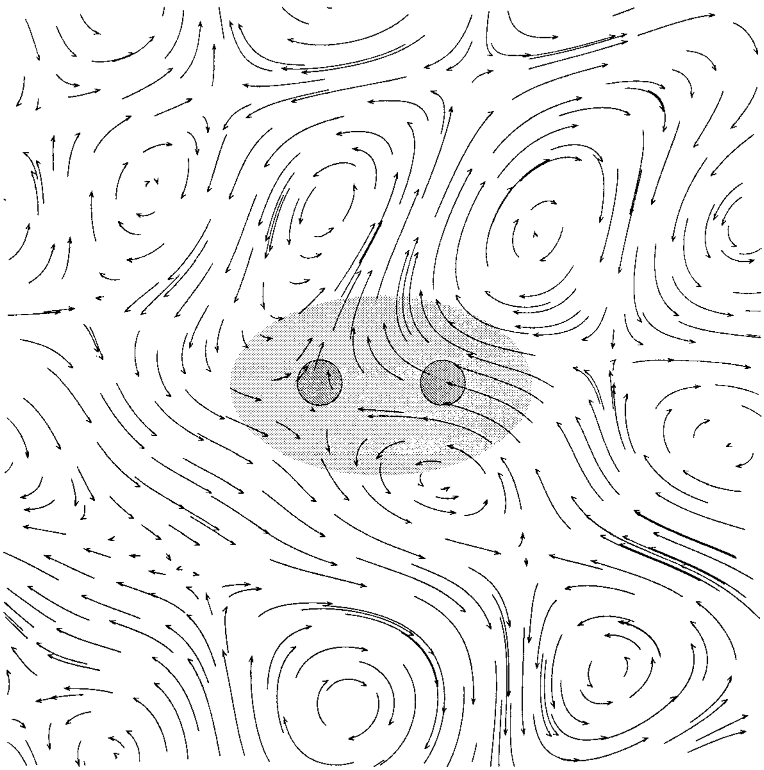


Fig. 1.1. Symbolic picture of a turbulent flow. Particles, or equivalently heat is injected in a finite range of size $L \sim 1/M$ (dark grey areas), whereas the turbulent flow grows up to scale $l \sim 1/m$, which finally shall be taken to infinity. This is possible, if the total number of particles, or the total heat, injected into the system is conserved. In that case, L and not l sets the largest scale in the problem, and multi-point correlation functions with an anomalous L-dependence will be observable in a domain of size L, here symbolically shaded in light grey. As will be shown below, they have anomalous corrections depending on L.

ordinary field-theories, like the ϕ^4-model. We also discuss, why the renormalization group can nevertheless be used to sum these divergences and leads to anomalous scaling of n-point correlation functions as e.g., $S^{2n}(x) := \langle [\Theta(x, t) - \Theta(0, t)]^{2n} \rangle$. To do so, we determine the full scaling dimension of the composite operators $\mathscr{S}_z^{(n, m)} := [(\nabla\Theta)^2]^n [z \nabla\Theta]^{2m}$, with $|z| = 1$. In a second step, we reformulate the problem in terms of a Langevin equation. This is interesting in its own, since it allows for a distinction between single-particle and multi-particle contributions, which is not obvious in the Focker–Planck treatment. It also gives an efficient algorithm to determine S^{2n} numerically, by measuring the diffusion of particles in a random velocity field. In a third and final step, we generalize the Langevin treatment of a particle to polymers and membranes, or more generally to an elastic object of inner dimension D with $0 \leqslant D \leqslant 2$. Our analysis will show that the interesting range for ε is $-2D/(2-D) < \varepsilon < 0$. For smaller ε, the advecting flow is irrelevant. For larger ε, the polymer or membrane is overstretched. This is the range, where already the particle, i.e., the center of mass of the polymer, shows anomalous diffusion. We also generalize these considerations to the case of self-avoiding polymers and membranes.

2. THE PASSIVE SCALAR

2.1. Model

The advection of a passive scalar field $\Theta(x, t)$ with $x \in \mathbb{R}^d$ the spatial coordinate and t the time, is described by the Focker–Planck type equation[4, 5]

$$[\partial_t + v(x, t)\, \nabla]\; \Theta(x, t) = \nu_0\, \Delta\Theta(x, t) + f(x, t) \tag{2.1}$$

The correlations of the advecting turbulent velocity field $v(x, t)$ are supposed to be Gaussian with zero mean and correlations which grow with the distance r as r^{ε}

$$\begin{aligned}\langle v^i(x, t)\, v^j(x', t')\rangle &= \mathrm{D}_v^{ij}(x-x', t-t') \\ &= \mathrm{D}_0\, \delta(t-t') \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, P^{ij}(k)\, \frac{\mathrm{e}^{ik(x-x')}}{(k^2+m^2)^{(d+\varepsilon)/2}}\end{aligned} \tag{2.2}$$

where

$$P^{ij}(k) := \delta^{ij} - \frac{k^i k^j}{k^2} \tag{2.3}$$

is the transversal projector and m some IR-regulator. The dimension of the coupling $u_0 := \mathrm{D}_0/\nu_0$ in units of m is

$$\varepsilon = [u_0]_m \tag{2.4}$$

We will see later that ε serves as a regulator. Eventually one is interested in the physically relevant case of $d=3$ and $\varepsilon = 4/3$ (Kolmogorov-scaling) or corrections thereto.[28]

f is a Gaussian scalar noise with zero mean and correlator

$$\langle f(x,t)\, f(x',t')\rangle = \delta(t-t')\, G_f(Mx, Mx') \equiv \delta(t-t')\, G_f^M(x,x') \tag{2.5}$$

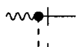
Often people use $G_f(Mx, Mx') = G_f(M\,|x-x'|)$, and its Fourier-transform $\widetilde{G}_f(k/M)$. However we will use the more general case for clearness of derivation. G_f is the source of correlation-functions of Θ, which otherwise would vanish. Physically, it may be viewed as source and sink of tracer-particles, or heat. We will see below when explicitly calculating expectation values that G_f sets the largest scale $L \equiv 1/M$ appearing in physical observables. Therefore, we demand that $G_f(s,s')$ rapidly decays to zero for s or s' larger than 1.

The analysis of Eq. (2.1) is most easily done by using a dynamic action[29, 30]

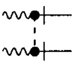
$$J[\Theta, \widetilde{\Theta}, f, v] = \int_{x,t} \widetilde{\Theta}(x,t)[\partial_t \Theta(x,t) - \nu_0\, \Delta\Theta(x,t) - v(x,t)\, \nabla\Theta(x,t) - f(x,t)] \tag{2.6}$$

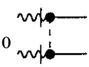
Expectation values are obtained by integrating $e^{-J[\Theta, \widetilde{\Theta}, f, v]}$ over Θ, $\widetilde{\Theta}$ and averaging over f and v. Since v and f are Gaussian, their average can be taken, leading to

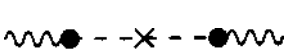
$$J[\Theta, \widetilde{\Theta}] = \int_{x,t} \widetilde{\Theta}(x,t)(\partial_t - \nu_0 \Delta)\, \Theta(x,t)$$

$- \int_{x,y,t} \Big[\frac{D_0}{2}$  $+ \frac{1}{2}$  $\Big]$ (2.7)

where

D_0  $= \widetilde{\Theta}(x,t)\, \nabla_i \Theta(x,t)\, \widetilde{\Theta}(y,t)\, \nabla_j \Theta(y,t)\, D_v^{ij}(x-y)$

$= \Theta(x,t)\, \nabla_i \widetilde{\Theta}(x,t)\, \Theta(y,t)\, \nabla_j \widetilde{\Theta}(y,t)\, D_v^{ij}(x-y) = D_0$ 

 $= \widetilde{\Theta}(x,t)\, \widetilde{\Theta}(y,t)\, G_f^M(x,y)$ (2.8)

Note that due to the transversal projector in the turbulent interaction, the partial integration from $\int_x \widetilde{\Theta}(x,t)\, \nabla_i \Theta(x,t)\, D^{ij}(x-y)$ to $-\int_x \nabla_i \widetilde{\Theta}(x,t)\, \Theta(x,t)\, D^{ij}(x-y)$ is possible since $\nabla_i D_v^{ij}(x-y) = 0$.

The free response and correlation functions read

$$\langle \tilde{\Theta}(k,\omega)\,\Theta(k',\omega')\rangle_0 = (2\pi)^{d+1}\,\delta(\omega+\omega')\,\delta^d(k+k')\,R(k,\omega), \qquad R(k,\omega)=\frac{1}{i\omega+\nu_0 k^2}$$

$$\langle \Theta(k,\omega)\,\Theta(k',\omega')\rangle_0 = (2\pi)^{d+1}\,\delta(\omega+\omega')\,\delta^d(k+k')\,C(k,\omega), \qquad C(k,\omega)=\frac{\tilde{G}_f^M(k)}{\omega^2+(\nu_0 k^2)^2} \tag{2.9}$$

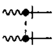
where in the last formula $G_f(x,x')$ was supposed to be of the form $G_f^M(x-x')$. Most convenient is a mixed time and k-dependent representation

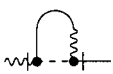
$$\langle \tilde{\Theta}(k,t)\,\Theta(k',t')\rangle_0 = (2\pi)^d\,\delta^d(k+k')\,R(k,t'-t), \qquad R(k,t)=\Theta(t)\,\mathrm{e}^{-\nu_0 k^2 t} \tag{2.10}$$

This also yields the response-function in position space

$$R(x,t)=\Theta(t)(4\pi\nu_0 t)^{-d/2}\,\mathrm{e}^{-x^2/4\nu_0 t} \tag{2.11}$$

2.2. Perturbative Corrections, Renormalization of the Dynamic Action *J*

We now study the renormalization of the model, i.e., we want to eliminate all UV-divergent terms at $\varepsilon=0$. It is important to notice that such divergences only come from the insertion of the turbulence-interaction $(\mathrm{D}_0/2)$  , but *not* from the insertion of the source of tracer-particles $\frac{1}{2}$. To first order in D_0, the only contribution is

$$\mathrm{e}^{-J} \rightarrow \frac{\mathrm{D}_0}{2}$$  $$\rightarrow -\mathrm{D}_0$$  (2.12)

The diagram is without the external legs in the most convenient mixed t and k representation of Eq. (2.10)

 $$= \int_{-\infty}^{\infty} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\Theta(t)\,\mathrm{e}^{-\nu_0 k^2 t}\,\frac{\delta_\eta(t)}{(k^2+m^2)^{(d+\varepsilon)/2}}\left(1-\frac{1}{d}\right) \tag{2.13}$$

In order to clarify the role of the factor $\delta(t)$ in  , we have to recall that this is an approximation for a sharply peaked but nevertheless smooth function around $t=0$. This is the reason why in Eq. (2.13), we have replaced the δ-distribution by a smoothened one $\delta_\eta(t)$, which in the limit of $\eta \to 0$ will reproduce $\delta(t)$. Integrating $\int \mathrm{d}t\, \theta(t)\, \mathrm{e}^{-k^2 v_0 t}\, \delta_\eta(t)$ and then taking the limit of $\eta \to 0$ thus simply yields a factor of $\frac{1}{2}$. Equation (2.13) becomes

$$= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^{(d+\varepsilon)/2}} \left(1 - \frac{1}{d}\right) = \frac{1}{2}\left(1-\frac{1}{d}\right) C_d \frac{m^{-\varepsilon}}{\varepsilon} \tag{2.14}$$

where C_d is defined as

$$\begin{aligned} C_d &:= \varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+1)^{(d+\varepsilon)/2}} \\ &= \frac{2\Gamma(1+\varepsilon/2)}{\Gamma((d+\varepsilon)/2)(4\pi)^{d/2}} \end{aligned} \tag{2.15}$$

This leads at leading order to a renormalization of v (denoting with subscript 0 bare quantities)

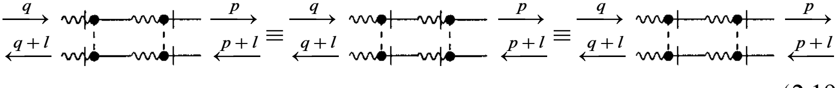
$$\begin{aligned} v_0 &= v Z_v \\ Z_v &= 1 - \frac{u}{\varepsilon}\left(1-\frac{1}{d}\right)\frac{C_d}{2} \end{aligned} \tag{2.16}$$

where we have introduced a coupling u_0 and its renormalized counterpart u through

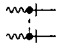
$$\mathrm{D}_0 = Z_\mathrm{D} \mathrm{D} \tag{2.17}$$

$$u_0 = \frac{\mathrm{D}_0}{v_0} = u m^\varepsilon \frac{Z_\mathrm{D}}{Z_v} \tag{2.18}$$

We now claim that Eqs. (2.16) and (2.17) are all renormalizations needed, and that even to all orders in perturbation theory. Let us first focus on the renormalization of D. There will appear diagrams like



(2.19)

We want to argue that due to the transversal projector in , this and all similar diagrams are finite. Up to an overall factor, and integrating over the time difference between the two vertices, they are

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{(k^2+m^2)^{(d+\varepsilon)/2}} \right]^2 \times \frac{[q(q+l)][(k+q)^2] - [q(k+q)][(q+l)(k+q)]}{(k+q)^2} \times \frac{[p(p+l)][(k+p)^2] - [p(k+p)][(p+l)(k+p)]}{(k+p)^2} \tag{2.20}$$

Since for large k

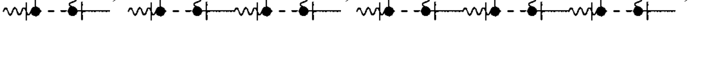
$$\begin{aligned} \frac{[q(q+l)][(k+q)^2] - [q(k+q)][(q+l)(k+q)]}{(k+q)^2} &= O(k^0) \\ \frac{[p(p+l)][(k+p)^2] - [p(k+p)][(p+l)(k+p)]}{(k+p)^2} &= O(k^0) \end{aligned} \tag{2.21}$$

the integral (2.20) scales for large k as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{(k^2+m^2)^{(d+\varepsilon)/2}} \right]^2 \sim \int \frac{\mathrm{d}k}{k} \frac{1}{k^{d+2\varepsilon}} \tag{2.22}$$

and is thus UV-convergent for any d and $\varepsilon > 0$. This means that $Z_{\mathrm{D}} = 1$. Note that the transversal projectors ensure that no additional divergences appear for $\varepsilon = 0$ at $d = 2$ or 4, since it allows to bring the derivatives always to the external legs. Moreover, no long-range interaction can be generated. This argument can be generalized to higher orders in perturbation theory, however only the absence of additional divergences for $d > 2$ is immediately apparent. We shall not elaborate on this question any longer, since it is not at the center of our analysis.

Let us now come back to counter-terms for v. By direct inspection, one sees that the only diverging diagrams are chains of bubbles, of the form

 (2.23)

However, these diagrams are already renormalized by Eq. (2.16). This is easily seen by directly summing the perturbative (geometric) series, as e.g., in ref. 31. The β-function to all orders in perturbation theory thus reads

$$\beta(u) := m \left.\frac{\partial}{\partial m}\right|_0 u = -\varepsilon u + \left(1 - \frac{1}{d}\right) \frac{C_d}{2} u^2 \tag{2.24}$$

(Note that we do not take C_d at $\varepsilon = 0$; this "minimal subtraction" is completely sufficient at 1-loop order, but unsufficient for the all order result (2.24).) This β-function has a fixed point at

$$u^* = \frac{2d}{(d-1)\, C_d}\, \varepsilon \tag{2.25}$$

One can now define the anomalous dimension γ_v of v as

$$\begin{aligned} \gamma_v(u) &:= m \frac{\partial}{\partial m} \ln Z_v \\ &= u \left(1 - \frac{1}{d}\right) \frac{C_d}{2} \end{aligned} \tag{2.26}$$

which at $u = u^*$ reads (to all orders in ε)

$$\gamma_v^* = \varepsilon \tag{2.27}$$

Since G_f is not renormalized, this leads to an anomalous dimension of Θ (in units of $x \sim 1/m$) to all orders in perturbation theory as

$$\eta^* = -\frac{\varepsilon}{2} \tag{2.28}$$

The full dimension of Θ thus is

$$[\Theta]_{x,\,\mathrm{f}} = 1 - \frac{\varepsilon}{2} \tag{2.29}$$

This simple scaling is only part of the whole story, as we shall see in the next section.

To summarize: we have constructed a renormalized action which is UV-finite in the limit of $\varepsilon \to 0$, and which gives the IR-scaling for $\varepsilon > 0$.

2.3. Observables and IR-Divergences

We now want to study correlation-functions as e.g.,

$$S^{2n}(x-y,t) := \langle [\Theta(x,t)-\Theta(y,t)]^{2n} \rangle \tag{2.30}$$

We always choose x and y inside the injection region, thus especially $L \gg |x-y|$. It will turn out that these observables are sensitive to the size $L = 1/M$ of the system, and demand new renormalizations. Since from the viewpoint of ϕ^4-theory this is rather strange, let us study an expectation value in the latter theory in order to see where the difference to the passive scalar problem lies. Suppose, one wants to evaluate the expectation value

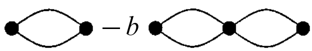
$$U(x,y) := \frac{1}{2} \langle \phi^2(x)\, \phi^2(y) \rangle \tag{2.31}$$

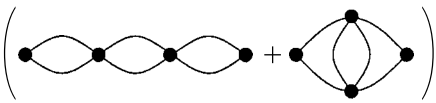
for the theory defined by the Hamiltonian in d dimensions

$$\mathcal{H}[\phi] = \int \mathrm{d}^d x\, \frac{1}{2} (\nabla\phi(x))^2 + b\phi^4(x) \tag{2.32}$$

(For the difference in definition between Eqs. (2.30) and (2.31) note that for correlation functions growing with the distance, definition (2.30) has to be used, whereas for decaying correlation functions, (2.31) is the correct one.)

Denoting expectation values in the free theory by $C(x-y) := \langle \phi(x)\, \phi(y) \rangle_0 \sim |x-y|^{2-d}$, the first contributions to $U(x,y)$ are (when setting to 0 self-contractions in the ϕ^4-interaction, and neglecting combinatorial factors)

$U(x, y) =$ 

$+\, b^2$  $+\, O(b^3)$ (2.33)

This formula is to be understood such that the outer points are always x and y and that one integrates over the inner points. Since $C(s) \sim s^{2-d}$ the term of order b scales as

 $\sim \int \mathrm{d}^d z\, |x-z|^{4-2d}\, |y-z|^{4-2d}$ (2.34)

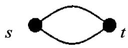
which for large z becomes

$$\int \mathrm{d}^d z\, |z|^{8-4d} \tag{2.35}$$

It is IR-convergent at least for d close to 4. Now still consider one of the terms of order b^2.

$$= \int \mathrm{d}^d s \, \mathrm{d}^d t \; C(x-s)^2 \, C(s-t)^2 \, C(t-y)^2 \tag{2.36}$$

Similar to what has happened in the last section, there is a logarithmic divergence at $\varepsilon = 0$ for small $s-t$, which has to be renormalized. Calculating directly the integral over $s-t$ in the regularized theory at $d<4$, this leads to $(z := s-t)$



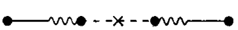
$$= \int \mathrm{d}^d z \; C(z)^2 \sim \frac{1}{4-d} L^{4-d} \tag{2.37}$$

where L is an effective IR-cutoff. The question now arises, what L is. Noting that the integral over the center of mass $(s+t)/2$ is IR-convergent with the identical argument that led to Eq. (2.35), the effective scale at which the integral (2.37) is cut off, is $L = |x-y|$. These kind of arguments can be continued to higher orders.[2] They show three things: First, expectation values of physical observables are IR-finite, i.e., boundaries of the system do not enter into their calculation. Second, the distances between the observable points set the largest scale L in the problem. Third, when varying these distances, L changes and thus the value of diverging subdiagrams as (2.37). This gives rise to an anomalous scaling of the observables. The latter is most comfortably taken care of by the renormalization group procedure, which also allows for a proof of the above statements.

Let us now turn back to the passive scalar problem, and consider

$$S^2(x-y, t=0) := \langle [\Theta(x,0) - \Theta(y,0)]^2 \rangle \tag{2.38}$$

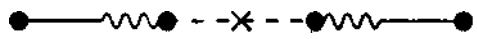
The order 0 contribution is



$$= \int_0^\infty \mathrm{d}t \int \mathrm{d}^d z \, \mathrm{d}^d z' [R(x-z,t) - R(y-z,t)] \times [R(x-z',t) - R(y-z',t)] \, G_f(M\,|z-z'|) \tag{2.39}$$

[2] A caveat is in order here: Calculating at small but finite values of $\varepsilon := 4-d$, there is always an IR-divergence at sufficiently high orders in perturbation theory. To see this take a long chain of bubbles, similar to that of Eq. (2.36). The standard way to circumvent this well-known (technical) problem of the massless theory is to take ε small enough inside each diagram.

Using Eq. (2.11) this can be written as



$$\sim \int \mathrm{d}^d z\, \mathrm{d}^d z' ([(x-z)^2+(x-z')^2]^{1-d} + [(y-z)^2+(y-z')^2]^{1-d} - [(x-z)^2+(y-z')^2]^{1-d} - [(y-z)^2+(x-z')^2]^{1-d})\, G_f(M\,|z-z'|) \tag{2.40}$$

Integrating over both z and z' large, the integral (2.40) scales as

$$\int \mathrm{d}^{2d} s\, s^{2(1-d)} s^{-2} G_f(Ms) \sim \int \frac{\mathrm{d}s}{s}\, G_f(s) \sim \ln L \tag{2.41}$$

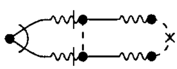
where the factor of s^{-2} is due to the differences in Eq. (2.40), and $L = 1/M$ is the scale at which G_f starts to decay rapidly. $S^2(x-y, t)$ thus *explicitly* depends on the largest distance $L > |x-y|$ in the problem, in the very contrast to the example of the ϕ^4-theory considered above.

We shall now show that perturbative corrections in D make S^2 depend even stronger on L, namely as $L^{n\varepsilon}$ at nth order in perturbation theory. To this aim consider the term of order D. Using the regularized theory, it appears at two places: First v is renormalized, and thus the resulting effective response-function $R(x, t)$ decays faster—this effect would render (2.41) IR-convergent. However, there is a second contribution, namely

D 

$$= \int \mathrm{d}^d z \int \mathrm{d}^d z' \int \mathrm{d}t\, \frac{\partial}{\partial z_i} [R(x-z, t) - R(y-z, t)] \times \frac{\partial}{\partial z'_j} [R(x-z', t) - R(y-z', t)]\, \mathrm{D}^{ij}_v(x-y) \tag{2.42}$$

This is only some part of the diagram, and in principle, it has to be closed through a , leading to



(2.43)

Since this diagram is again plagued by an IR-divergence, the leading contribution in Eq. (2.42) will come from the domain of large z and z'. In that limit, $|x-y|$ is much smaller than both z and z' and consequently, $\Theta(x) - \Theta(y)$ can be replaced by $(x-y)\, \nabla\Theta((x+y)/2)$. Let us again stress that this is valid in the domain, where $|x| \ll L$ and $|y| \ll L$, or when using $G_f(M\,|x-y|)$ where $|x-y| \ll L$. Due to that replacement, we can now use a very powerful trick: Instead of analyzing the IR-divergences of $\langle [\Theta(x,t)-\Theta(y,t)]^2 \rangle$, or more generally of $S^{2n}(x-y,t) := \langle [\Theta(x,t)-\Theta(y,t)]^{2n} \rangle$ we can analyse the *UV-divergences* of the *composite operator* $[(x-y)\,\nabla\Theta((x+y)/2)]^{2n}$. The latter however is a standard task in perturbative renormalization. We will see in the next section, that this leads to a whole family of operators and anomalous dimensions; the operator with the smallest dimension will then give the term which most sensitively depends on L. In order to avoid confusions, let us already note that the second moment $S^2(r)$ discussed above does not depend on L, since the contribution to the response-function and that in Eq. (2.43) just cancel. This can also be obtained exactly.[5]

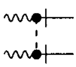
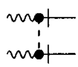
2.4. The Scaling of S^{2n} and Renormalization of Composite Operators

As discussed in the last subsection, we now have to study the renormalization of $[z\,\nabla\Theta]^{2n}$. It will turn out that under renormalization this term generates $[z\,\nabla\Theta]^{2n-2}\,z^2[(\nabla\Theta)^2]$. In a second step, $[z\,\nabla\Theta]^{2n-2}\,z^2[(\nabla\Theta)^2]$ generates $[z\,\nabla\Theta]^{2n-4}\,z^4[(\nabla\Theta)^2]^2$ a.s.o. until also a term of the form $z^{2n}[(\nabla\Theta)^2]^n$ is generated. All these operators will mix under renormalization. The eigen-operator with the smallest dimension will give the term which most strongly depends on L.

We now treat the general case. Define

$$\mathcal{S}^{(n,m)} := z^{2n}[(\nabla\Theta)^2]^n\,[z\,\nabla\Theta]^{2m} \tag{2.44}$$

We first observe that the operator product expansion (denoted by $\blacklozenge$) is

$$\mathcal{S}^{(n,m)} \blacklozenge \frac{\mathrm{D}}{2}$$

$$= T^{ij}\left[\,(\nabla^i\Theta)(\nabla^j\Theta) \blacklozenge \frac{\mathrm{D}}{2}\right.$$

$$\left.\vphantom{\frac{D}{2}}\right] \tag{2.45}$$

with

$$\begin{aligned} T^{ij} &= \frac{1}{2}\frac{\partial}{\partial(\nabla^i\Theta)}\frac{\partial}{\partial(\nabla^j\Theta)}\{z^{2n}[(\nabla\Theta)^2]^n\,(z\,\nabla\Theta)^{2m}\} \\ &= n\delta^{ij}z^{2n}[(\nabla\Theta)^2]^{n-1}\,(z\,\nabla\Theta)^{2m} \\ &\quad + 2n(n-1)\,z^{2n}(\nabla^i\Theta)(\nabla^j\Theta)[(\nabla\Theta)^2]^{n-2}\,(z\,\nabla\Theta)^{2m} \\ &\quad + 2nmz^{2n}[z^i\,\nabla^j\Theta + z^j\,\nabla^i\Theta][(\nabla\Theta)^2]^{n-1}\,(z\,\nabla\Theta)^{2m-1} \\ &\quad + m(2m-1)\,z^{2n}z^iz^j[(\nabla\Theta)^2]^n\,(z\,\nabla\Theta)^{2m-2} \end{aligned} \tag{2.46}$$

Equation (2.45) then reads

$$D\int_0^\infty \mathrm{d}t \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, T^{ij} p^i p^j R^2(p,t)\, \frac{1}{(p^2+m^2)^{(d+\varepsilon)/2}}\left[(\nabla\Theta)^2 - \frac{(p\,\nabla\Theta)^2}{p^2}\right] + O(\varepsilon^2) \tag{2.47}$$

Since $R(p,t) = \mathrm{e}^{-\nu_0 p^2 t}\Theta(t)$, integration over t yields (up to finite terms)

$$\frac{D}{2\nu_0}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, T^{ij}\, \frac{p^i p^j}{p^2}\, \frac{1}{(p^2+m^2)^{(d+\varepsilon)/2}}\left[(\nabla\Theta)^2 - \frac{(p\,\nabla\Theta)^2}{p^2}\right] \tag{2.48}$$

T^{ij} in Eq. (2.46) has the form

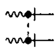
$$T^{ij} = A^2\delta^{ij} + B^i C^j \tag{2.49}$$

Inserting this into Eq. (2.48) and using the formulas from Appendix A.1 and Eq. (2.15) yields

$$\frac{u}{2}\, C_d\, \frac{m^{-\varepsilon}}{\varepsilon}\left[A^2(\nabla\Theta)^2\left(1-\frac{1}{d}\right) + (\nabla\Theta)^2\,(BC)\,\frac{1}{d} - \frac{1}{d(d+2)}\,(2(B\,\nabla\Theta)(C\,\nabla\Theta) + (BC)(\nabla\Theta)^2)\right] \tag{2.50}$$

Specifying T^{ij} in Eq. (2.49) to its value of Eq. (2.46) gives

$$\frac{u}{2}\, C_d\, \frac{m^{-\varepsilon}}{\varepsilon}\, \frac{1}{d(d+2)} \times \{[n(d-1)(d+2n+4m) - 2m(2m-1)]\, z^{2n}[(\nabla\Theta)^2]^n\, [z\,\nabla\Theta]^{2m} + m(2m-1)(d+1)\, z^{2n+2}[(\nabla\Theta)^2]^{n+1}\, [z\,\nabla\Theta]^{2m-2}\} \tag{2.51}$$

The final result when contracting $\mathscr{S}^{(n,m)}$ with $(D/2)$  is

$\mathscr{S}^{(n,m)} \blacklozenge \frac{D}{2}$ 

$$= \mathscr{S}^{(n,m)}\, \frac{u}{2}\frac{C_d}{\varepsilon}\, m^{-\varepsilon}\, \frac{1}{d(d+2)} \times [n(d-1)(d+2n+4m) - 2m(2m-1)] + \mathscr{S}^{(n+1,m-1)}\, \frac{u}{2}\frac{C_d}{\varepsilon}\, m^{-\varepsilon}\, \frac{d+1}{d(d+2)}\, m(2m-1) \tag{2.52}$$

This shows that $\mathscr{S}^{(0,n)}, \mathscr{S}^{(1,n-1)},..., \mathscr{S}^{(n,0)}$ mix under renormalization. The first two eigen-operators are $\mathscr{S}^{(n,0)}$ and $\bar{\mathscr{S}}^{(n-1,1)} := \mathscr{S}^{(n-1,1)} - (1/d)\, \mathscr{S}^{(n,0)}$, with eigen-values

$\mathscr{S}^{(n,0)} \blacklozenge \dfrac{D}{2}$  $= \mathscr{S}^{(n,0)} \dfrac{u}{2} \dfrac{C_d}{\varepsilon} m^{-\varepsilon} \dfrac{n(d-1)(d+2n)}{d(d+2)} \qquad (2.53)$

$\bar{\mathscr{S}}^{(n-1,1)} \blacklozenge \dfrac{D}{2}$  $= \bar{\mathscr{S}}^{(n-1,1)} \dfrac{u}{2} \dfrac{C_d}{\varepsilon} m^{-\varepsilon} \dfrac{(n-1)(d-1)(d+2n+2)-2}{d(d+2)} \qquad (2.54)$

More importantly, due to the triangular form of the matrix, the eigen-values can just be read off from the diagonal. Therefore the eigen-operators $\bar{\mathscr{S}}^{(n,m)}$ are multiplicatively renormalizable (again denoting by index $_0$ bare quantities)

$$\bar{\mathscr{S}}_0^{(n,m)} = Z^{(n,m)} \bar{\mathscr{S}}^{(n,m)}$$
$$Z^{(n,m)} = 1 - \frac{u}{2} \frac{C_d}{\varepsilon} \frac{1}{d(d+2)} [n(d-1)(d+2n+4m) - 2m(2m-1)] \tag{2.55}$$

This yields the anomalous scaling-function $\gamma^{(n,m)}(u)$ of $\bar{\mathscr{S}}_0^{(n,m)}$ in units of x as

$$\begin{aligned} \gamma^{(n,m)}(u) &:= -m \frac{\partial}{\partial m} \ln Z^{(n,m)} \\ &= -\frac{u}{2} C_d \frac{1}{d(d+2)} [n(d-1)(d+2n+4m) - 2m(2m-1)] \end{aligned} \tag{2.56}$$

At the IR fixed point $u = u^*$ from Eq. (2.25), this is

$$\gamma^{(n,m)} = -\frac{\varepsilon}{(d-1)(d+2)} [n(d-1)(d+2n+4m) - 2m(2m-1)] \tag{2.57}$$

Taking care of the naive perturbative contribution to the scaling of $\bar{\mathscr{S}}_0^{(n,m)}$ from Eq. (2.28), we finally obtain the full scaling-dimension $\Delta^{(n,m)}$ of $\bar{\mathscr{S}}_0^{(n,m)}$

$$\begin{aligned} \Delta^{(n,m)} &= \gamma^{(n,m)} - 2(n+m)\,\eta^* \\ &= \gamma^{(n,m)} + (n+m)\,\varepsilon \\ &= -\varepsilon \frac{2n(n-1+2m)}{(d+2)} + \varepsilon \frac{m(4m-1) + d^2 + d}{(d+2)(d-1)} \end{aligned} \tag{2.58}$$

These results have already been obtained in refs. 7, 12, and 16, where only the case $m=0$ was given. The case $m>0$ can be found in ref. 32.

The exponents satisfy the inequality

$$\Delta^{(n,m)} < \Delta^{(n-1,m+1)} \tag{2.59}$$

such that indeed $\Delta^n := \Delta^{(n,0)}$ is the smallest of all exponents $\Delta^{(n-m,m)}$ and dominates the L-dependence of $S^{2n}(r,t)$ as stated above. Explicitly,

$$S^{2n}(r,t) \sim r^{n(2-\varepsilon)} \left(\frac{r}{L}\right)^{\Delta^n}, \qquad \Delta^n = -\varepsilon \frac{2n(n-1)}{d+2} \tag{2.60}$$

Only the second moment does not depend on L. As demonstrated in ref. 5, this is the consequence of a conservation law, which allows for an exact calculation of the second moment.

These results have been tested numerically with different methods in refs. 10, 21, 23, and 27.

The other question one might ask is why only one of the two factors in Eq. (2.60) depends on L, and whether the r-dependence comes out correctly. To understand this point, we recall that the first factor is due to the renormalization of v, and thus contributes to the anomalous dimension of Θ, irrespective of the boundary conditions. The second factor stems from the anomalous dimension of the composite operator $S^{(n,0)}$, which was associated to the IR-divergence, i.e., L-dependence of $S^{2n}(r,t)$, and which has two contributions: the proper renormalization of $S^{(n,0)}$ as given by $Z^{(n,0)}$ or $\gamma^{(n,0)}$, and the renormalization of v; these add up to $\Delta^{(n,0)}$ as given in Eq. (2.58). Only the combination of these terms contribute to the L-dependence of $S^{2n}(r,t)$.

Also note that the exponents with $m>0$ are also observable, and correspond to observables of different symmetries.

3. LANGEVIN-DESCRIPTION OF THE PASSIVE SCALAR

3.1. Model and Basic Properties

Let us now turn to a Langevin-description of the passive scalar problem. We start from Eq. (2.1)

$$[\partial_t + v(x,t)\nabla]\,\Theta(x,t) = \nu_0\,\Delta\Theta(x,t) + f(x,t) \tag{3.1}$$

Without the source $f(x,t)$, this can easily be converted into a Langevin equation. The question arises, how the additional term $f(x,t)$ can be incorporated. We will see below that it corresponds to the creation and annihilation of particles, and that it can indeed be formulated within a Langevin description: However, this is a question of marginal interest, since we had seen in the last section that the whole renormalization procedure can be performed without ever specifying the correlations of $f(x, t)$, just knowing that they will deliver some IR-cutoff L. We therefore start our analysis by studying Eq. (3.1) with $f(x, t) \equiv 0$. Using standard arguments,[33] it is transformed into a Langevin equation for the motion of a particle with position $r(t) \in \mathbb{R}^d$. Since it will turn out later, that to reproduce all expectation values, one has to introduce N particles, we will already give the corresponding generalization here.

$$\begin{aligned} \partial_t r^i_\alpha(t) &= v^i(r_\alpha(t), t) - \zeta^i_\alpha(t) \\ \langle \zeta^i_\alpha(t)\, \zeta^j_\beta(t') \rangle &= 2\nu_0 \delta^{ij} \delta_{\alpha\beta} \delta(t-t') \end{aligned} \tag{3.2}$$

The vv correlations are the same as in Eq. (2.2). The dynamic action which enforces the Langevin-equation to be satisfied reads

$$J[r, \tilde{r}, v, \zeta] = \sum_{\alpha=1}^{N} \int \mathrm{d}t\, \tilde{r}^i_\alpha(t) [\partial_t r^i_\alpha(t) - v^i(r_\alpha(t), t) + \zeta^i_\alpha(t)] \tag{3.3}$$

Averaging $\mathrm{e}^{-J[r, \tilde{r}, v, \zeta]}$ over ζ and v leads to

$$\begin{aligned} J[r, \tilde{r}] &= \int \mathrm{d}t \sum_{\alpha=1}^{N} [\tilde{r}^i_\alpha(t)\, \partial_t r^i_\alpha(t) - \nu_0 \tilde{r}^i_\alpha(t)^2] \\ &\quad - \int \mathrm{d}t\, \frac{\mathrm{D}_0}{2} \sum_{\alpha, \beta=1}^{N} \tilde{r}^i_\alpha(t) \left[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, P^{ij}(k)\, \frac{\mathrm{e}^{ik[r_\alpha(t) - r_\beta(t)]}}{(k^2+m^2)^{(d+\varepsilon)/2}} \right] \tilde{r}^j_\beta(t) \end{aligned} \tag{3.4}$$

Symbolically, this is written as

$$J[r, \tilde{r}] = \int \mathrm{d}t \left(\sum_{\alpha=1}^{N} [\ \rightsquigarrow\!\bullet\!-\ _\alpha - \nu_0\ \rightsquigarrow\!\bullet\!\rightsquigarrow\ _\alpha] - \frac{\mathrm{D}_0}{2} \sum_{\alpha, \beta=1}^{N}\ _\alpha \rightsquigarrow\!\bullet\!-\,-\,-\,-\,-\!\bullet\!\rightsquigarrow\ _\beta \right) \tag{3.5}$$

Free response and correlation functions are

$$\begin{aligned} R(\omega) &= \frac{1}{i\omega}, \qquad R(t) = \Theta(t) \\ C(\omega) &= \frac{2\nu_0}{\omega^2}, \qquad C(t) := \frac{1}{2d} \langle [r(t) - r(0)]^2 \rangle_0 = \nu_0\, |t| \end{aligned} \tag{3.6}$$

3.2. Renormalization of the Dynamic Action

In Subsection 2.2 we have, seen that in the *dynamic action* (2.7) only v_0 demands a renormalization. How does this renormalization show up in the formulation as a Langevin equation? To answer this question, write down the first order term in D_0 from the expansion of e^{-J}:

$$\frac{D_0}{2} \sum_{\alpha, \beta = 1}^{N}$$  (3.7)

with


$$= \tilde{r}_\alpha^i(t) \left[\int \frac{d^d k}{(2\pi)^d} P^{ij}(k) \frac{e^{ik[r_\alpha(t) - r_\beta(t)]}}{(k^2 + m^2)^{(d+\varepsilon)/2}} \right] \tilde{r}_\beta^j(t) \tag{3.8}$$

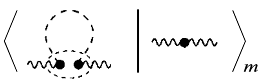
We now have to analyze short-distance divergences, i.e., what happens if the two points come close together. This is most easily done using the techniques of multilocal operator product expansion, introduced in refs. 34 and 35, further developed in refs. 36–39 and reviewed in ref. 40. By the dashed line which encircles the two fat points, we indicate points which come close together. We now want to express $e^{ik[r_\alpha(t) - r_\beta(t)]}$ through its normal-ordered version $:e^{ik[r_\alpha(t) - r_\beta(t)]}:$, which does not contain any self-contractions. This is

$$e^{ik[r_\alpha(t) - r_\beta(t')]} = :e^{ik[r_\alpha(t) - r_\beta(t')]}: e^{-k^2(1/2d)\langle [r_\alpha(t) - r_\beta(t')]^2 \rangle_0} \tag{3.9}$$

This leads to a drastic simplification: Since $(1/2d)\langle [r_\alpha(t) - r_\beta(t)]^2 \rangle_0$ equals infinity except for $\alpha = \beta$ for which it vanishes, Eq. (3.8) gives


$$\begin{aligned} &= \delta_{\alpha\beta} \tilde{r}_\alpha^i(t)\, \tilde{r}_\alpha^j(t) \left[\int \frac{d^d k}{(2\pi)^d} P^{ij}(k) \frac{1}{(k^2 + m^2)^{(d+\varepsilon)/2}} \right] \\ &= \delta_{\alpha\beta} \tilde{r}_\alpha(t)^2 \left(1 - \frac{1}{d}\right) C_d \frac{m^\varepsilon}{\varepsilon} \end{aligned} \tag{3.10}$$

where C_d is defined in Eq. (2.15). Symbolically, the diagram is written as


$$= \left(1 - \frac{1}{d}\right) C_d \frac{m^\varepsilon}{\varepsilon} \tag{3.11}$$

which reminds of Feynman's bra and ket notation. We have added an index m to indicate the IR-cutoff. Equation (3.11) leads to the same renormalization for v as given in Eq. (2.16) for the Focker–Planck formulation of the problem.

Even though equivalent, the treatment in terms of the Langevin equation reveals one important property: The only renormalization of the dynamic action comes from the divergence in a *single particle* trajectory. We will see later, that the renormalization of S^n is due to multiparticle diagrams.

3.3. Simple Expectation Values and Translation Table

In this section, we make more explicit the relation between the two formulations of the problem. What we want to calculate are expectation values of $\Theta(x, t)$. We first observe that in the limit of $N \to \infty$

$$\Theta(x, t) \leftrightarrow \Theta_{\mathrm{L}}(x, t) := \frac{1}{N} \sum_{\alpha=1}^{N} \delta^d(r_\alpha(t) - x) \tag{3.12}$$

If we do not know where the particles started, then obviously

$$\langle \Theta_{\mathrm{L}}(x, t) \rangle_0 = 0 \tag{3.13}$$

This is also true for higher moments

$$\langle \Theta_{\mathrm{L}}(x, t)^n \rangle_0 = 0 \tag{3.14}$$

It is important to note that the limit of $N \to \infty$ is necessary in order to suppress correlations coming from the same particle, which in the expectation value (3.14) are of order $1/N$.

Equivalently in momentum representation, when defining the Fourier-transform as

$$f(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} f(k)\, \mathrm{e}^{ikx} \tag{3.15}$$

Eq. (3.12) will read:

$$\Theta(k, t) \leftrightarrow \Theta_{\mathrm{L}}(k, t) = \frac{1}{N} \sum_{\alpha=1}^{N} \mathrm{e}^{-ikr_\alpha(t)} \tag{3.16}$$

We now check that the free response-function (2.10) of the Focker–Planck formulation is correctly reproduced. In contrast to the Focker–Planck formulation, here we have to solve the initial time problem explicitly, i.e., suppose that the particles start at position x_0 at time t_0.

$$
\begin{aligned}
&R_{\mathrm{L}}(x-x_0, t-t_0)\\
&\quad := \langle \Theta_{\mathrm{L}}(x,t)\rangle_0 \big|_{r_\alpha(t_0)=x_0} \Theta(t-t_0)\\
&\quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\langle \frac{1}{N} \sum_{\alpha=1}^{N} \mathrm{e}^{ik[r_\alpha(t)-x]} \right\rangle_0 \Bigg|_{r_\alpha(t_0)=x_0} \Theta(t-t_0)\\
&\quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\langle \frac{1}{N} \sum_{\alpha=1}^{N} \mathrm{e}^{ik[r_\alpha(t)-r_\alpha(t_0)]} \right\rangle_0 \Bigg|_{r_\alpha(t_0)=x_0} \mathrm{e}^{ik[x_0-x]} \Theta(t-t_0)\\
&\quad = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \,\mathrm{e}^{-k^2 v_0 (t-t_0)} \mathrm{e}^{ik[x_0-x]} \Theta(t-t_0)\\
&\quad \equiv R_{\mathrm{FP}}(x-x_0, t-t_0)
\end{aligned} \tag{3.17}
$$

as given in Eq. (2.11). Note that this response-function is a single-particle function, i.e., only the response of a single particle α to a change in its (earlier) trajectory contributes.

Injecting particles with a rate $f(x, t)$ at time t at position x, models the corresponding term in Eq. (2.1), and finally leads to the same correlation function as in the Focker–Planck representation

$$
\begin{aligned}
C_{\mathrm{L}}(x, t; y, t') &= \int \mathrm{d}t_0 \int \mathrm{d}t'_0 \int \mathrm{d}^d x_0 \int \mathrm{d}^d y_0\, R_{\mathrm{L}}(x-x_0, t-t_0)\, R_{\mathrm{L}}(y-y_0, t'-t'_0)\\
&\qquad \times \langle f(x_0, t_0)\, f(y_0, t'_0)\rangle\\
&= \int \mathrm{d}t_0 \int \mathrm{d}^d x_0 \int \mathrm{d}^d y_0\, R_{\mathrm{L}}(x-x_0, t-t_0)\, R_{\mathrm{L}}(y-y_0, t'-t_0)\\
&\qquad \times G_f(Mx_0, My_0)\\
&\equiv C_{\mathrm{FP}}(x, t; y, t')
\end{aligned} \tag{3.18}
$$

By comparing the terms of order $v(x, t)$ in Eqs. (2.6) and (3.3), we also obtain the equivalence

$$
\Theta(x,t)\, \nabla^i \tilde{\Theta}(x,t) \leftrightarrow \Theta_{\mathrm{L}}(x,t)\, \nabla^i \tilde{\Theta}_{\mathrm{L}}(x,t) := \sum_{\alpha=1}^{N} \tilde{r}^i_\alpha(t)\, \delta^d(r_\alpha(t)-x) \tag{3.19}
$$

3.4. The Scaling of S^{2n} and Renormalization of Composite Operators

We now have to reproduce the results of Section 2.4. This will be done in two steps. First, we will show that the OPE of $\mathcal{O}^i_{\mathrm{L}}(x, t) := \Theta_{\mathrm{L}}(x, t) \times$

$\nabla^i \tilde{\Theta}_{\mathrm{L}}(x,t) = \sum_{\alpha=1}^{N} \tilde{r}^i_\alpha(t)\, \delta^d(r_\alpha(t)-x)$ with the nth power of $\Theta_{\mathrm{L}}(y,t') = (1/N)\sum_{\beta=1}^{N} \delta^d(r_\beta(t')-y)$ is as in the formulation via a Focker–Planck equation encoded in the contraction of $\mathcal{O}^i_{\mathrm{L}}(x,t)$ with a single $\Theta_{\mathrm{L}}(y,t')$:

$$\mathcal{O}^i_{\mathrm{L}}(x,t) \blacklozenge \Theta_{\mathrm{L}}(y,t')^n = n\Theta_{\mathrm{L}}(y,t')^{n-1}\,[\mathcal{O}^i_{\mathrm{L}}(x,t) \blacklozenge \Theta_{\mathrm{L}}(y,t')] \tag{3.20}$$

The reason is that when contracting $\tilde{r}^i_\alpha(t)$ in $\mathcal{O}^i_{\mathrm{L}}(x,t)$ with $\Theta_{\mathrm{L}}(y,t')^n$ singles out a particle α in one of the $\Theta_{\mathrm{L}}(y,t')$. Correlations of this particle α with $r_\beta(t')$ in another of the factors $\Theta_{\mathrm{L}}(y,t')$ only exist for $\alpha=\beta$ which is suppressed by a factor of $1/N$.

Therefore, it is sufficient to show that $\mathcal{O}^i(x,t) \blacklozenge \Theta(y,t')$ is the same in both the Focker–Planck and the Langevin formulations. In the Focker–Planck formulation, we have

$$\mathcal{O}^i_{\mathrm{FP}}(x,t) \blacklozenge \Theta_{\mathrm{FP}}(y,t') = \Theta_{\mathrm{FP}}(x,t)\, \nabla^i R_{\mathrm{FP}}(x-y, t-t') \tag{3.21}$$

In the Langevin-formulation, the same expression reads

$$\begin{aligned}
&\mathcal{O}^i_{\mathrm{L}}(x,t) \blacklozenge \Theta_{\mathrm{L}}(y,t')\\
&\quad= \sum_{\alpha=1}^{N} \tilde{r}^i_\alpha(t)\, \delta^d(r_\alpha(t)-x) \blacklozenge \frac{1}{N}\sum_{\beta=1}^{N} \delta^d(r_\beta(t')-y)\\
&\quad= \Theta(t'-t)\,\frac{1}{N}\sum_{\alpha=1}^{N}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}\,(ip)^i\, e^{ik[r_\alpha(t)-x]} \blacklozenge e^{ip[r_\alpha(t')-y]}\\
&\quad= \Theta(t'-t)\,\frac{1}{N}\sum_{\alpha=1}^{N}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}\,(ip)^i :e^{ik[r_\alpha(t)-x]}e^{ip[r_\alpha(t')-y]}:\, e^{kp\,|t-t'|}
\end{aligned} \tag{3.22}$$

where the normal-order sign ":" indicates that contractions between the included operators are factored out as $e^{kp\,|t-t'|}$. This is useful since the normal-ordered operator $:e^{ik[r_\alpha(t)-x]}e^{ip[r_\alpha(t')-y]}:$ is free of divergences when approaching x and y. For an introduction and review of these techniques, see ref. 40.

In the next step, the integration variable k is shifted to $k \to k-p$

$$\begin{aligned}
&\Theta(t'-t)\,\frac{1}{N}\sum_{\alpha=1}^{N}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}\,(ip)^i\\
&\qquad\times :e^{ik[r_\alpha(t)-x]}e^{ip[r_\alpha(t')-r_\alpha(t)]}:\, e^{(k-p)\,p\,|t-t'|}e^{ip[x-y]}
\end{aligned} \tag{3.23}$$

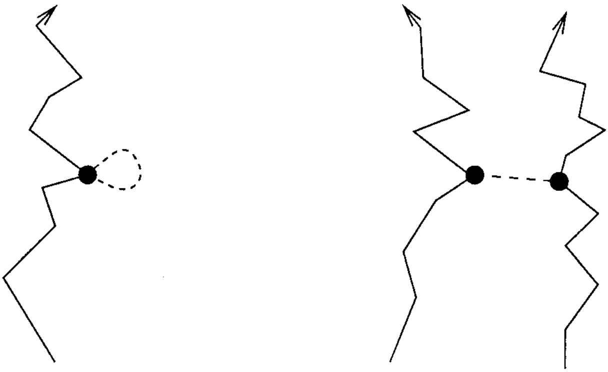


Fig. 3.1. The two different types of diagrams, involved in the renormalization of (a) v (left) and (b) S^{2n} (right). In both cases we have drawn the particle trajectories, as well as one interaction.

Partially undoing the normal-order procedure finally leads to

$$\Theta(t'-t)\frac{1}{N}\sum_{\alpha=1}^{N}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}(ip)^i \times :e^{ik[r_\alpha(t)-x]}::e^{ip[r_\alpha(t')-r_\alpha(t)]}:e^{-p^2|t-t'|}e^{ip[x-y]} \tag{3.24}$$

Note that this is not the standard procedure (as described in ref. 40) but is particularly useful for our purposes. Two routes of argument are now open: The first one consists in the observation that in the desired limit of $t\to t'$, $:e^{ip[r_\alpha(t')-r_\alpha(t)]}:$ is approximately 1, and thus the integrals over k and p factorize, leading to

$$\Theta_L(x,t)\,\nabla^i R_L(x-y,t-t') \tag{3.25}$$

which using Eq. (3.17) is the same result as in Eq. (3.21).

We finally want to argue that the above result becomes exact, when specifying the boundary conditions. To that purpose, suppose that we start at time τ at position 0. The condition that the particle be at position 0 is again expressed as a δ-function and using the Fourier-representation $\delta^d(r_\alpha(\tau))=\int(d^dl/(2\pi)^d)\,e^{ilr_\alpha(\tau)}$, we are led to study the expectation value of

$$\left\langle e^{ilr_\alpha(\tau)}\int\frac{d^dk}{(2\pi)^d}e^{ik[r_\alpha(t)-x]}\right\rangle_0=\int\frac{d^dk}{(2\pi)^d}e^{kl|t-\tau|}e^{-ikx}\langle e^{ilr_\alpha(\tau)+ikr_\alpha(t)}\rangle_0 \tag{3.26}$$

Since $\langle e^{ilr_\alpha(\tau)+ikr_\alpha(t)}\rangle_0=(2\pi)^d\,\delta^d(k+p)$, only expectation values of exponentials

$$:e^{i\sum_{j=1}^{n}k_jr_\alpha(t_j)}:$$

survive for which global "charge neutrality" $\sum_{j=1}^{n} k_j = 0$ holds. Therefore, in Eq. (3.24), we have to supply an additional factor $\mathrm{e}^{-ikr_\alpha(\tau)}$ at initial time τ, leading inside the integral to

$$:\mathrm{e}^{ik[r_\alpha(t) - r_\alpha(\tau)]}: :\mathrm{e}^{ip[r_\alpha(t') - r_\alpha(t)]}: \mathrm{e}^{-p^2 |t - t'| - k^2 |t - \tau|} \tag{3.27}$$

(Note that we have already normal-ordered the first exponential, leading to the factor of $\mathrm{e}^{-k^2 |t - \tau|}$.) The key-observation now is that

$$:\mathrm{e}^{ik[r_\alpha(t) - r_\alpha(\tau)]}: :\mathrm{e}^{ip[r_\alpha(t') - r_\alpha(t)]}: \equiv :\mathrm{e}^{ik[r_\alpha(t) - r_\alpha(\tau)]} \mathrm{e}^{ip[r_\alpha(t') - r_\alpha(t)]}: \tag{3.28}$$

as long as $\tau < t$ and (by assumption) $\tau < t'$. This factorization property is one of the essential simplifications for polymers (which are nothing but a random walk), see e.g., Section 10 of ref. 40. For our case, it shows that the above stated *approximate* equivalence between the Focker–Planck and Langevin descriptions is indeed exact, as one expects from the equivalence of the two equations.

In conclusion: Since the above arguments show (at least at leading order) the equivalence of the both perturbation theories for the renormalization of S^{2n}, we obtain the same results as in Subsection 2.4.

Let us still give some remarks on the class of diagrams involved in the renormalization of S^{2n}. Whereas the diagrams which contribute to the renormalization of v are single-particle diagrams, i.e., diagrams where one particle interacts with itself, the diagrams which contribute to the renormalization of $S^{2n}(x - y)$ are multiple-particle diagrams, i.e., diagrams where particles which finally end at x interact with other particles, which finally end at y.

3.5. Interpretation in Terms of Particle Trajectories Only

First of all, one can determine the single particle motion, which is super-diffusive. By means of a complete RG-analysis, or faster using the method of exact exponent identities,[40] we obtain due to the non-renormalization of the terms proportional to $\tilde{r}\, \partial_t r$ and D_0 the exact identity

$$\langle (r_\alpha(t) - r_\alpha(0))^2 \rangle \sim |t|^{2/(2-\varepsilon)} \tag{3.29}$$

which is the analogue of Eq. (2.27) ff.

More interestingly, $S^{2n}(x - y, t)$ can also be obtained in terms of particle trajectories only, following refs. 27 and 41. We first note that due to Eq. (3.12), $\Theta_{\mathrm{L}}(x, t)$ is the contribution from particles which are created by

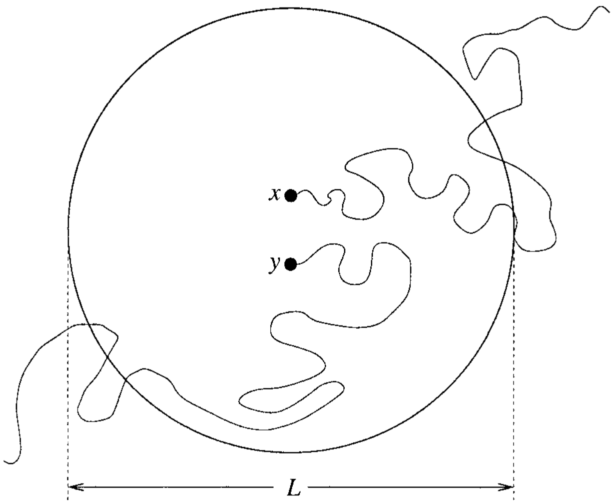


Fig. 3.2. Intuitive interpretation of $S^{2n}(x-y)$ via particles being advected by the turbulent flow. Drown is one configuration which contributed to t_{xy} of Eq. (3.28). Note that particles may return into the box, such that one has to wait long enough until the probability of return tends to 0. For details cmp. the main text.

$f(x', t')$ at time $t' < t$ and position x' and arrive at time t at position x:

$$\Theta_{\rm L}(x, t) = \frac{1}{N} \sum_{\alpha=1}^{N} \int_{-\infty}^{t} {\rm d}t' \; f(r_\alpha(t'), t') \big|_{r_\alpha(t)=x} \tag{3.30}$$

This gives a simple method to evaluate moments of Θ within a Monte Carlo simulation. The following observation helps to render the derivation more transparent:

For given ζ and v, the process is deterministic. Since ζ and v are Gaussian and uncorrelated in time, they are time-reversal invariant. We therefore write $\Theta_{\rm L}(x, t)$ as integral over all trajectories, starting at time t at x and ending at time $t' > t$ at x', where they are “created” by $f(x', t')$:

$$\Theta_{\rm L}(x, t) = \frac{1}{N} \sum_{\alpha=1}^{N} \int_{t}^{\infty} {\rm d}t' \; f(r^x_\alpha(t'), t') \tag{3.31}$$

where $r^x_\alpha(t)$ satisfies the equation of motion and boundary condition

$$\partial_{t'} r^x_\alpha(t') = v(r^x_\alpha(t'), t') + \zeta_\alpha(t), \qquad r^x_\alpha(t) = x \tag{3.32}$$

We now turn to the evaluation of higher correlation functions. Let us demonstrate the principle on the example of the second moment, keeping in mind that finally $N \to \infty$:

$$\begin{aligned}
&\langle [\Theta(x,t) - \Theta(y,t)]^2 \rangle \\
&\quad = \langle\langle\langle [\Theta_{\mathrm{L}}(x,t) - \Theta_{\mathrm{L}}(y,t)]^2 \rangle_\zeta \rangle_f \rangle_v \\
&\quad = \frac{1}{N^2} \sum_{\alpha,\beta=1}^{N} \left\langle \left\langle \left\langle \int_t^\infty \mathrm{d}t_1 \int_t^\infty \mathrm{d}t_2 [f(r_\alpha^x(t_1), t_1) - f(r_\alpha^y(t_1), t_1)] \right.\right.\right. \\
&\qquad \left.\left.\left. \times [f(r_\alpha^x(t_2), t_2) - f(r_\beta^y(t_2), t_2)] \right\rangle_\zeta \right\rangle_f \right\rangle_v \\
&\quad = \frac{1}{N^2} \sum_{\alpha,\beta=1}^{N} \int_t^\infty \mathrm{d}t' \langle\langle G_f^M(r_\alpha^x(t'), r_\beta^x(t')) \\
&\qquad + G_f^M(r_\alpha^y(t'), r_\beta^y(t')) - 2G_f^M(r_\alpha^x(t'), r^y{}_\beta(t')) \rangle_\zeta \rangle_v
\end{aligned} \tag{3.33}$$

where we used that

$$\langle f(r,t)\, f(r',t') \rangle = \delta(t-t')\, G_f(Mr, Mr') \equiv \delta(t-t')\, G_f^M(r,r') \tag{3.34}$$

is δ-correlated in time. In principle, $G_f(x, y)$ has to fulfill four conditions

(i) $G_f(x, y) = G_f(y, x)$.

(ii) $\int_x G_f(x, y) = 0$ as a consequence of $\int_r f(r, t) = 0$ in order to have no global heating (particle conservation). This condition is necessary to reach a steady state.

(iii) $G_f(x, y) \xrightarrow{x \to \infty} 0$. We take it at least exponentially decaying at scale 1.

(iv) G_f has to be realizable as a stochastic process, such that as a consequence of $\langle (f(x,t) - f(y,t))^2 \rangle \geqslant 0$ one must have $G_f(x, x) + G_f(y, y) - 2G(x, y) \geqslant 0$.

A possible choice for $G_f(x, y)$ that satisfies the above conditions is

$$G_f(x, y) := \Delta_x \Delta_y \, \mathrm{e}^{-x^2 - y^2} = 4\mathrm{e}^{-x^2 - y^2}(2x^2 - d)(2y^2 - d) \tag{3.35}$$

Constraint (iv) is satisfied since

$$\begin{aligned}
&G_f(x, x) + G_f(y, y) - 2G_f(x, y) \\
&\qquad = 4\mathrm{e}^{-2(x^2 + y^2)} [d(\mathrm{e}^{x^2} - e^{y^2}) + 2\mathrm{e}^{y^2} x^2 - 2\mathrm{e}^{x^2} y^2]^2 \geqslant 0
\end{aligned} \tag{3.36}$$

In simulations, people[27, 41] use $G_f(x, y) = \mathbf{\Theta}(|x-y|<1)$ (with $\mathbf{\Theta}$ being the step function), which violates condition (ii), but appearantly leads to good results. (Particle conservation seems to be no problem, when working at fixed particle number. For another argument see ref. 27.) As explained in Section 2.1, we make the ansatz that $G_f(x, y)$ depends both on x and y and not only on the difference $x-y$. Similar in spirit to refs. 27 and 41 would thus be $G_f(x, y) = \mathbf{\Theta}(|x|<1)\,\mathbf{\Theta}(|y|<1)$. Equation (3.33) then acquires the form

$$\begin{aligned}&\langle [\Theta(x,t)-\Theta(y,t)]^2\rangle\\&\quad=\frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}\int_t^\infty \mathrm{d}t'\langle\langle[\mathbf{\Theta}(|r_\alpha^x(t')|<L)-\mathbf{\Theta}(|r_\alpha^y(t')|<L)]\\&\qquad\times[\mathbf{\Theta}(|r_\beta^x(t')|<L)-\mathbf{\Theta}(|r_\beta^y(t')|<L)]\rangle_\zeta\rangle_v\end{aligned}\tag{3.37}$$

We see that this is the v-average of a quantity which can be interpreted as a time. Let us in general define

$$\begin{aligned}t_{xy} := \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}\int_t^\infty \mathrm{d}t'\langle\, & G_f^M(r_\alpha^x(t'), r_\beta^x(t'))\\&+G_f^M(r_\alpha^y(t'), r_\beta^y(t'))-2G_f^M(r_\alpha^x(t'), r^y{}_\beta(t'))\rangle_\zeta\end{aligned}\tag{3.38}$$

Then $\langle[\Theta(x,t)-\Theta(y,t)]^2\rangle$ is the v-average of t_{xy}:

$$\langle [\Theta(x,t)-\Theta(y,t)]^2\rangle = \langle t_{xy}\rangle_v\tag{3.39}$$

Analogously, for the fourth moment we can write using that f is Gaussian and $N\to\infty$

$$\begin{aligned}&\langle [\Theta(x,t)-\Theta(y,t)]^4\rangle\\&\quad=\frac{3}{N^4}\sum_{\alpha,\beta,\gamma,\delta=1}^{N}\int_t^\infty \mathrm{d}t'\int_t^\infty \mathrm{d}t''\\&\qquad\times\langle\langle[G_f^M(r_\alpha^x(t'), r_\beta^x(t'))+G_f^M(r_\alpha^y(t'), r_\beta^y(t'))-2G_f^M(r_\alpha^x(t'), r_\beta^y(t'))]\\&\qquad\times[G_f^M(r_\gamma^x(t''), r_\delta^x(t''))+G_f^M(r_\gamma^y(t''), r_\delta^y(t''))-2G_f^M(r_\gamma^x(t''), r_\delta^y(t''))]\rangle_\zeta\rangle_v\\&\quad=3\langle(t_{xy})^2\rangle_v\end{aligned}\tag{3.40}$$

In general, the $2n$th moment is

$$\langle [\Theta(x,t)-\Theta(y,t)]^{2n}\rangle = \frac{(2n)!}{2^n n!}\langle (t_{xy})^n\rangle_v\tag{3.41}$$

One can now easily check the deviation from Gaussian behaviour by analyzing connected expectation values. Equation (3.41) also tells us that the anomalous behaviour of S^{2n} comes from rare large events, i.e., rare large t_{xy}. Moreover, the measured ensemble of t_{xy} not only contains the information about the moments, but even the complete probability distribution function.

Let us give some comments on simulations:[41, 27] In the above formulation, we have always averaged over N replicas for the particles and their corresponding thermal noises, finally taking $N \to \infty$. For the $2n$th moment, it is indeed sufficient to keep $2n$ different particles. Moreover, the thermal noise can be dropped for particles which are not starting at the same point. Finally, by going to the relative coordinate system, and using G_f of the form $G_f(x-y)$, only the $2n-1$ relative coordinates have to be propagated.

Also note that with a little bit of work, using Eq. (3.29) and the anomalous dimension of $(\nabla\Theta)^{2n}$, one can again obtain the anomalous scaling behaviour as given in Eq. (2.60).

4. GENERALIZATION TO POLYMERS AND MEMBRANES

4.1. Construction of the Generalized Model

We are now in a position to generalize the above considerations to polymers and polymerized (tethered) membranes.

To this aim, we introduce a polymer or polymerized tethered membrane[42, 43] with coordinates

$$\mathbf{x} \in \mathbb{R}^D \to r(\mathbf{x}) \in \mathbb{R}^d \tag{4.1}$$

where we think of D between 0 and 2, and in particular of $D=0$ for a particle, $D=1$ for a polymer and $D=2$ for a membrane. For polymers, $\mathbf{x}$ measures the length along the polymer; for membranes, $\mathbf{x}$ belongs to a 2-dimensional coordinate system. $r(\mathbf{x})$ is the position of the monomer $\mathbf{x}$ in imbedding space. The standard model for polymers is due to Edwards[44] and reads generalized to membranes[42, 43]

$$\mathscr{H}[r] = \int \mathrm{d}^D\mathbf{x}\, \frac{1}{2}\, (\nabla r(\mathbf{x}))^2 + b \int \mathrm{d}^D\mathbf{x} \int \mathrm{d}^D\mathbf{y}\; \delta^d(r(\mathbf{x}) - r(\mathbf{y})) \tag{4.2}$$

The second term punishes self-intersections of the membrane, making the membrane self-avoiding. In what follows, we shall study phantom membranes, i.e., drop the term proportional to b. We shall discuss the general case in Section 4.5.

As for particles, we introduce N copies, labeled by lower Greek indices. For the αth polymer, the equation of motion for the monomer $\mathbf{x}$ at time t and with coordinates $r_\alpha(\mathbf{x}, t)$ then reads

$$\partial_t r_\alpha^i(\mathbf{x}, t) = \Delta_{\mathbf{x}} r_\alpha^i(\mathbf{x}, t) + v^i(r_\alpha(\mathbf{x}, t), t) + \zeta_\alpha^i(\mathbf{x}, t) \tag{4.3}$$

Note that the elasticity of the polymer or membrane has been scaled to 1. The vv-correlations are the same as in Eq. (2.2), and we already note that $\varepsilon = 0$ is *not* the marginal case. The thermal noise is Gaussian with zero mean and correlations

$$\langle \zeta_\alpha^i(\mathbf{x}, t)\, \zeta_\beta^j(\mathbf{y}, t') \rangle = 2\nu_0 \delta^{ij} \delta_{\alpha\beta} \delta(t - t')\, \delta^D(\mathbf{x} - \mathbf{y}) \tag{4.4}$$

The dynamic action which enforces the Langevin-equation to be satisfied reads in analogy to Eq. (3.3)

$$J[r, \tilde{r}, v, \zeta] = \sum_{\alpha=1}^{N} \int \mathrm{d}t\, \mathrm{d}^D\mathbf{x}\, \tilde{r}_\alpha^i(\mathbf{x}, t) \times [\partial_t r_\alpha^i(\mathbf{x}, t) - \Delta_{\mathbf{x}} r_\alpha^i(\mathbf{x}, t) - v^i(r_\alpha(\mathbf{x}, t), t) - \zeta_\alpha^i(\mathbf{x}, t)] \tag{4.5}$$

Averaging $\mathrm{e}^{-J[r, \tilde{r}, v, \zeta]}$ over ζ and v leads to

$$\begin{aligned} J[r, \tilde{r}] = &\int \mathrm{d}t\, \mathrm{d}^D\mathbf{x} \sum_{\alpha=1}^{N} [\tilde{r}_\alpha^i(\mathbf{x}, t)\, \partial_t r_\alpha^i(\mathbf{x}, t) - \tilde{r}_\alpha^i(\mathbf{x}, t)\, \Delta_{\mathbf{x}} r_\alpha^i(\mathbf{x}, t) - \nu_0 \tilde{r}_\alpha^i(\mathbf{x}, t)^2] \\ &- \frac{\mathrm{D}_0}{2} \int \mathrm{d}t\, \mathrm{d}^D\mathbf{x}\, \mathrm{d}^D\mathbf{y} \sum_{\alpha, \beta=1}^{N} \tilde{r}_\alpha^i(\mathbf{x}, t) \\ &\times \left[\int \frac{\mathrm{d}^d k}{(2\pi)^d} P^{ij}(k) \frac{\mathrm{e}^{ik[r_\alpha(\mathbf{x}, t) - r_\beta(\mathbf{y}, t)]}}{(k^2 + m^2)^{(d+\varepsilon)/2}} \right] \tilde{r}_\beta^j(\mathbf{y}, t) \end{aligned} \tag{4.6}$$

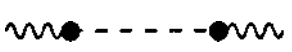
Symbolically, this is written as

$J[r, \tilde{r}] = \int \mathrm{d}t\, \mathrm{d}^D\mathbf{x} \left(\sum_{\alpha=1}^{N} [\right.$ $_\alpha +$  $_\alpha - \nu_0$  $_\alpha \left.] \right)$

$- \frac{\mathrm{D}_0}{2} \int \mathrm{d}t\, \mathrm{d}^D\mathbf{x}\, \mathrm{d}^D\mathbf{y} \sum_{\alpha, \beta=1}^{N}$ $_\alpha$ $_\beta$ (4.7) where we used the abbreviations

 $_{\alpha}=\tilde{r}_\alpha(\mathbf{x},t)\,\partial_t r_\alpha(\mathbf{x},t),$  $_{\alpha}=\tilde{r}_\alpha(\mathbf{x},t)(-\Delta_{\mathbf{x}})\,r_\alpha(\mathbf{x},t)$

 $_{\alpha}=\tilde{r}_\alpha(\mathbf{x},t)^2$

$_{\alpha}$  $_{\beta}=\tilde{r}^i_\alpha(\mathbf{x},t)\left[\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,P^{ij}(k)\,\frac{\mathrm{e}^{ik[r_\alpha(\mathbf{x},t)-r_\beta(\mathbf{y},t)]}}{(k^2+m^2)^{(d+\varepsilon)/2}}\right]\tilde{r}^j_\beta(\mathbf{y},t)$ (4.8)

The free single-membrane response- and correlation-functions are

$$R(\mathbf{k},\omega)=\frac{1}{i\omega+\mathbf{k}^2},\qquad R(\mathbf{k},t)=\Theta(t)\,\mathrm{e}^{-t\mathbf{k}^2}$$

$$R(\mathbf{x},t)=\Theta(t)(4\pi t)^{-D/2}\,\mathrm{e}^{-\mathbf{x}^2/4t},\qquad C(\mathbf{k},\omega)=\frac{2\nu_0}{\omega^2+(\mathbf{k}^2)^2}$$

$$C(\mathbf{k},t):=\frac{\nu_0}{\mathbf{k}^2}\,\mathrm{e}^{-|t|\,\mathbf{k}^2} \tag{4.9}$$

At equal times, the free correlation-function has the simple form

$$C(\mathbf{x}-\mathbf{y},0):=\frac{1}{2d}\,\langle(r(\mathbf{x},t)-r(\mathbf{y},t))^2\rangle_0=\frac{\nu_0}{(2-D)\,S_D}\,|\mathbf{x}-\mathbf{y}|^{2-D} \tag{4.10}$$

All other free correlations vanish.

4.2. Renormalization of the Dynamic Action

As in Section 3.2, we now have to analyze possible renormalizations of the dynamic action. As in Eq. (3.7), we start from

$\frac{\mathrm{D}_0}{2}\sum_{\alpha,\beta=1}^{N}$ $_{\alpha}$  $_{\beta}$ (4.11)

which is more explicitly

$_{\alpha}$  $_{\beta}=\int\mathrm{d}^D\mathbf{y}\,\tilde{r}^i_\alpha(\mathbf{x},t)\left[\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,P^{ij}(k)\,\frac{\mathrm{e}^{ik[r_\alpha(\mathbf{x},t)-r_\alpha(\mathbf{y},t)]}}{(k^2+m^2)^{(d+\varepsilon)/2}}\right]\tilde{r}^j_\beta(\mathbf{y},t)$ (4.12)

We will see below that contracting a response-field will give no contribution. We are thus left to normal-order the r.h.s. of Eq. (4.12) by virtue of a generalization of Eq. (3.9)

$$\mathrm{e}^{ik[r_\alpha(\mathbf{x},t)-r_\beta(\mathbf{y},t')]}=:\mathrm{e}^{ik[r_\alpha(\mathbf{x},t)-r_\beta(\mathbf{y},t')]}:\,\mathrm{e}^{-k^2(1/2d)\langle[r_\alpha(\mathbf{x},t)-r_\beta(\mathbf{y},t')]^2\rangle_0} \tag{4.13}$$

Only the term with $\alpha = \beta$ gives a non-zero contribution:

$$\int \mathrm{d}^D\mathbf{y}\, \tilde{r}_\alpha^i(\mathbf{x}, t) \left[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, P^{ij}(k) \times \frac{:\mathrm{e}^{ik[r_\alpha(\mathbf{x}, t) - r_\alpha(\mathbf{y}, t)]}:\, \mathrm{e}^{-\nu_0/((2-D)\, S_D)\, k^2\, |\mathbf{x}-\mathbf{y}|^{2-D}}}{(k^2+m^2)^{(d+\varepsilon)/2}} \right] \tilde{r}_\alpha^j(\mathbf{y}, t) \tag{4.14}$$

The leading UV-divergence is obtained by expanding $:\mathrm{e}^{ik[r_\alpha(\mathbf{x}, t) - r_\alpha(\mathbf{y}, t)]}:$ and keeping the term of order 1. (Note that such an expansion is justified since the normal-ordered product itself does not contain any divergence.) Using the rotational invariance of the integration over k yields

$$\tilde{r}_\alpha(\mathbf{x}, t)^2 \left(1 - \frac{1}{d}\right) \int \mathrm{d}^D\mathbf{y} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{\mathrm{e}^{-\nu_0/((2-D)\, S_D)\, k^2)|\mathbf{x}-\mathbf{y}|^{2-D}}}{(k^2+m^2)^{(d+\varepsilon)/2}} \tag{4.15}$$

Performing the integrals over $\mathbf{y}$ and k finally gives the contraction of  and projection onto :

$$\left\langle \quad \right. $$  $$\Big|$$  $$\left. \quad \right\rangle_m = \left(1 - \frac{1}{d}\right) C_d^D\, \frac{m^{-\delta}}{\delta}\, \nu_0^{-D/(2-D)} \tag{4.16}$$

$$\delta = \varepsilon + \frac{2D}{2-D} \tag{4.17}$$

$$C_d^D = \frac{2S_D}{2-D} \left[(2-D)\, S_D\right]^{D/(2-D)} \times \frac{\Gamma(D/(2-D))\, \Gamma(d/2 - D/(2-D))\, \Gamma(1+\delta/2)}{(4\pi)^{d/2}\, \Gamma((d+\varepsilon)/2)\, \Gamma(d/2)}$$

$$\xrightarrow{\delta \to 0} \frac{2S_D}{2-D} \left[(2-D)\, S_D\right]^{D/(2-D)} \frac{\Gamma(D/(2-D))}{(4\pi)^{d/2}\, \Gamma(d/2)} \tag{4.18}$$

This is the generalization of Eq. (3.11) to membranes. We have put the index m to the diagram to remind the reader, that m acts as regulator. Note that now the dimensional regularization parameter is δ instead of ε.

Other renormalizations for the dynamic action do not appear: First, contracting in Eq. (4.12) one of the response-fields, say $\tilde{r}_\alpha^i(\mathbf{x}, t)$ leads to a factor of k^i, which together with $P^{ij}(k)$ gives 0. This argument is generalized to all orders in perturbation theory upon remarking that the same factor always appears in the MOPE for the interaction which is most advanced in time. One can also show that the emerging diagrams themselves also vanish, due to a response-function at equal times. Thus, only the term proportional to $\tilde{r}_\alpha(x, t)^2$ is renormalized.

Let us now turn back to the renormalization-group functions. Bare and renormalized couplings are

$$\nu_0 = Z_\nu \nu, \qquad u_0 = \frac{\mathrm{D}_0}{\nu_0^{2/(2-D)}} = u \frac{Z_\mathrm{D}}{Z_\nu^{2/(2-D)}} m^\delta \tag{4.19}$$

$$Z_\nu = 1 - \frac{u}{\delta}\left(1 - \frac{1}{d}\right)\frac{C_d^D}{2}, \qquad Z_\mathrm{D} = 1 \tag{4.20}$$

In analogy to Eq. (2.24), the β-function reads

$$\beta(u) := m \frac{\partial}{\partial m}\bigg|_0 u = -\delta u + \frac{C_d^D}{2-D}\left(1 - \frac{1}{d}\right) u^2 + O(u^3) \tag{4.21}$$

It has a non-trivial IR-attractive fixed point at

$$u^* = \frac{(2-D)\, d}{C_d^D (d-1)}\, \delta \tag{4.22}$$

We can as in Eq. (2.26) define the anomalous dimension γ_ν of ν as

$$\begin{aligned} \gamma_\nu(u) &:= m \frac{\partial}{\partial m} \ln Z_\nu \\ &= u \left(1 - \frac{1}{d}\right)\frac{C_d^D}{2} \end{aligned} \tag{4.23}$$

which at $u = u^*$ reads

$$\gamma_\nu^* = \frac{2-D}{2}\,\delta \tag{4.24}$$

A new exponent is associated to the equal-time inner-membrane correlation function

$$\begin{aligned} C_\mathrm{m}(\mathbf{x}-\mathbf{y}) &:= \frac{1}{2d} \langle (r(\mathbf{x}, t) - r(\mathbf{y}, t))^2 \rangle \\ &\sim |\mathbf{x}-\mathbf{y}|^{2\kappa^*} \end{aligned} \tag{4.25}$$

From Eq. (4.24) we obtain the result to order δ

$$\kappa^* = \frac{2-D}{2}\left(1 + \frac{2-D}{4}\,\delta + O(\delta^2)\right) \tag{4.26}$$

Using that neither the term $\int \tilde{r}\dot{r}$, nor $\int \tilde{r}(-\Delta)\, r$, nor the interaction proportional to $\mathrm{D}_0 = \mathrm{D}$ is renormalized, the scaling dimension of r can as in Eq. (3.29) be obtained exactly:

$$\langle (r_\alpha(\mathbf{x}, t) - r_\alpha(\mathbf{x}, t'))^2 \rangle \sim |t - t'|^{2/(2-\varepsilon)} \tag{4.27}$$

Since $\mathbf{x}^2 \sim t$

$$\langle (r_\alpha(\mathbf{x}, t) - r_\alpha(\mathbf{y}, t))^2 \rangle \sim |\mathbf{x} - \mathbf{y}|^{2\kappa^*}, \qquad \kappa^* = \frac{2}{2-\varepsilon} = \frac{2}{4/(2-D) - \delta} \tag{4.28}$$

In the limit of $\delta \to 0$, this reproduces Eq. (4.26). It is important to note that for $\varepsilon > 0$, the exponent κ^* is larger than 1, thus the membrane overstretched, and the description of a membrane via Eq. (4.2) with only a harmonic elastic term $\mathscr{H}[r] = \int \mathrm{d}^D\mathbf{x}\, \frac{1}{2}[\nabla r(\mathbf{x})]^2$ breaks down. This coincides with the range of ε, for which already a single particle, hence the center of mass of the membrane, exhibits anomalous diffusion. This range will not be described by our model. In experiments one has indeed observed destruction of polymers by a turbulent flow.[45, 46]

4.3. Higher Moments: The Scaling of S^{2n}

Let us now address the question of higher correlation functions. To this aim, we first have to generalize the particle-density $\Theta_{\mathrm{L}}(x, t)$ defined in Eq. (3.12) to polymers and membranes. Be $\mathscr{V} := \int \mathrm{d}^D\mathbf{x}$ the volume of a membrane, then define

$$\Theta_{\mathrm{m}}(x, t) := \frac{1}{N\mathscr{V}} \sum_{\alpha=1}^{N} \int \mathrm{d}^D\mathbf{x}\, \delta^d(r_\alpha(\mathbf{x}, t) - x) \tag{4.29}$$

Note that $\Theta_{\mathrm{m}}(x, t)$ has a well-defined limit both for $N \to \infty$ and $\mathscr{V} \to \infty$. Even though we average over all monomers, it will later turn out that $\Theta_{\mathrm{m}}(x, t)$ can be interpreted in terms of a single monomer, in analogy of the discussion in Section 3.5.

Let us again consider $S^{2n}(x, y) := \langle [\Theta_{\mathrm{m}}(x, t) - \Theta_{\mathrm{m}}(y, t)]^{2n} \rangle$. As in Eq. (2.45), we have to study the contraction of $T^{ij}\, \nabla^i \Theta_{\mathrm{m}}(y, t)\, \nabla^j \Theta_{\mathrm{m}}(y, t)$ with

$$J_{\mathrm{int}} := \frac{\mathrm{D}_0}{2} \int \mathrm{d}t\, \mathrm{d}^D\mathbf{x}\, \mathrm{d}^D\mathbf{x}' \sum_{\alpha, \beta=1}^{N} \tilde{r}^a_\alpha(\mathbf{x}, t) \times \left[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, P^{ab}(k)\, \frac{\mathrm{e}^{ik[r_\alpha(\mathbf{x}, t) - r_\beta(\mathbf{x}', t)]}}{(k^2 + m^2)^{(d+\varepsilon)/2}} \right] \tilde{r}^b_\beta(\mathbf{x}', t)$$

This reads

$$
\begin{aligned}
&T^{ij}\,\nabla^i\Theta_{\mathrm{m}}(y,t)\,\nabla^j\Theta_{\mathrm{m}}(y,t)\;\blacklozenge\;J_{\mathrm{int}}\\
&\quad= -\left(\frac{1}{N\mathscr{V}}\right)^2 \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int \mathrm{d}^D\mathbf{y}\int \mathrm{d}^D\mathbf{y}'\int \frac{\mathrm{d}^d(p-k)}{(2\pi)^d}\\
&\qquad\times\int \frac{\mathrm{d}^d(l+k)}{(2\pi)^d}\,\mathrm{e}^{i(p-k)[r_\alpha(\mathbf{y},t)-y]}\mathrm{e}^{i(l+k)[r_\beta(\mathbf{y}',t)-y]}\\
&\qquad\times T^{ij}(p-k)^i\,(l+k)^j\\
&\qquad\blacklozenge\;\frac{\mathrm{D}_0}{2}\int \mathrm{d}\tau\,\mathrm{d}^D\mathbf{x}\,\mathrm{d}^D\mathbf{x}'\sum_{\gamma,\delta=1}^{N}\tilde{r}^a_\gamma(\mathbf{x},t-\tau)\\
&\qquad\times\left[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,P^{ab}(k)\,\frac{\mathrm{e}^{ik[r_\gamma(\mathbf{x},t-\tau)-r_\delta(\mathbf{x}',t-\tau)]}}{(k^2+m^2)^{(d+\varepsilon)/2}}\right]\tilde{r}^b_\delta(\mathbf{x}',t-\tau)
\end{aligned}
\tag{4.30}
$$

where we have already shifted the l- and p-integration for later convenience. Following the partial normal ordering procedure of Section 3.4 and noting $\mathbf{z}=\mathbf{x}-\mathbf{y}$, $\mathbf{z}'=\mathbf{x}'-\mathbf{y}'$ leads to

$$
\begin{aligned}
&T^{ij}\,\nabla^i\Theta_{\mathrm{m}}(y,t)\,\nabla^j\Theta_{\mathrm{m}}(y,t)\;\blacklozenge\;J_{\mathrm{int}}\\
&\quad= \mathrm{D}_0\left(\frac{1}{N\mathscr{V}}\right)^2 \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int \mathrm{d}^D\mathbf{y}\int \mathrm{d}^D\mathbf{y}'\int \mathrm{d}^D\mathbf{z}\\
&\qquad\times\int \mathrm{d}^D\mathbf{z}'\int \mathrm{d}\tau\int \frac{\mathrm{d}^d p}{(2\pi)^d}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\\
&\qquad\times T^{ij}(p-k)^i\,(k+l)^j\,\frac{(p-k)^a\,P^{ab}(k)(k+l)^b}{(k^2+m^2)^{(d+\varepsilon)/2}}\\
&\qquad\times :\mathrm{e}^{i(p-k)[r_\alpha(\mathbf{y},t)-r_\alpha(\mathbf{x},t-\tau)]}:\\
&\qquad\times \mathrm{e}^{ip[r_\alpha(\mathbf{x},t-\tau)-x]}\mathrm{e}^{i(x-y)(p-k)-(p-k)^2C(\mathbf{z},\tau)}R(\mathbf{z},\tau)\\
&\qquad\times :\mathrm{e}^{i(k+l)[r_\beta(\mathbf{y}',t)-r_\beta(\mathbf{x}',t-\tau)]}:\\
&\qquad\times \mathrm{e}^{il[r_\beta(\mathbf{y},t-\tau)-x]}\mathrm{e}^{i(x-y)(k+l)-(k+l)^2C(\mathbf{z}',\tau)}R(\mathbf{z}',\tau)
\end{aligned}
\tag{4.31}
$$

This expression allows for one exact simplification, namely

$$
(p-k)^a\,P^{ab}(k)(k+l)^b = p^aP^{ab}(k)\,l^b \tag{4.32}
$$

Similar to what has been done after Eq. (3.24), we use the approximations

$$
\begin{aligned}
&:\mathrm{e}^{i(p-k)[r_\alpha(\mathbf{y},t)-r_\alpha(\mathbf{x},t-\tau)]}:\approx 1\\
&:\mathrm{e}^{i(k+l)[r_\beta(\mathbf{y}',t)-r_\beta(\mathbf{x}',t-\tau)]}:\approx 1
\end{aligned}
\tag{4.33}
$$

which were exact for the particle (see the discussion in Section 3.4).

Next, since we are searching for the leading pole in $\delta=\varepsilon+2D/(2-D)$, which comes from the region, where k becomes large and where τ, $\mathbf{z}$ and $\mathbf{z}'$ become small simultaneously, we can replace $k+l$ by k and $p-k$ by $-k$. This leads to

$$
\begin{aligned}
&T^{ij}\,\nabla^i\Theta_{\mathrm{m}}(y,t)\,\nabla^j\Theta_{\mathrm{m}}(y,t)\ \blacklozenge\ J_{\mathrm{int}}\\
&\quad\approx -\mathrm{D}_0\left(\frac{1}{N\mathscr{V}}\right)^2\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int \mathrm{d}^D\mathbf{y}\,\mathrm{d}^D\mathbf{y}'\,\mathrm{d}^D\mathbf{z}\,\mathrm{d}^D\mathbf{z}'\,\mathrm{d}\tau\,\frac{\mathrm{d}^dp}{(2\pi)^d}\\
&\quad\quad\times\frac{\mathrm{d}^dl}{(2\pi)^d}\,\frac{\mathrm{d}^dk}{(2\pi)^d}\,T^{ij}k^ik^j\,\frac{p^aP^{ab}(k)\,l^b}{(k^2+m^2)^{(d+\varepsilon)/2}}\\
&\quad\quad\times \mathrm{e}^{ip[r_\alpha(\mathbf{x},t-\tau)-x]}\mathrm{e}^{il[r_\beta(\mathbf{y},t-\tau)-x]}\mathrm{e}^{-k^2[C(\mathbf{z},\tau)+C(\mathbf{z}',\tau)]}R(\mathbf{z},\tau)\,R(\mathbf{z}',\tau)\\
&\quad=\mathrm{D}_0\int\mathrm{d}^D\mathbf{z}\int\mathrm{d}^D\mathbf{z}'\int\mathrm{d}\tau\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,T^{ij}k^ik^j\\
&\quad\quad\times\frac{P^{ab}(k)}{(k^2+m^2)^{(d+\varepsilon)/2}}\,\mathrm{e}^{-k^2[C(\mathbf{z},\tau)+C(\mathbf{z}',\tau)]}R(\mathbf{z},\tau)\,R(\mathbf{z}',\tau)\\
&\quad\quad\times\nabla^a\Theta_{\mathrm{m}}(x,t-\tau)\,\nabla^b\Theta_{\mathrm{m}}(x,t-\tau)
\end{aligned}
\tag{4.34}
$$

One now has to evaluate the integral over k, $\mathbf{z}$, $\mathbf{z}'$ and τ, with m as an IR-regulator. The calculation of the leading pole in δ is substantially simplified by moving the regulator from the k-integration to the τ-integration. For the leading pole, we have (setting $v_0=1$ for calculational convenience)

$$
\begin{aligned}
&\mathrm{D}_0\int\mathrm{d}^D\mathbf{z}\int\mathrm{d}^D\mathbf{z}'\int\mathrm{d}\tau\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,T^{ij}k^ik^j\\
&\quad\times\frac{P^{ab}(k)}{(k^2+m^2)^{(d+\varepsilon)/2}}\,\mathrm{e}^{-k^2[C(\mathbf{z},\tau)+C(\mathbf{z}',\tau)]}R(\mathbf{z},\tau)\,R(\mathbf{z}',\tau)\,\Theta_m^a\Theta_m^b\\
&\approx\mathrm{D}_0\int\mathrm{d}^D\mathbf{z}\int\mathrm{d}^D\mathbf{z}'\int_0^{m^{-4/(2-D)}}\mathrm{d}\tau\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,T^{ij}k^ik^j\\
&\quad\times\frac{P^{ab}(k)}{|k|^{d+\varepsilon}}\,\mathrm{e}^{-k^2[C(\mathbf{z},\tau)+C(\mathbf{z}',\tau)]}R(\mathbf{z},\tau)\,R(\mathbf{z}',\tau)\,\Theta_m^a\Theta_m^b
\end{aligned}
\tag{4.35}
$$

where we have abbreviated $\Theta_m^a := \nabla^a \Theta_{\mathrm{m}}(x, t-\tau)$. Since the integral scales like $m^{-\delta}$, it can equivalently be written as

$$-\frac{\mathrm{D}_0}{\delta}\frac{m\partial}{\partial m}\int \mathrm{d}^D\mathbf{z}\int \mathrm{d}^D\mathbf{z}'\int_0^{m^{-4/(2-D)}} \mathrm{d}\tau \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, T^{ij}k^i k^j$$

$$\times \frac{P^{ab}(k)}{|k|^{d+\varepsilon}}\, e^{-k^2[C(\mathbf{z},\tau)+C(\mathbf{z}',\tau)]} R(\mathbf{z},\tau)\, R(\mathbf{z}',\tau)\, \Theta_m^a \Theta_m^b$$

$$= \frac{4\mathrm{D}_0}{2-D}\frac{m^{-\delta}}{\delta}\int \mathrm{d}^D\mathbf{z}\int \mathrm{d}^D\mathbf{z}'\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, T^{ij}k^i k^j$$

$$\times \frac{P^{ab}(k)}{|k|^{d+\varepsilon}}\, e^{-k^2[C(\mathbf{z},1)+C(\mathbf{z}',1)]} R(\mathbf{z},1)\, R(\mathbf{z}',1)\, \Theta_m^a \Theta_m^b \tag{4.36}$$

In order to proceed, we specify T^{ij} as

$$T^{ij} = A^2\delta^{ij} + B^i C^j \tag{4.37}$$

The integral over k can still be performed, leading to

$$\mathrm{D}_0 \frac{4}{2-D}\frac{m^{-\delta}}{\delta}\frac{\Gamma(1-\varepsilon/2)}{\Gamma(d/2)(4\pi)^{d/2}}\int \mathrm{d}^D\mathbf{z}$$

$$\times \int \mathrm{d}^D\mathbf{z}'[C(\mathbf{z},1)+C(\mathbf{z}',1)]^{\varepsilon/2-1} R(\mathbf{z},1)\, R(\mathbf{z}',1)$$

$$\times \left[A^2(\nabla\Theta_{\mathrm{m}})^2\left(1-\frac{1}{d}\right) + \frac{d+1}{d(d+2)}(BC)(\nabla\Theta_{\mathrm{m}})^2 \right.$$

$$\left. -\frac{2}{d(d+2)}(\nabla\Theta_{\mathrm{m}}B)(\nabla\Theta_{\mathrm{m}}C) \right] \tag{4.38}$$

We now introduce the abbreviation

$$I^D := \frac{1}{C_d^D}\frac{8}{2-D}\frac{\Gamma(1-\varepsilon/2)}{\Gamma(d/2)(4\pi)^{d/2}}$$

$$\times \int \mathrm{d}^D\mathbf{z}\int \mathrm{d}^D\mathbf{z}'[C(\mathbf{z},1)+C(\mathbf{z}',1)]^{\varepsilon/2-1} R(\mathbf{z},1)\, R(\mathbf{z}',1) \tag{4.39}$$

which is understood to be evaluated at $v_0 = 1$. In the limit of $\delta \to 0$ it reads

$$I^D = \frac{4\Gamma(2/(2-D))}{S_D[(2-D)\,S_D]^{D/(2-D)}\,\Gamma(D/(2-D))} \int \mathrm{d}^D\mathbf{z} \times \int \mathrm{d}^D\mathbf{z}'[C(\mathbf{z},1) + C(\mathbf{z}',1)]^{-2/(2-D)}\,R(\mathbf{z},1)\,R(\mathbf{z}',1)$$

$$= \frac{4DS_D}{(2-D)[(2-D)\,S_D]^{D/(2-D)}} \int \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}\,\mathbf{z}^D \times \int \frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'}\,(\mathbf{z}')^D\,[C(\mathbf{z},1) + C(\mathbf{z}',1)]^{-2/(2-D)}\,R(\mathbf{z},1)\,R(\mathbf{z}',1) \quad (4.40)$$

Note that I^D only depends on D, but not on d. For $D \to 0$, the integrals over $\mathbf{z}$ and $\mathbf{z}'$ get localized at $\mathbf{z} = \mathbf{z}' = 0$, and since in that limit $C(\mathbf{z}, 1) \to 1$ and $R(\mathbf{z}, 1) \to 1$, $I^D \to 1$ and we recover our earlier results of Section 2.4. For general D, I^D is smaller than 1. Its explicit value is plotted in Fig. 4.1.

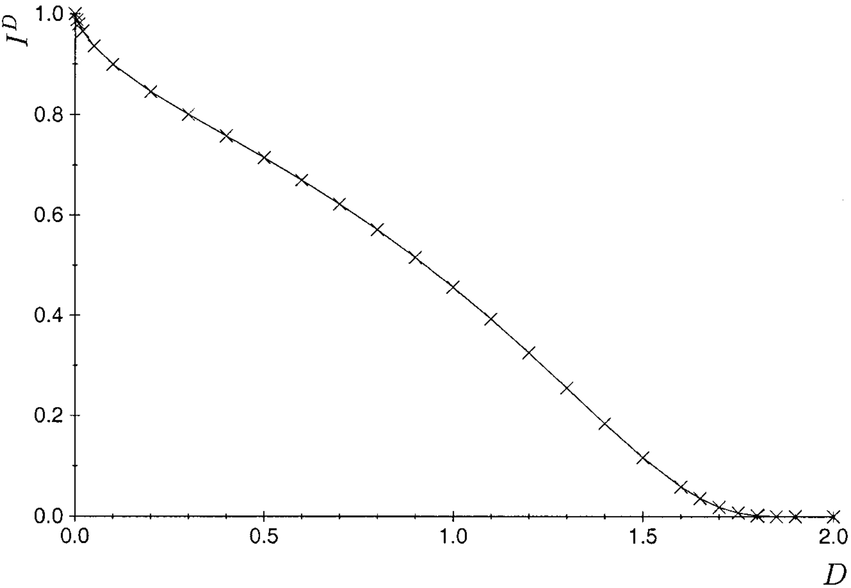


Fig. 4.1. I^D as defined in Eq. (4.40). The crosses have been obtained numerically, the solid line interpolates between theses values.

With this abbreviation, Eq. (4.38) becomes

$$\frac{\mathrm{D}_0}{2}\frac{m^{-\delta}}{\delta}C_d^D I^D\left[A^2(\nabla\Theta_{\mathrm{m}})^2\left(1-\frac{1}{d}\right)\right.$$
$$\left.+\frac{d+1}{d(d+2)}(BC)(\nabla\Theta_{\mathrm{m}})^2-\frac{2}{d(d+2)}(\nabla\Theta_{\mathrm{m}}B)(\nabla\Theta_{\mathrm{m}}C)\right] \tag{4.41}$$

Setting

$$\mathscr{S}_{\mathrm{m}}^{(n,m)}:=z^{2n}[(\nabla\Theta_{\mathrm{m}})^2]^n(z\,\nabla\Theta_{\mathrm{m}})^{2m} \tag{4.42}$$

yields for T^{ij} as in Eq. (2.46)

$$\begin{aligned}T^{ij}&=\frac{1}{2}\frac{\partial}{\partial(\nabla^i\Theta_{\mathrm{m}})}\frac{\partial}{\partial(\nabla^j\Theta_{\mathrm{m}})}\{z^{2n}[(\nabla\Theta_{\mathrm{m}})^2]^n(z\,\nabla\Theta_{\mathrm{m}})^{2m}\}\\&=n\,\delta^{ij}z^{2n}[(\nabla\Theta_{\mathrm{m}})^2]^{n-1}(z\,\nabla\Theta_{\mathrm{m}})^{2m}\\&\quad+2n(n-1)\,z^{2n}(\nabla^i\Theta_{\mathrm{m}})(\nabla^j\Theta_{\mathrm{m}})[(\nabla\Theta_{\mathrm{m}})^2]^{n-2}(z\,\nabla\Theta_{\mathrm{m}})^{2m}\\&\quad+2nmz^{2n}[z^i\,\nabla^j\Theta_{\mathrm{m}}+z^j\,\nabla^i\Theta_{\mathrm{m}}][(\nabla\Theta_{\mathrm{m}})^2]^{n-1}(z\,\nabla\Theta_{\mathrm{m}})^{2m-1}\\&\quad+m(2m-1)\,z^{2n}z^iz^j[(\nabla\Theta_{\mathrm{m}})^2]^n(z\,\nabla\Theta_{\mathrm{m}})^{2m-2}\end{aligned} \tag{4.43}$$

Insertion into Eq. (4.41) gives

$$\frac{\mathrm{D}_0}{2}\frac{m^{-\delta}}{\delta}C_d^D I^D\frac{1}{d(d+2)}\{[n(d-1)(d+2n+4m)-2m(2m-1)]\,\mathscr{S}_{\mathrm{m}}^{(n,m)}$$
$$+m(2m-1)(d+1)\,\mathscr{S}_{\mathrm{m}}^{(n+1,m-1)}\} \tag{4.44}$$

Reestablishing the necessary factors of v_0 gives the final result

$$\begin{aligned}&\mathscr{S}_{\mathrm{m}}^{(n,m)}\;\blacklozenge\;J_{\mathrm{int}}\\&\quad=\mathscr{S}_{\mathrm{m}}^{(n,m)}\frac{u}{2}C_d^D I^D\frac{m^{-\delta}}{\delta}\frac{1}{d(d+2)}[n(d-1)(d+2n+4m)-2m(2m-1)]\\&\quad\quad+\mathscr{S}_{\mathrm{m}}^{(n+1,m-1)}\frac{u}{2}C_d^D I^D\frac{m^{-\delta}}{\delta}\frac{d+1}{d(d+2)}m(2m-1)\end{aligned} \tag{4.45}$$

Equation (4.45) is formally equivalent to Eq. (2.52) upon replacing C_d by $C_d^D I^D$. As in Eq. (2.55) this yields for the eigen-operators $\bar{\mathscr{S}}_0^{(n,m)}$

$$\begin{aligned}&\bar{\mathscr{S}}_0^{(n,m)}=Z^{(n,m)}\bar{\mathscr{S}}^{(n,m)}\\&Z^{(n,m)}=1-\frac{u}{2}\frac{C_d^D I^D}{\delta}\frac{1}{d(d+2)}[n(d-1)(d+2n+4m)-2m(2m-1)]\end{aligned} \tag{4.46}$$

and with the help of Eq. (4.45), we can evaluate the anomalous exponents $\gamma^{(n,m)}$ as defined in Eqs. (2.56) and (2.57):

$$\begin{aligned}\gamma_{\mathrm{m}}^{(n,m)}(u) &:= -m\frac{\partial}{\partial m}\ln Z^{(n,m)} \\ &= -\frac{u}{2}C_d^D I^D \frac{1}{d(d+2)}[n(d-1)(d+2n+4m)-2m(2m-1)]\end{aligned} \tag{4.47}$$

At the IR fixed point $u = u^*$ from Eq. (4.22), this reads

$$\gamma_{\mathrm{m}}^{(n,m)} = -\frac{\delta I^D}{(d-1)(d+2)}[n(d-1)(d+2n+4m)-2m(2m-1)] \tag{4.48}$$

The full dimension $\Delta_{\mathrm{m}}^{(n,m)}$ of the operator $\bar{\mathscr{F}}^{(n,m)}$ then is

$$\begin{aligned}\Delta_{\mathrm{m}}^{(n,m)} &= (n+m)\,\varepsilon + \gamma_{\mathrm{m}}^{(n,m)} \\ &= -(n+m)\frac{2D}{2-D} + \delta\left(n+m-\frac{\delta I^D}{(d-1)(d+2)}\right. \\ &\quad \left.\times[n(d-1)(d+2n+4m)-2m(2m-1)]\right)\end{aligned} \tag{4.49}$$

The term $(n+m)\,\varepsilon$ is equivalent to the corresponding term in Eq. (2.58). It will be derived in the next section. The contribution to S^{2n} is due to the term for $m = 0$

$$\Delta_{\mathrm{m}}^{(n,0)} = n\left[\frac{-2D}{2-D} + \delta\left(1 - I^D\frac{d+2n}{d+2}\right)\right] \tag{4.50}$$

This gives the final result for $S^{(2n)}(r)$ with $r := |x-y|$

$$\begin{aligned}S^{(2n)}(r) &= \langle[\Theta(x,t)-\Theta(y,t)]^{2n}\rangle \\ &\sim r^{n(4/(2-D)-\delta)}\left(\frac{r}{L}\right)^{n[-2D/(2-D)+\delta(1-I^D(d+2n)/(d+2))]}\end{aligned} \tag{4.51}$$

Note that for $D > 0$, $I^D < 1$ and already the second moment ($n = 1$) has an anomalous contribution at order δ

$$\Delta_{\mathrm{m}}^{(1,0)} = -\frac{2D}{2-D} + \delta(1 - I^D) \tag{4.52}$$

4.4. Physical Interpretation

In Section 3.5 we have interpreted equal-time correlation-functions of moments of Θ as expectation values of moments of the time t_{xy} which is constructed from the motion of particles starting at x and y. This discussion in terms of particle-trajectories was quite general and can immediately be carried over to polymers or membranes. In generalisation of Eqs. (3.31) and (4.29) we write

$$\Theta_{\mathrm{m}}(x, t) = \frac{1}{N\mathscr{V}} \sum_{\alpha=1}^{N} \int \mathrm{d}^D\mathbf{x} \int_t^{\infty} \mathrm{d}t' \, f(r_\alpha^x(\mathbf{x}, t'), t') \tag{4.53}$$

Again, it is to be understood that $r_\alpha^x(\mathbf{x}, t')$ satisfies the equation of motion (4.3) and that $r_\alpha^x(\mathbf{x}, t) = x$. Then define t_{xy} as

$$\begin{aligned} t_{xy} := \frac{1}{\mathscr{V}^2} \frac{1}{N^2} \sum_{\alpha, \beta=1}^{N} \int \mathrm{d}^D\mathbf{z} \int \mathrm{d}^D\mathbf{z}' \int_t^{\infty} \mathrm{d}t' \\ \times \langle G_f^M(r_\alpha^x(\mathbf{z}, t'), r_\beta^x(\mathbf{z}', t')) + G_f^M(r_\alpha^y(\mathbf{z}, t'), r_\beta^y(\mathbf{z}', t')) \\ - 2G_f^M(r_\alpha^x(\mathbf{z}, t'), r_\beta^y(\mathbf{z}', t')) \rangle_\zeta \end{aligned} \tag{4.54}$$

Note that the expectation value is independent of $\mathbf{z}$ and $\mathbf{z}'$ such that the average over $\mathbf{z}$ and $\mathbf{z}'$ can be dropped. We thus can alternatively define

$$\begin{aligned} t_{xy} := \frac{1}{N^2} \sum_{\alpha, \beta=1}^{N} \int_t^{\infty} \mathrm{d}t' \langle G_f^M(r_\alpha^x(\mathbf{z}, t'), r_\beta^x(\mathbf{z}', t')) + G_f^M(r_\alpha^y(\mathbf{z}, t'), r_\beta^y(\mathbf{z}', t')) \\ - 2G_f^M(r_\alpha^x(\mathbf{z}, t'), r_\beta^y(\mathbf{z}', t')) \rangle_\zeta \end{aligned} \tag{4.55}$$

This object is the analog of Eq. (3.38), with a single particle α replaced by an arbitrarily chosen monomor $\mathbf{z}$ on the membrane α.

Let us now come to the evaluation of observables. In Eq. (4.27), we have seen that the scaling of time and space is related by $t \sim r^{2-\varepsilon}$, such that

$$\Theta^2 \sim t \sim r^{2-\varepsilon} = r^{4/(2-D)-\delta} \tag{4.56}$$

This result is identical to the particle case, such that without any proper renormalization, $(\nabla\Theta)^{2n}$ would have dimension $-n\varepsilon$, establishing the first term in Eq. (4.49). Equation (4.56) also yields the non L-dependent term in Eq. (4.51).

Finally, in a computer-experiment, one can then measure ($r := |x - y|$)

$$S^{(2n)}(r) = \langle [\Theta(x, t) - \Theta(y, t)]^{2n} \rangle$$
$$\sim r^{n(4/(2-D)-\delta)} \left(\frac{r}{L}\right)^{n[-2D/(2-D)+\delta(1-I^D(d+2n)/(d+2))]} \sim \langle (t_{xy})^n \rangle_v \tag{4.57}$$

Let us state explicitly the result for polymers, which are probably easier to simulate than membranes. Using that the integral I^D in Eq. (4.40) for $D = 1$ gives $I^1 = 0.456143$, we obtain

$$S^{(2n)}_{\text{Polymer}}(r) = \langle [\Theta(x, t) - \Theta(y, t)]^{2n} \rangle$$
$$\sim r^{n(4-\delta)} \left(\frac{r}{L}\right)^{n[-2+\delta(1-0.456143(d+2n)/(d+2))]} \sim \langle (t_{xy})^n \rangle_v \tag{4.58}$$

We also want to give some practical hints:

One should best use a box of size L and let particles or monomers start at position $\pm r/2$, with $r \ll L$, where the box extends to $L/2$ in every direction. Also note that the evaluation of the $2n$-th moment demands to propagate $2n$ membranes, a number which may be reduced to $2n - 1$ when going to the relative coordinate system.

Another point is whether one should best measure the L- or the r-dependence. For particles, the best observable is the L-dependence, since it directly gives the multi-scaling exponent. For polymers, one may instead measure what happens when $r \to \lambda r$ and $L \to \lambda^{-1} L$, since for that resealing

$$S^{(2n)}_{\text{Polymer}}(r)\big|_{r \to \lambda r;\, L \to L/\lambda} \sim \lambda^{n\delta(1-0.912286(d+2n)/(d+2))} \tag{4.59}$$

4.5. Self-Avoidance

Physical membranes are always self-avoiding, i.e., they are described by model (4.2) with $b > 0$. One would therefore like to have a combined treatment of self-avoidance and passive advection. In general, such a treatment is impossible, since the upper critical dimensions are different and not for both couplings exists a small control parameter. In the present problem however, the control parameters, i.e., dimensions of the couplings are

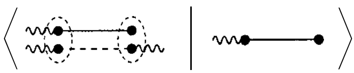
$$\begin{aligned} \delta &:= [u_0]_m = \frac{2D}{2-D} + \varepsilon \\ \gamma &:= [b_0]_\mu = 2D - \frac{2-D}{2}\, d \end{aligned} \tag{4.60}$$

where we have introduced a scale $\mu \sim 1/\mathbf{x}$ which is common for self-avoidance and which is related to m by

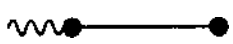
$$m = \mu^{(2-D)/2} \tag{4.61}$$

By choosing the range ε of the turbulent advection and the dimension of imbedding space d, both δ and γ can be set to zero. This is the common expansion point. Note that this expansion point is for $\varepsilon = -d/2$, such that the advecting turbulent field is indeed long-range correlated. Short-range correlated turbulent disorder is principally different, since under renormalization it generates potential disorder, and physics is described by a new universal fixed point. At least this has been observed for static disorder in ref. 39.

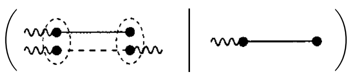
Let us now turn to a diagrammatic analysis. The most complicated diagrams come form the correction to self-avoidance by the turbulent advection and vice versa. First of all, the turbulent advection is long-range correlated and thus not corrected by self-avoidance, which is short-range. On the other hand, the turbulent advection can correct self-avoidance through the diagram


(4.62)

where we have denoted with

 $= 2 \int \mathrm{d}^D\mathbf{x} \int \mathrm{d}^D\mathbf{y} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \tilde{r}(\mathbf{x}, t)(ik)\, \mathrm{e}^{ik[r(\mathbf{x}, t) - r(\mathbf{y}, t)]}$ (4.63)

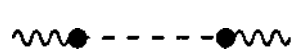
the self-avoidance interaction. Explicitly calculating the diagram shows that due to the transversal projector in the turbulent advection, the MOPE coefficient

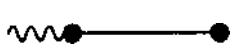

(4.64)

identically vanishes. This substantially simplifies the analysis, since now only the correction to the fields and to the scaling of time intervenes in the RG-analysis. We therefore introduce renormalized fields r, $\tilde{r}$ and a renormalized time. To this aim, we replace throughout this article $r \to r_0$ and $\tilde{r} \to \tilde{r}_0$, as well as $t \to \lambda_0 t$. This procedure is more formal than that employed in the rest of this article, but necessary to obtain the renormalization group β-functions. Define (again denoting with subscript $_0$ bare quantities)

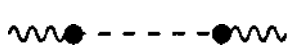
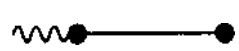
$$r_0 = \sqrt{Z}\, r$$
$$\tilde{r}_0 = \sqrt{\tilde{Z}}\, \tilde{r} \qquad (4.65)$$
$$\lambda_0 = Z_\lambda \lambda$$

This gives the dynamic action (with summation over the replicas and integration variables suppressed, setting also $v_0 \to 1$), for which we first give the bare, and second the renormalized version:

$J[r, \tilde{r}] = \int$  $_0 + \lambda_0$  $_0 - \lambda_0$  $_0 - \frac{u_0 \lambda_0}{2}$  $_0$

$+ b_0 \lambda_0$  $_0$

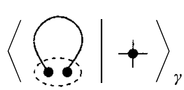
$= \int \sqrt{Z\tilde{Z}}$  $+ \sqrt{Z\tilde{Z}}\, Z_\lambda \lambda$  $- Z_\lambda \tilde{Z} \lambda$ 

$- \frac{u\lambda}{2} Z_u$  $+ b\lambda Z_b$  (4.66)

Since there is no counter-term for ,

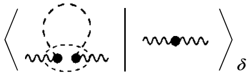
$$\tilde{Z} Z = 1 \qquad (4.67)$$

Second, from the term proportional to  which is corrected by self-avoidance, we obtain

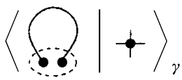
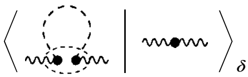
$Z_\lambda = Z_\lambda \sqrt{\tilde{Z} Z} = 1 - \frac{b}{\gamma}(2 - D)$  (4.68)

where we have used the static notation,[36, 40] since the dynamic diagrams involved in the renormalization of self-avoidance can all be reduced to static ones.[37, 40] We recall the notation (see e.g., ref. 40) that $\langle$  $|$  $\rangle_\gamma$ means the residue of the pole in $1/\gamma$ of the diagram $\langle$  $|$  $\rangle_L$. The latter is defined as the integral over the MOPE-coefficient ( $|$ ), cut off at scale L. The diagram and Z-factor is as defined in ref. 36, where a different normalization was used. However the final result is only sensitive to the ratio of diagrams; moreover since as discussed above there is no diagram mixing b and u, the result only depends on the ratio of the two diagrams involved in the renormalization of pure self-avoidance. We shall therefore in the following treat overall normalizations rather sloppily.

The term proportional to ~•~ is only renormalized by the turbulent advection[37, 40] and reads

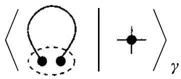
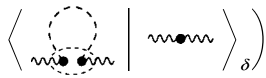
$$Z_\lambda \tilde{Z} = 1 - \frac{1}{2}\frac{u}{\delta}$$  (4.69)

These relations can be solved for Z

$$Z = \frac{1}{\tilde{Z}} = \frac{Z_\lambda}{\tilde{Z} Z_\lambda} = 1 - \frac{b}{\gamma}(2-D)$$  $$+\frac{1}{2}\frac{u}{\delta}$$  (4.70)

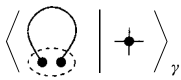
The relation between bare and renormalized coupling for the turbulent advection are in the absence of a proper renormalization of u, i.e., $Z_u = 1$

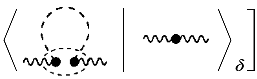
$$u_0 = u m^\delta \tilde{Z}^{-1} Z^{-\varepsilon/2} Z_\lambda^{-1} = u m^\delta Z^{1-\varepsilon/2} Z_\lambda^{-1}$$

$$= u m^\delta \Bigg(1 - \frac{b}{\gamma} D$$  $$+ \frac{u}{\delta}\frac{1}{2-D}$$  $$\Bigg)$$ (4.71)

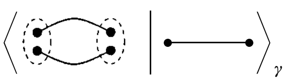
leading to the β_u-function

$$\beta_u(b, u) := m \frac{\partial}{\partial m}\bigg|_0 u$$

$$= u \bigg[-\delta - b \frac{2D}{2-D}$$ 

$$+ u \frac{1}{2-D}$$  $$\bigg]$$ (4.72)

Self-avoidance is renormalized by

$$Z_b = 1 + \frac{b}{\gamma}$$  (4.73)

b_0 and b are thus related by

$$b_0 = b\mu^\gamma Z_b Z_\lambda^{-1} Z^{(d+1)/2} \tilde{Z}^{-1/2}$$

$$= b\mu^\gamma \left(1 + \frac{b}{\gamma} \left\langle \quad \middle| \quad \right\rangle_\gamma - \frac{b}{\gamma} \frac{2-D}{2} d \left\langle \quad \middle| \quad \right\rangle_\gamma + \frac{u}{\delta} \frac{d+2}{4} \left\langle \quad \middle| \quad \right\rangle_\delta \right) \tag{4.74}$$

This gives the β_b-function

$$\beta_b(b, u) := \mu \frac{\partial}{\partial \mu}\bigg|_0 b$$

$$= b \left[-\gamma + b \left\langle \quad \middle| \quad \right\rangle_\gamma - b \frac{2-D}{2} d \left\langle \quad \middle| \quad \right\rangle_\gamma + u \frac{2-D}{2} \frac{d+2}{4} \left\langle \quad \middle| \quad \right\rangle_\delta \right] \tag{4.75}$$

Equations (4.75) and (4.72) determine the critical point, and (4.70) then gives the size-exponent

$$\kappa(b, u) := \frac{2-D}{2} - \frac{1}{2} \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z \tag{4.76}$$

$$= \frac{2-D}{2} \left[1 - b \left\langle \quad \middle| \quad \right\rangle_\gamma + \frac{u}{4} \left\langle \quad \middle| \quad \right\rangle_\delta \right] \tag{4.77}$$

at this critical point.

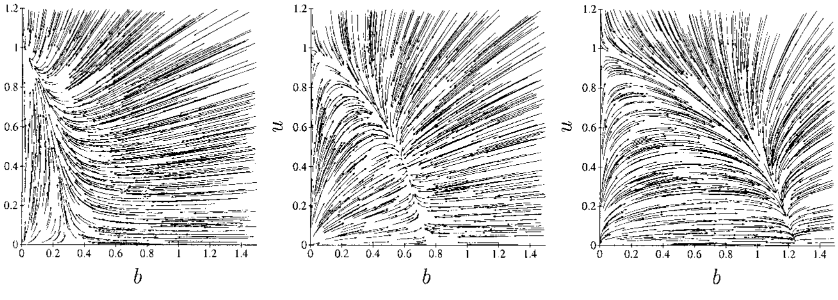


Fig. 4.2. Flow of Eqs. (4.78) for $\delta = 1$, and $\gamma = 0.6$, $\gamma = 1.5$, $\gamma = 2.5$ respectively.

For the remainder of this section, and to allow for a simpler analysis, we will specify to polymers. We also use our freedom in reparametrization of u to set $\langle$  $|$ $\rangle_\delta = 1$. This gives the system of equations

$$\begin{aligned} \beta_u(b, u) &= u[-\delta + b + u] \\ \beta_b(b, u) &= b\left[-\gamma + 2b + \frac{3}{4}u\right] \\ \kappa(b, u) &= \frac{1}{2}\left[1 + \frac{b}{2} + \frac{u}{4}\right] \end{aligned} \tag{4.78}$$

We can distinguish three fixed points:

(i) The pure turbulence fixed point $u^* = \delta$ and $b^* = 0$ is stable for $\gamma < \frac{3}{4}\delta$. The value of κ is $\kappa^* = \frac{1}{2} + \delta/8$.

(ii) The pure self-avoidance fixed point $b^* = \gamma/2$ and $u^* = 0$ is stable for $\gamma > 2\delta$. The value of κ is $\kappa^* = \frac{1}{2} + \gamma/8$.

(iii) The mixed fixed point $b^* = \frac{4}{5}\gamma - \frac{3}{5}\delta$, $u^* = \frac{8}{5}\delta - \frac{4}{5}\gamma$ is stable for $\frac{3}{4}\delta < \gamma < 2\delta$. The value of κ at the fixed point is $\kappa^* = \frac{1}{2} + \delta/20 + \gamma/10$.

Note that (i) and (ii) reproduce the result of the preceding sections and for self-avoiding polymers respectively. This completes the discussion of properties of a single self-avoiding polymer (or membrane) in a turbulent flow.

The next question is how multiple membrane properties, especially the scaling functions S^{2n} are modified. Two routes may be taken: either one considers real physical membranes which are mutually self-avoiding. However, then already the expectation value of $\Theta_{\mathrm{m}}(x, t)^2$ would vanish, since never two monomers can arrive at the same point x at time t, due to self-avoidance. Interesting expectation values are $\Theta_{\mathrm{m}}(x, t)\, \Theta_{\mathrm{m}}(y, t)$. In the case of no turbulent advection, they are known as contact exponents;[47, 35, 40] they are also related to the scaling dimension of operators in scalar field-theory.

The other possible generalization, which is less physical, is to impose self-avoidance only between monomers of the same membrane. Then, the diagrams evaluated in Section 4.3 are complete and one simply has to use the modified expressions for κ^* and for the fixed-point value of u in order to obtain the new multiscaling exponents.

It should also be possible to take into account the back-reaction of the membrane on itself, at least approximately. A treatment à la Zimm[48] would in generalization to ref. 38 lead to a triple ε-expansion, namely in $d-4$, $D-1$ and δ.

We leave the exploration of these ideas for future research.

5. CONCLUSIONS

In this article, we have shown how the multiscaling found in the passive scalar problem carries over to extended elastic objects as polymers and membranes. This was possible by first reformulating the problem in terms of the advection of particles, which then allowed for a generalization to polymers and polymerized membranes. Similar to the passive scalar case, we have calculated the anomalous exponents to first order in a perturbative expansion. We have also discussed, how these quantities can be measured numerically, by studying the drift of two monomers, which sit on different polymers or membranes.

A. APPENDICES

A.1. Some Integrals

For a d-dimensional rotationally invariant integral over p, which may be $\int_p = \int \mathrm{d}^d p\, \mathrm{e}^{-\Sigma_{i=1}^d \lambda_i p_i^2}$, we have with some constant C

$$\int_p = \int \mathrm{d}^d p\, \mathrm{e}^{-\Sigma_{i=1}^d\, \lambda_i p_i^2} = C \frac{1}{\sqrt{\prod_{i=1}^d \lambda_i}} \tag{A.1}$$

In the case that all λ_i equal λ, the latter reads with the same constant C

$$\int \mathrm{d}^d p\, \mathrm{e}^{-\lambda p^2} = C \lambda^{-d/2} \tag{A.2}$$

Using this, we obtain by differentiating with respect to λ_i moments of p_i; e.g.,

$$\left[\int_p p^2\right]^{-1} \int_p p_1^2 = \frac{1}{d} \tag{A.3}$$

$$\left[\int_p (p^2)^2\right]^{-1} \int_p (p_1)^4 = \frac{3}{d(d+2)} \tag{A.4}$$

$$\left[\int_p (p^2)^2\right]^{-1} \int_p p_1^2 p_2^2 = \frac{1}{d(d+2)} \tag{A.5}$$

Another intelligent way of doing this is to study

$$\int_p \mathrm{e}^{-p^2/2}\, \mathrm{e}^{\lambda p} = C'\, \mathrm{e}^{-\lambda^2/2} \tag{A.6}$$

This yields e.g.,

$$\left[\int_p (p^2)^2\right]^{-1} \int_p (ap)(bp)(cp)(dp) = \frac{1}{d(d+2)}\,[(ab)(cd)+(ac)(bd)+(ad)(bc)] \tag{A.7}$$

where the global prefactor is most easily checked by setting $a=b=c=d$ and comparing with Eq. (A.4). We need

$$\left[\int_p (p^2)^2\right]^{-1} \int_p (xp)(yp)(lp)^2 = \frac{1}{d(d+2)}\,((xy)\,l^2+2(lx)(ly)) \tag{A.8}$$

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