Nonstationary dynamics of the Alessandro-Beatrice-Bertotti-Montorsi model

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We obtain an exact solution for the motion of a particle driven by a spring in a Brownian random-force landscape, the Alessandro-Beatrice-Bertotti-Montorsi (ABBM) model. Many experiments on quasistatic driving of elastic interfaces (Barkhausen noise in magnets, earthquake statistics, shear dynamics of granular matter) exhibit the same universal behavior as this model. It also appears as a limit in the field theory of elastic manifolds. Here we discuss predictions of the ABBM model for monotonous, but otherwise arbitrary, time-dependent driving. Our main result is an explicit formula for the generating functional of particle velocities and positions. We apply this to derive the particle-velocity distribution following a quench in the driving velocity. We also obtain the joint avalanche size and duration distribution and the mean avalanche shape following a jump in the position of the confining spring. Such nonstationary driving is easy to realize in experiments, and provides a way to test the ABBM model beyond the stationary, quasistatic regime. We study extensions to two elastically coupled layers, and to an elastic interface of internal dimension d, in the Brownian force landscape. The effective action of the field theory is equal to the action, up to one-loop corrections obtained exactly from a functional determinant. This provides a connection to renormalization-group methods.

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I. INTRODUCTION

The motion of domain walls in soft magnets [1–3], fluid contact lines on a rough surface [4–6], or strike-slip faults in geophysics [7–9] can all be described on a mesoscopic level as motion of elastic interfaces driven through a disordered environment. Their response to external driving is not smooth, but exhibits discontinuous *jumps* or *avalanches*. Physically, these are seen, e.g., as pulses of Barkhausen noise in magnets [10,11], or slip instabilities leading to earthquakes on geological faults [12–14]. While the microscopic details of the dynamics are specific to each system, some large-scale features are universal [15]. The most prominent example are the exponents of the power-law distributions of avalanche sizes (for earthquakes, the well-known Gutenberg-Richter distribution [16–18]) and durations.

The Alessandro-Beatrice-Bertotti-Montorsi (ABBM) model [1] is a mean-field model for the dynamics of an interface in a disordered medium. It approximates a d-dimensional interface in a $(d+1)$-dimensional system, defined by a height function $u(x,t)$, by a single degree of freedom, its average height $u(t) = \frac{1}{L^d}\int \mathrm{d}^d x\, u(x,t)$. It satisfies the equation of motion

$$\partial_t u(t) = F(u(t)) - m^2[u(t) - w(t)]. \tag{1}$$

$w(t)$ is the external driving and $F(u)$ is an *effective* random force, sum of the local pinning forces. In [1], it was postulated to be a Gaussian with the correlations of a Brownian motion,

$$\overline{[F(u_1) - F(u_2)]^2} = 2\sigma|u_1 - u_2| \,, \tag{2}$$

where $\sigma > 0$ characterizes the disorder strength.

This model has been analyzed in depth for the case of a constant driving velocity, i.e., $w(t) = vt$ [1,3,19–24]. The distribution of avalanche sizes and durations was obtained by mapping (1) to a Fokker-Planck equation [1,3]. The mean shape of an avalanche was also computed using this mapping [22–24]. These results agree well with numerous experiments on systems with long-range elastic interactions, realized, e.g., in certain classes of soft magnets, or in geological faults [3,7, 21,26,27].

However, long-range-correlated disorder as in (2) is *a priori* an unphysical assumption for materials where the true microscopic disorder is, by nature, short ranged. Hence in realistic systems, it can only arise as a model for the *effective disorder* felt by the interface. This guess, originally made by ABBM based on experiments, turns out to be very judicious.

In [21], it was shown that the effective disorder for an interface with infinite-range elastic interactions is indeed given by (2). This led to the wide belief that the ABBM model is a universal model for the center-of-mass of an interface in dimension d at or above a certain upper critical dimension d_c depending on the range of the elastic interactions in the system [28]. Much of the popularity of the ABBM model is owed to this presumed universality. However, only recently this assumption was proven for short-ranged microscopic disorder using the functional renormalization group (FRG) [23,24], a method well suited to study interfaces (see [29] for an introduction and a short review). This proof required quasistatic driving $w(t) = vt$ with $v = 0^+$. Whether this property also holds for finite driving velocity $v > 0$, and in that case up to which scale, requires further investigation. The same question for nonstationary driving also remains open.

There are some hints that nonstationary dynamics may require a different treatment. For example, avalanche size and duration exponents seem to vary over the hysteresis loop [30–32].

Related is the question of static avalanches, i.e., jumps in the order parameter of the *ground state* upon variation of an external control parameter, as, e.g., the magnetic field. This has been studied for elastic manifolds via functional RG

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methods [33–35], and for spin glasses using replica symmetry breaking [36,37].

In this paper, we discuss the results given by the ABBM model when the driving $w(t)$ is a monotonous but otherwise arbitrary function of time. While this misses important and interesting physics of ac driving and the hysteresis loop [38], it is much more general than the cases treated so far. We will give an analytic solution for arbitrary driving, and then specialize to examples such as the relaxation of the velocity $\dot{u}(t)$ after the driving is stopped and the response to finite-size "kicks" in the driving force, $\dot{w}(t) = w_0\delta(t)$. This should allow to clarify the range of the ABBM universality class by comparing these predictions to experiments and further theoretical work. Such nonstationary driving can easily be realized, e.g., in Barkhausen noise experiments, where $w(t)$ is the external magnetic field, and can be tuned as desired.

This paper is structured as follows. In Sec. II, we review the approach to the ABBM model through the Martin-Siggia-Rose (MSR) formalism. The MSR formalism maps disorder averages over solutions of the stochastic differential equation (1) to correlation functions in a field theory. In [23,24], this method was used to compute the Laplace transform of the p-point probability distribution of the velocity in the ABBM model, via the solution of a nonlinear "instanton" equation. From it, the avalanche shape and duration distributions were obtained for quasistatic driving, in agreement with the results of [22,25]. Here we extend the method of Refs. [23,24] and show that it is even more powerful: For any monotonous (but not necessarily stationary) driving $w(t)$, the resulting field theory can be solved exactly. We give an explicit formula for the generating functional of the particle velocity $\dot{u}$. In Sec. III, we apply this solution to several examples. In particular, we derive the law for the decay of the velocity after the driving is stopped, which may easily be tested in experiments. In Sec. IV, we extend the method to variants of the ABBM model with additional spatial degrees of freedom. This includes the generalization of the ABBM model to a d-dimensional interface submitted to a quenched random force with the correlations of the Brownian motion, a model whose statics was studied in [34]. For this more general model, under monotonous driving, we show that the action of the field theory is not renormalized in any spatial dimension d. In Sec. V, we compute the generating functional for the particle position u, which is more subtle than the one for the velocity $\dot{u}$. In Secs. VI and VII, we summarize the results and mention possible extensions. In particular, we explain why *nonmonotonous* motion requires a separate treatment, and does not follow from the present results.

II. SOLUTION OF THE NONSTATIONARY ABBM MODEL

To understand the physics of (1), one would like to know the joint probability distribution for arbitrary sets of velocities $\dot{u}(t_1)\cdots\dot{u}(t_n)$, averaged over all realizations of the random force F. This is encoded in the generating functional

$$G[\lambda,w] = \overline{e^{\int_t \lambda(t)\dot{u}(t)}}, \tag{3}$$

where $\overline{\cdots}$ denotes disorder averaging. One then recovers, e.g., the generating function $\overline{e^{\lambda\dot{u}(t_0)}}$ of the distribution of $\dot{u}(t_0)$ by setting $\lambda(t) = \lambda\delta(t-t_0)$, and similarly for n-time correlation functions.

Our main result is an explicit formula for G in the case of monotonous but nonstationary motion. Given the distribution of velocities $P_0(\dot{u}_i)$ at an initial time t_i, we claim that $G_{t_i} := \overline{e^{\int_{t_i}^\infty \mathrm{d}t\,\lambda(t)\dot{u}(t)}}$ is

$$G_{t_i}[\lambda,w] = e^{m^2\int_{t_i}^\infty \mathrm{d}t\,\tilde{u}(t)\dot{w}(t)}\int_0^\infty \mathrm{d}\dot{u}_i\,P_0(\dot{u}_i)e^{\tilde{u}(t_i)\dot{u}_i}. \tag{4}$$

Here $\tilde{u}(t)$ is the solution of an instanton equation [23,24]:

$$\partial_t\tilde{u}(t) - m^2\tilde{u}(t) + \sigma\tilde{u}(t)^2 = -\lambda(t). \tag{5}$$

Boundary conditions are $\tilde{u}(\infty) = 0$; $\lambda(t)$ is assumed to vanish at infinity. Note that $\tilde{u}(t)$ only depends on $\lambda(t)$, i.e., the type of observable one is interested in, but not on the driving $w(t)$. The latter only enters in (4).

In the following, we are mostly interested in the case when the initial time $t_i \to -\infty$. Our observables will be local in time, so that $\lambda(t)$ decays quickly for $t \to \pm\infty$. Then, $\tilde{u}(t_i) \to 0$ and (4) becomes independent of initial conditions,

$$G[\lambda,w] = e^{m^2\int_t \tilde{u}(t)\dot{w}(t)}. \tag{6}$$

To prove (6), we first discuss how a closed equation for the velocity variable can be formulated. We then use the Martin-Siggia-Rose formalism to transform it to a field theory, and evaluate the resulting path integral to obtain (6). Both steps use crucially the assumption of monotonous motion.

A. Velocity in the ABBM model

The equation of motion for the velocity $\dot{u}(t)$ is obtained by differentiating (1):

$$\partial_t\dot{u}(t) = \partial_t F(u(t)) - m^2[\dot{u}(t) - \dot{w}(t)]. \tag{7}$$

A priori, to determine the probability distribution of $\dot{u}(t)$, one needs $\dot{u}(0)$ and $u(0)$, since the random force depends on the trajectory $u(t)$ and not just on $\dot{u}$. However, under the assumption that all trajectories are monotonous [$\dot{u}(t) \geqslant 0$ for all times t], the probability distribution of $\dot{u}(t)$ is independent of $u(0)$. Indeed, under this assumption, one can replace $\partial_t F(u(t))$ by a multiplicative Gaussian noise which only depends on $\dot{u}(t)$. More precisely, we can set $\partial_t F(u(t)) = \sqrt{\dot{u}(t)}\xi(t)$, where $\overline{\xi(t)\xi(t')} = 2\sigma\delta(t-t')$. To see this explicitly, consider the generating functional

$$\begin{aligned} H[\lambda] &= \overline{e^{\int_t \lambda(t)\partial_t F(u(t))}} \\ &= e^{-\int_{t,t'}\lambda(t)\lambda(t')\frac{\sigma}{2}\partial_t\partial_{t'}|u(t)-u(t')|}. \end{aligned} \tag{8}$$

Since $\dot{u}(t) \geqslant 0$ at all times, we know that [39]

$$\begin{aligned} \partial_t\partial_{t'}|u(t)-u(t')| &= \dot{u}(t)\partial_{t'}\mathrm{sgn}[u(t)-u(t')] \\ &= \dot{u}(t)\partial_{t'}\mathrm{sgn}(t-t') = -2\dot{u}(t)\delta(t-t'), \end{aligned} \tag{9}$$

and hence

$$H[\lambda] = e^{\int_t \lambda(t)^2\sigma\dot{u}(t)} = \overline{e^{\int_t \lambda(t)\sqrt{\dot{u}(t)}\xi(t)}}. \tag{10}$$

Note that for monotonous driving, the monotonicity assumption $\dot{u}(t) \geqslant 0$ is enforced automatically if it is at $t = t_0$ [40]:

$$\begin{aligned} &\dot{u}(t_0) \geqslant 0, \dot{w}(t) \geqslant 0 \quad \text{for all} \quad t \geqslant t_0 \\ &\Rightarrow \dot{u}(t) \geqslant 0 \quad \text{for all} \quad t \geqslant t_0. \end{aligned} \tag{11}$$

In this way, we see that for monotonous motion, (7) is a closed stochastic differential equation for the velocity $\dot{u}(t)$. Given an initial velocity distribution $P(\dot{u}(0))$, it can be solved without knowledge of the position $u(0)$.

B. MSR field theory for the ABBM velocity

The Martin-Siggia-Rose (MSR) approach allows us to express (3), averaged over all realizations of F in (7) in a path-integral formalism, following [19,20,23,24,41,42].

Introducing the Wick-rotated MSR response field $\tilde{u}(t)$ and averaging over the disorder, one gets:

$$\begin{aligned} G[\lambda,w] &= \int \mathcal{D}[\dot{u},\tilde{u}] e^{-S[\dot{u},\tilde{u}]+\int_t \lambda(t)\dot{u}(t)}, \\ S[\dot{u},\tilde{u}] &= \int_t \tilde{u}(t)\{\partial_t \dot{u}(t) + m^2[\dot{u}(t)-\dot{w}(t)]\} \\ &\quad + \frac{\sigma}{2}\int_{t,t'} \partial_t\partial_{t'}|u(t)-u(t')|\tilde{u}(t)\tilde{u}(t'). \end{aligned} \tag{12}$$

Since we consider only paths where $\dot{u}(t) \geqslant 0$ at all times, using (9) we can rewrite the action as

$$\begin{aligned} S[\dot{u},\tilde{u}] = \int_t &(\tilde{u}(t)\{\partial_t \dot{u}(t) + m^2[\dot{u}(t)-\dot{w}(t)]\} \\ &- \sigma\dot{u}(t)\tilde{u}(t)^2). \end{aligned} \tag{13}$$

The key observation which allows us to evaluate this exactly was first noted in [23,24]: The action is linear in $\dot{u}(t)$. This means that the path integral over $\dot{u}$ can be evaluated, giving a δ-functional. Instead of using this in the limit of $v \to 0$ as in [23,24], one can write more generally:

$$\begin{aligned} G[\lambda,w] &= \int \mathcal{D}[\tilde{u},\dot{u}] e^{m^2\int_t \tilde{u}(t)\dot{w}(t)} \\ &\quad \times e^{\int_t \dot{u}(t)[\partial_t\tilde{u}(t) - m^2\tilde{u}(t) + \sigma\tilde{u}(t)^2 + \lambda(t)]} \\ &= \int \mathcal{D}[\tilde{u}] e^{m^2\int_t \tilde{u}(t)\dot{w}(t)} \\ &\quad \times \delta[\partial_t\tilde{u}(t) - m^2\tilde{u}(t) + \sigma\tilde{u}(t)^2 + \lambda(t)]. \end{aligned}$$

This then reduces to (6) with $\tilde{u}(t)$ given by (5). Note that the Jacobian from evaluating the δ-functional is independent of $w(t)$. We assume in the following that for $\dot{w}(t) = 0$ we have $\dot{u} = 0$ and hence $G[\lambda,\dot{w}=0] = \overline{e^{\int \lambda(t)\dot{u}(t)}} = 1$ for any λ. Thus (6) is correctly normalized.

For the more rigorously minded reader, another derivation of (4) and (5) is presented in Appendix A. It avoids the use of path integrals with unclear convergence properties and takes into account the initial condition.

III. EXAMPLES

A. Stationary velocity distribution and propagator

As a first application, let us rederive the well-known probability distribution for the velocity in the case of stationary driving, $w(t) = vt$.

To obtain the generating function of the velocity distribution at t_0, we set $\lambda(t) = \lambda\delta(t-t_0)$ in (3). The solution of (5) is [43]

$$\tilde{u}(t) = \frac{\lambda}{\lambda + (1-\lambda)e^{-(t-t_0)}}\theta(t < t_0). \tag{14}$$

As already derived in [23], for $\dot{w}(t) = v$ one gets

$$\int_t \tilde{u}(t)\dot{w}(t) = -v\ln(1-\lambda), \tag{15}$$

and hence $G(\lambda) = (1-\lambda)^{-v}$. This generating function yields the probability distribution

$$P(\dot{u}) = \frac{e^{-\dot{u}}\dot{u}^{-1+v}}{\Gamma(v)}, \tag{16}$$

which is the well-known result for the stationary velocity distribution [1,3].

Using the same method, we can obtain the two-time velocity probability distribution. For $\lambda(t) = \lambda_1\delta(t-t_1) + \lambda_2\delta(t-t_2)$, with $t_1 < t_2$, the solution of (5) is

$$\tilde{u}(t) = \begin{cases} 0, & t > t_2, \\ \frac{1}{1+\frac{1-\lambda_2}{\lambda_2}e^{t_2-t}}, & t_1 < t < t_2, \\ \frac{1}{1-\frac{\lambda_1\lambda_2 e^{t_1} - (1-\lambda_1)(1-\lambda_2)e^{t_2}}{(1+\lambda_1)\lambda_2 e^{t_1} + \lambda_1(1-\lambda_2)e^{t_2}}e^{t_1-t}}, & t < t_1. \end{cases} \tag{17}$$

As already derived in [23], for $\dot{w}(t) = v$ one gets

$$\int_t \tilde{u}(t)\dot{w}(t) = -v\ln\{1-\lambda_1-\lambda_2+\lambda_1\lambda_2[1-e^{-(t_2-t_1)}]\},$$

and using (6),

$$G(\lambda_1,\lambda_2) = \{1-\lambda_1-\lambda_2+\lambda_1\lambda_2[1-e^{-(t_2-t_1)}]\}^{-v}. \tag{18}$$

Taking the inverse Laplace transform, we obtain the two-time velocity distribution

$$P(\dot{u}_1,\dot{u}_2) = \frac{\sqrt{\dot{u}_1\dot{u}_2}^{-1+v}}{2\Gamma(v)\sinh\frac{\tau}{2}} I_{-1+v}\left(\frac{\sqrt{\dot{u}_1\dot{u}_2}}{\sinh\frac{\tau}{2}}\right) e^{\frac{v}{2}\tau - \frac{\dot{u}_1+\dot{u}_2}{1-e^{-\tau}}},$$

where $\dot{u}_1 := \dot{u}(t_1)$, $\dot{u}_2 := \dot{u}(t_2)$, $\tau := t_2 - t_1 > 0$, and I_α is the modified Bessel function. This formula generalizes the quasistatic result of [23] to arbitrary v. Dividing by the one-point distribution $P(\dot{u}_1)$ given in (16), one obtains a closed formula for the ABBM propagator for velocity $v > 0$:

$$P(\dot{u}_2|\dot{u}_1) = \frac{\sqrt{\frac{\dot{u}_2}{\dot{u}_1}}^{-1+v}}{2\sinh\frac{\tau}{2}} I_{-1+v}\left(\frac{\sqrt{\dot{u}_1\dot{u}_2}}{\sinh\frac{\tau}{2}}\right) e^{\frac{v}{2}\tau - \frac{\dot{u}_1 e^{-\tau}+\dot{u}_2}{1-e^{-\tau}}}. \tag{19}$$

Using this result and the Markov property of Eq. (7), n-point correlation functions of the velocity can be expressed in closed form as products of Bessel functions.

B. Velocity distribution after a quench in the driving speed

Now let us consider a nonstationary situation. Assume that the domain wall is driven with a constant velocity $v_1 > 0$ for $t < 0$, which is changed to $v_2 \geqslant 0$ for $t > 0$. One expects that the velocity distribution interpolates between the stationary distribution for v_1 at $t = 0$ and the stationary distribution for v_2 for $t \to \infty$. In this subsection, we will compute its exact form for all times.

For the one-time velocity distribution, $\lambda(t) = \lambda\delta(t-t_0)$ and the solution of (5) is unchanged, given by (14).

Now, using $\dot{w}(t) = (v_1 - v_2)\theta(-t) + v_2$, one gets

$$\int_t \tilde{u}(t)\dot{w}(t)$$

$$= \int_{-\infty}^{0} \frac{v_1 \lambda\, dt}{\lambda + (1-\lambda)e^{-(t-t_0)}} + \int_0^{t_0} \frac{v_2 \lambda\, dt}{\lambda + (1-\lambda)e^{-(t-t_0)}}$$

$$= (v_1 - v_2)\ln\left(1 + \frac{\lambda}{1-\lambda}e^{-t_0}\right) - v_2 \ln(1-\lambda).$$

Thus, with the help of (6),

$$G(\lambda) = \overline{e^{\lambda \dot{u}(t_0)}} = [1 - \lambda(1 - e^{-t_0})]^{v_1 - v_2}(1-\lambda)^{-v_1}. \tag{20}$$

Inverting the Laplace transform, one obtains

$$P(\dot{u}(t_0)) = \frac{e^{-\dot{u}}\dot{u}^{-1+v_2}(1-e^{-t_0})^{v_1-v_2}}{\Gamma(v_2)} \times {}_1F_1\left(v_2 - v_1, v_2, \frac{\dot{u}}{1-e^{t_0}}\right). \tag{21}$$

An interesting special case is when the driving is turned off at $t = 0$, i.e., $v_1 = v$ and $v_2 = 0$. According to (11), the particle will continue to move forward until it encounters the first zero of $\dot{u} = F(u) - m^2[u - w(0)]$. Correspondingly, we expect that the velocity distribution decays from the stationary probability distribution at $t \leqslant 0$ to a δ distribution at zero at $t \to \infty$. The explicit calculation for $\dot{u} := \dot{u}(t_0)$ yields

$$P(\dot{u}) = (1 - e^{-t_0})^v \delta(\dot{u}) + e^{-\dot{u}-t_0 v}(e^{t_0} - 1)^{-1+v} v\, {}_1F_1\left(1 - v, 2, \frac{\dot{u}}{1-e^{t_0}}\right). \tag{22}$$

The $\delta(\dot{u})$ term gives the probability that the motion has stopped at time t_0,

$$P[\dot{u}(t_0) = 0] = (1 - e^{-t_0})^v. \tag{23}$$

As expected, this is zero at $t_0 = 0$ and tends to 1 as $t_0 \to \infty$. Correspondingly, the distribution for the relaxation time T, i.e., the time for the particle to stop moving from the stationary driving state at velocity v, is given by

$$P(T) = \left.\frac{\partial}{\partial t_0}\right|_{t_0 = T} P[\dot{u}(t_0) = 0] = v e^{-T}(1 - e^{-T})^{-1+v}.$$

The term in (22) not proportional to the δ function (once normalized) gives the conditional distribution of velocities assuming the particle is still moving. Its form compares well to simulations; see Fig. 1.

Using (20), one also sees that the mean velocity interpolates exponentially between the old and the new value of the driving speed,

$$\overline{\dot{u}_{t_0}} = \partial_\lambda|_{\lambda=0} G(\lambda) = v_2 + (v_1 - v_2)e^{-t_0}. \tag{24}$$

These results are valuable since they provide a tool to test the validity of the ABBM model in different experimental protocols. In application to Barkhausen noise, one could perform experiments where the driving by the external magnetic field is stopped at some time. This would allow to verify, e.g., (22) experimentally, since the velocity in our model is the induced voltage in a Barkhausen experiment. This would be one of the first checks on whether the good agreement between the ABBM theory and experiments persists in the nonstationary case.

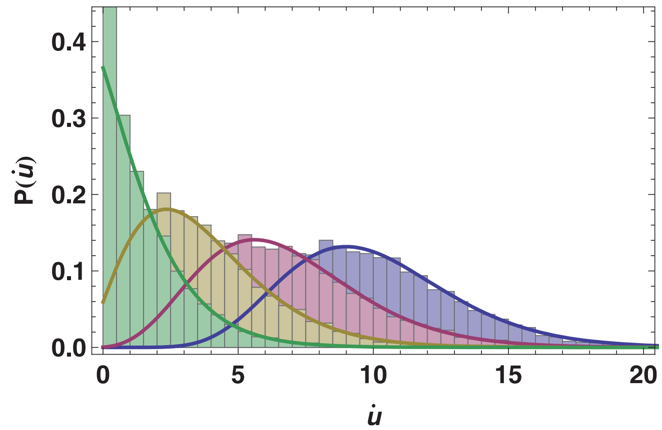

FIG. 1. (Color online) Decay of velocity distribution after driving was stopped. Curves (from right to left): Results for $t_0 = 0, 0.4, 1, 2$ from (22). Bar charts: Corresponding simulation results, averaged over 10^4 trajectories. Initial driving velocity (for $t < 0$) was $v = 10$.

C. Nonstationary avalanches

Using similar techniques, one can treat the case of a finite jump from 0 to w in the location of the confining harmonic well in (1), $w(t) = w\theta(t)$ equivalent to a "kick" $\dot{w}(t) = w\delta(t)$. For $t < 0$ the particle is at rest, and the quench at $t = 0$ triggers exactly one avalanche. Its size is given by $S = \int_0^\infty \dot{u}(t)\mathrm{d}t$ and its duration T by the first time when $\dot{u}(T) = 0$. Note that this avalanche occurs as the nonstationary response to a kick of arbitrary size, a problem *a priori* different from the stationary avalanches studied previously [3,23,24] for small constant drive $\dot{w}(t) = v = 0^+$. In this section, we will derive the distribution of avalanche sizes and durations for arbitrary kick sizes w.

1. *Preparation of the initial condition*

The assumption $G[\lambda, \dot{w} = 0] = 1$ which we made in Sec. II B implies that the initial condition at t_i, which is the lower limit of all time integrals in the action and in (6), is $\dot{u}(t_i) = 0$. This means that the particle is exactly at rest for $t \geqslant t_i$ if $\dot{w}(t) = 0$ for $t \geqslant t_i$. Furthermore, to assure that the particle will not revisit part of the trajectory, we demand $u(t) \leqslant u(t_i)$ for all $t < t_i$. One protocol with which this can be enforced is as follows: Start at some time $t_1 \ll t_i$ at an arbitrary position $u(t_1) \ll 0$, and take $w(t) = 0$ for all $t \in [t_1, t_i]$. Then $\dot{u}(0)$ will be almost surely positive. Thus, between t_1 and t_i, the particle will move forward until it reaches the smallest u where $F(u) - m^2 u = 0$. Since $t_1 \ll t_i$, almost surely it will reach this point before t_i and thus be at rest at t_i. This choice of initial condition is equivalent to choosing a random configuration from the steady state for quasistatic driving at $v = 0^+$.

2. *Duration distribution*

First, let us derive the exact distribution of avalanche durations following a kick. The generating function for

$P(\dot{u}(t_0))$ at time $t_0 > 0$ is obtained as in the previous section as

$$G(\lambda) = \overline{e^{\lambda \dot{u}(t_0)}} = \exp\left(\frac{w\lambda}{\lambda + (1-\lambda)e^{t_0}}\right). \tag{25}$$

Laplace inversion gives, denoting $\dot{u} := \dot{u}(t_0)$,

$$P(\dot{u}) = e^{-\frac{w+e^{t_0}\dot{u}}{e^{t_0}-1}}\left[\delta(\dot{u}) + \frac{1}{2\sinh\frac{t_0}{2}}\sqrt{\frac{w}{\dot{u}}}I_1\left(\frac{\sqrt{\dot{u}w}}{\sinh\frac{t_0}{2}}\right)\right]. \tag{26}$$

The mean velocity

$$\overline{\dot{u}(t_0)} = \partial_\lambda|_{\lambda=0}G(\lambda) = we^{-t_0} \tag{27}$$

decays in the same way as in (24) for stopped driving. However, the probability distributions of $\dot{u}(t_0)$ are different, as can be seen by comparing (26) and (22). The probability that $\dot{u}(t_0) = 0$, i.e., that the avalanche has terminated at time $T < t_0$, is obtained by taking the limit $\lambda \to -\infty$ in (25), which gives the δ-function piece in (26),

$$P[\dot{u}(t_0) = 0] = P(T \leqslant t_0) = \exp\left(-\frac{w}{e^{t_0}-1}\right). \tag{28}$$

Note that this procedure requires $P(\dot{u} < 0) = 0$, which is the case here.

Correspondingly, the probability density for the avalanche duration T is given by

$$P(T) = \frac{\partial}{\partial t_0}\bigg|_{t_0=T} P[\dot{u}(t_0) = 0] = \frac{w\exp\left(-\frac{w}{e^T-1}\right)}{\left(2\sinh\frac{T}{2}\right)^2}. \tag{29}$$

We observe that for infinitesimally small quenches w, one recovers—up to a normalization factor—the distribution obtained in [23,24] for avalanches at stationary, quasistatic driving, with the universal power law T^{-2} for small times [3]:

$$\rho(T) := \partial_w|_{w=0}P(T) = \frac{1}{\left(2\sinh\frac{T}{2}\right)^2}. \tag{30}$$

Hence, the nonstationary character is not important in that limit.

For finite $w > 0$, the mean avalanche duration is obtained from (28),

$$\overline{T}(w) = \gamma_E - e^w\mathrm{Ei}(-w) + \log(w) \overset{w\to\infty}{\sim} \log w.$$

It behaves as $\overline{T}(w) \sim w\ln(1/w)$ at small w and diverges logarithmically for large w. In the latter limit, the distribution of $\tilde{T} := T - \ln w$ approaches a Gumbel distribution

$$P(\tilde{T}) \approx e^{-\tilde{T}}e^{-e^{-\tilde{T}}}$$

on the interval $\tilde{T} \in [-\infty,\infty]$, as if the duration were given by the maximum of w independent random variables.

3. *Joint size and duration distribution*

One can now proceed to a more general case, and compute the joint distribution of avalanche durations and sizes. We again calculate the generating function

$$G(\lambda_1,\lambda_2) = \overline{e^{\lambda_1 S + \lambda_2 \dot{u}(t_0)}},$$

where $S := \int_0^\infty \dot{u}(t)\mathrm{d}t$ is the avalanche size. The solution of (5) for $\lambda(t) = \lambda_1 + \lambda_2\delta(t-t_0)$ is given by

$$\begin{aligned}\tilde{u}(t) &= \frac{1}{2}(1-\sqrt{1-4\lambda_1}) \\ &\quad + \frac{e^{\sqrt{1-4\lambda_1}(t-t_0)}\sqrt{1-4\lambda_1}\lambda_2\theta(t_0-t)}{\sqrt{1-4\lambda_1}-\lambda_2[1-e^{\sqrt{1-4\lambda_1}(t-t_0)}]}.\end{aligned}$$

Since the driving is $\dot{w}(t) = w\delta(t)$, we obtain from (6)

$$G(\lambda_1,\lambda_2) = e^{wZ(\lambda_1,\lambda_2)}, \tag{31}$$

$$\begin{aligned}Z(\lambda_1,\lambda_2) &= \frac{1}{2}(1-\sqrt{1-4\lambda_1}) \\ &\quad + \frac{e^{-\sqrt{1-4\lambda_1}t_0}\sqrt{1-4\lambda_1}\lambda_2}{\sqrt{1-4\lambda_1}-\lambda_2(1-e^{-\sqrt{1-4\lambda_1}t_0})}.\end{aligned} \tag{32}$$

For $\lambda_2 = 0$, this gives the distribution of avalanche sizes S for arbitrary kick size w,

$$P(S) = \frac{w}{2\sqrt{\pi}S^{\frac{3}{2}}}\exp\left(-\frac{w^2}{4S} - \frac{S}{4} + \frac{w}{2}\right). \tag{33}$$

As it should, this coincides with the distribution obtained for quasistatic driving, $v = 0^+$ [44].

In the case of a nonstationary kick, we can obtain more information on the avalanche dynamics by considering the joint distribution of avalanche sizes S and durations T. As above, the probability that $\dot{u}(t_0) = 0$ and hence the probability that the duration T of the avalanche lies in the interval $]0;t_0[$, is given by the limit $\lambda_2 \to -\infty$. Thus, the joint probability density $P(S,T)$ of sizes S and durations T satisfies

$$\begin{aligned}&\int_0^\infty \mathrm{d}S \int_0^{t_0} \mathrm{d}T\, e^{\lambda_1 S}P(S,T) \\ &\quad = \exp\left(\frac{w}{2}(1-\sqrt{1-4\lambda_1}) - w\frac{\sqrt{1-4\lambda_1}e^{-\sqrt{1-4\lambda_1}t_0}}{1-e^{-\sqrt{1-4\lambda_1}t_0}}\right).\end{aligned}$$

Deriving with respect to t_0, we obtain

$$\int_0^\infty \mathrm{d}S\, e^{\lambda S}P(S,T) = \frac{w(1-4\lambda)e^{\frac{w}{2}(1-\sqrt{1-4\lambda}\coth\frac{T}{2}\sqrt{1-4\lambda})}}{\left(2\sinh\frac{T}{2}\sqrt{1-4\lambda}\right)^2}, \tag{34}$$

which for $\lambda = 0$ reproduces (29). This implies the scaling form [45]:

$$P(S,T) = e^{-\frac{S}{4}}f(S/T^2), \tag{35}$$

$$f(x) = \mathrm{LT}^{-1}_{s\to x}\frac{we^{\frac{w}{2}}se^{-\frac{w}{T^2}\sqrt{s}\coth\sqrt{s}}}{T^4(\sinh\sqrt{s})^2}. \tag{36}$$

Although no formula to invert the Laplace transform in a closed form is evident, one can, for example, calculate the mean avalanche size for a fixed value of the avalanche duration,

$$\begin{aligned}\overline{S}(T) &= \frac{\int_0^\infty \mathrm{d}S\, S\, P(S,T)}{\int_0^\infty \mathrm{d}S\, P(S,T)} \\ &= \frac{4 - wT - 4\cosh T + (2T+w)\sinh T}{\cosh T - 1}.\end{aligned} \tag{37}$$

As $w \to 0$, this has a well-defined limit

$$\overline{S}(T) = 2T\coth\frac{T}{2} - 4. \tag{38}$$

Equation (38) reproduces the expected scaling behavior [3,21], $\overline{S}(T) \sim T^2$, for small avalanches. This is apparent in (35), since the $e^{-\frac{S}{4}}$ factor can be neglected for small S. The new result in Eq. (38) predicts the deviations of large avalanches from this scaling, and shows that they obey $S \sim T$ instead. This is in qualitative agreement with experimental observations on Barkhausen noise in polycrystalline FeSi materials [3,11,25]. It would be interesting to test the quantitative agreement of (38) with experiments as well.

We can also obtain the large-T behavior at fixed S (fixed λ) since in that limit

$$\int_0^\infty dS\, e^{\lambda S} P(S,T) \approx w(1-4\lambda) e^{w/2} e^{-\frac{1}{2}(w+2T)\sqrt{1-4\lambda}}. \tag{39}$$

This implies

$$P(S,T) \approx \frac{w(2T+w)[(2T+w)^2 - 6S] e^{\frac{w}{2} - \frac{(2T+w)^2}{4S} - \frac{S}{4}}}{2\sqrt{\pi} S^{7/2}}. \tag{40}$$

Note that (39) is also valid at fixed T and large negative λ, hence (40) also gives the behavior for $S \ll T^2$ at fixed T. One notes some resemblance with (33).

We now consider the limit of a small kick $w \to 0$. Equation (34) gives

$$P(S,T) = w\rho(S,T) + O(w^2), \tag{41}$$

where $\rho(S,T)$ can be interpreted as an avalanche size and duration "density," satisfying

$$\int_0^\infty dS\, e^{\lambda S} \rho(S,T) = \frac{(1-4\lambda)}{\left(2\sinh \frac{T}{2}\sqrt{1-4\lambda}\right)^2}. \tag{42}$$

This Laplace transform can be inverted:

$$\rho(S,T) = e^{-S/4} \frac{1}{T^4} g(S/T^2), \tag{43}$$

$$g(x) = \mathrm{LT}^{-1}_{s\to x} \frac{s}{(\sinh\sqrt{s})^2} = \frac{d}{dx} h(x),$$

$$h(x) = \sum_{n=-\infty}^{+\infty} (1 - 2\pi^2 n^2 x) e^{-n^2\pi^2 x} = \sum_{m=-\infty}^{\infty} \frac{2m^2 e^{-\frac{m^2}{x}}}{\sqrt{\pi} x^{3/2}}. \tag{44}$$

We have used $\sum_{n=-\infty}^{\infty} \frac{s-n^2\pi^2}{(s+n^2\pi^2)^2} = 1/(\sinh\sqrt{s})^2$. Note that $\rho(S,T)$, as a size density, is normalized to $\int_0^\infty dS\, \rho(S,T) = \rho(T)$, given in (30), since a fixed duration T acts as a small avalanche-size cutoff. The total size density $\rho(S) = \int dT\, \rho(S,T) = \frac{1}{2\sqrt{\pi}S^{3/2}} \exp(-\frac{S}{4})$ is not normalized, since w, which acts as a small-scale cutoff in (33), has been set to 0.

Finally, note that (34) allows one to go further and compute any moment as well as, by numerical Laplace inversion, the full joint distribution $P(S,T)$. This is shown in Fig. 2.

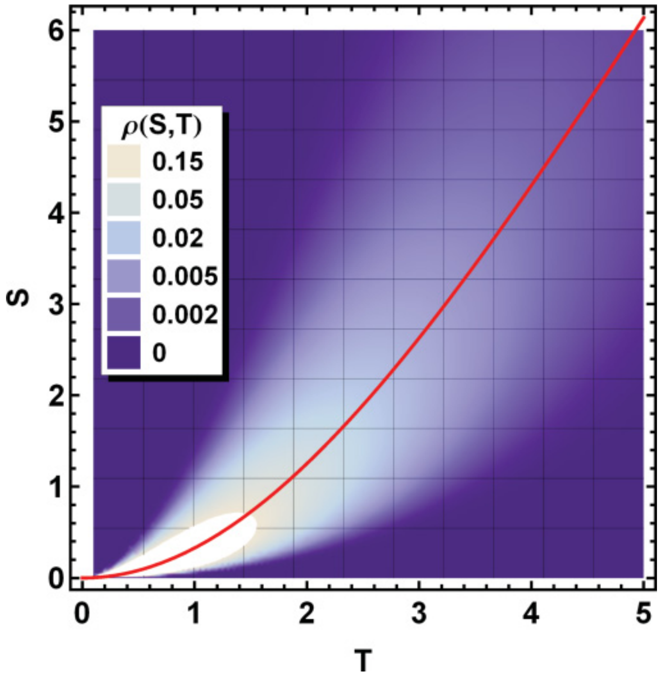

FIG. 2. (Color online) Joint density $\rho(S,T)$ of avalanche sizes S and durations T in the ABBM model, obtained by numerical Laplace inversion of (42)–(44). The red line is the mean size $\bar{S}(T)$ for a fixed duration T given in (38).

4. Avalanche shape following a pulse

We consider now the joint probability of velocities at two times $0 < t_1 < t_2$ following a pulse at time $t = 0$. By (6), its generating function is

$$\overline{e^{\lambda_1 \dot u(t_1) + \lambda_2 \dot u(t_2)}} = e^{w\tilde{u}(0)},$$

where $\tilde{u}(0)$ is the two-time solution (17). We are interested in $P(\dot u(t_1), \dot u(t_2) = 0)$ obtained by taking $\lambda_2 \to -\infty$:

$$\int d\dot u_1 e^{\lambda_1 \dot u_1} P(\dot u_1, 0) = \exp\left(\frac{w}{1 - \frac{\lambda_1 e^{t_1} + (1-\lambda_1)e^{t_2}}{(1+\lambda_1)e^{t_1} - \lambda_1 e^{t_2}} e^{t_1}} \right).$$

We use that $\mathrm{LT}^{-1}_{s\to u} e^{d + \frac{a}{b+s}} = e^d[\sqrt{\frac{a}{u}} I_1(2\sqrt{au}) e^{-bu} + \delta(u)]$ with $d = -\frac{w}{e^{t_1}-1}$, $a = we^{t_1}/(e^{t_1}-1)^2$, and $b = \frac{1}{e^{t_2-t_1}-1} + \frac{1}{1-e^{-t_1}}$. Taking ∂_{t_2} and setting $t_2 = T$, we find the joint probability distribution of the avalanche duration T and the velocity $\dot u(t_1) = \dot u_1$,

$$\begin{aligned} P(\dot u_1, T) &= -\partial_{t_2} b e^d \sqrt{a\dot u_1} I_1(2\sqrt{a\dot u_1}) e^{-b\dot u_1}|_{t_2=T} \\ &= \frac{1}{\left[2\sinh\left(\frac{T-t_1}{2}\right)\right]^2} \frac{\sqrt{w\dot u_1}}{2\sinh\frac{t_1}{2}} I_1\left(\frac{\sqrt{w\dot u_1}}{\sinh\frac{t_1}{2}}\right) \\ &\quad \times e^{-\frac{w}{e^{t_1}-1} - \left(\frac{1}{e^{T-t_1}-1} + \frac{1}{1-e^{-t_1}}\right)\dot u_1}. \end{aligned} \tag{45}$$

Dividing by $P(T)$ given in (29), we find the conditional probability for the velocity distribution at t_1 for an avalanche of duration T. In particular, we get the average avalanche shape,

$$\overline{\dot u(t_1)}_T = \frac{4\sinh(\frac{t_1}{2})\sinh(\frac{T-t_1}{2})}{\sinh(\frac{T}{2})} + w\left[\frac{\sinh(\frac{T-t_1}{2})}{\sinh(\frac{T}{2})}\right]^2. \tag{46}$$

For $w \to 0$, one recovers the stationary avalanche shape obtained in [22,23]. On the other hand, avalanches following a pulse of size $w > 0$ have an asymmetric shape, since $\dot u(t = 0^+) = w$. This should provide an elegant way to discriminate between the two situations experimentally.

D. Power spectral density and distribution of Fourier modes

In signal analysis, an important observable used to characterize a time series is the *power spectral density* $P(\omega)$ defined as

$$P(\omega) := \lim_{T\to\infty} \frac{1}{T} \overline{\left| \int_{-T/2}^{T/2} e^{i\omega t}[\dot{u}(t) - \overline{\dot{u}(t)}]\mathrm{d}t \right|^2}. \quad (47)$$

This gives a measure for the abundance of the frequency component ω in the time series $\dot{u}(t)$. For a stationary signal where the two-time velocity correlation function only depends on the time difference, (47) is equal to its Fourier transform:

$$P(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\, \overline{\dot{u}(0)\dot{u}(t)}^c\, \mathrm{d}t. \quad (48)$$

For driving with constant velocity $w(t) = vt$, one knows [3,46] $\overline{\dot{u}(0)\dot{u}(t)}^c = ve^{-|t|}$, and hence the power spectrum for the velocity in the ABBM model is

$$P(\omega) = \frac{2v}{1+\omega^2}. \quad (49)$$

We can now proceed further and obtain the probability density of each Fourier component. We consider (6) with $\lambda(t) = \lambda \cos\omega t\, \theta(T-t)\theta(t)$, where T is a large-time cutoff. To solve (5) with this choice of λ, we substitute $\tilde{u}(t) = \frac{1}{2} + \frac{\phi'(t)}{\phi(t)}$ giving *Mathieu's equation*,

$$\phi''(t) - \left(\frac{1}{4} - \lambda\cos\omega t\right)\phi(t) = 0.$$

This is to be solved with the boundary condition $\tilde{u}(T) = 0$, i.e., $\phi'(T) = -\frac{1}{2}\phi(T)$.

The general solution is a linear combination of two *Floquet* solutions

$$\phi(t) = e^{\mu t}P_1(t) + e^{-\mu t}P_2(t), \quad (50)$$

where $P_{1,2}(t)$ are periodic functions. $\mu = \mu(\lambda,\omega)$ is related to the conventionally defined *Mathieu characteristic exponent* $\nu(a,q)$ (in the notation of [47]) by

$$\mu = \frac{\omega}{2i}\,\nu\left(-\frac{1}{\omega^2},\frac{2\lambda}{\omega^2}\right).$$

When λ is real and close to 0, μ is real, has the same sign as λ, and is odd in λ. Thus, for $0 < t \ll T$, the solution $\phi(t)$ given in (50) is dominated by the exponentially decaying term

$$\phi(t) \approx e^{-\mu(|\lambda|,\omega)t}P(t),$$

with $P(t) = P_{1,2}(t)$, depending on the sign of λ. Thus, for $0 < t \ll T$ we have

$$\tilde{u}(t) = \frac{1}{2} - \mu(|\lambda|,\omega) + \frac{P'(t)}{P(t)}. \quad (51)$$

In order to evaluate (6), one needs to integrate $\tilde{u}(t)$ over t from 0 to T. Since $P(t)$ is periodic, its contribution vanishes for each period,

$$\int_s^{s+\frac{2\pi}{\omega}} \tilde{u}(t)\,\mathrm{d}t = \frac{2\pi}{\omega}\left[\frac{1}{2} - \mu(|\lambda|,\omega)\right], \quad 0 < s \ll T. \quad (52)$$

For constant driving, $w(t) = vt$ and $T \gg \frac{2\pi}{\omega}$, one thus obtains using (6)

$$\overline{e^{\lambda\int_0^T \dot{u}(t)\cos\omega t}} = e^{v\int_0^T \tilde{u}(t)} = e^{vT[\frac{1}{2}+i\frac{\omega}{2}\nu(-\frac{1}{\omega^2},\frac{2|\lambda|}{\omega^2})]+o(T)}. \quad (53)$$

As expected by symmetry, this is an even function in λ. It remains real as long as the Mathieu exponent ν is purely imaginary, which is the case for $|\lambda| < \lambda_c(\omega)$. One can interpret the corresponding Mathieu functions as Schrödinger wave functions in the periodic potential,

$$V(x) = \tfrac{1}{4} - \lambda\cos(\omega x).$$

The region $|\lambda| < \lambda_c(\omega)$ is the region where the energy $E = 0$ is outside the energy band(s) of this potential, and all wave functions are evanescent. At $\lambda = \pm\lambda_c$, one has $\nu = 0$, and for $|\lambda| > \lambda_c$, i.e., outside the "band gap," the expectation value on the left-hand side of (53) does not exist. This indicates that the distribution of $\int_0^T \dot{u}(t)\cos\omega t$ has exponential tails for any $\omega > 0$. The exponent of this tail can be computed in terms of the so-called Mathieu characteristic values a_r and b_r [47].

Furthermore, from (53) one observes the scaling behavior of the cumulants,

$$\overline{\left(\int_0^T \dot{u}(t)\cos\omega t\right)^n}^c \sim T, \quad (54)$$

which is reminiscent of the central limit theorem.

Taking two derivatives of (53) with respect to λ, and using $\partial_q^2\big|_{q=0^+}\nu(-b^2,q) = \frac{-i}{2b(b^2+1)}$, one verifies once more (49). However, (53) goes beyond that and gives the full probability distribution of each frequency component of the time series $\dot{u}(t)$.

With this, we conclude our examples on the "classical" ABBM model and move to generalizations which can be treated by our method as well.

IV. ABBM MODEL WITH SPATIAL DEGREES OF FREEDOM

An interesting generalization of the ABBM model (1) is a model with spatial degrees of freedom (e.g., an extended elastic interface in dimension $d > 0$), but subject to the same kind of disorder as in the ABBM model, i.e., a pinning force correlated as a random walk.

An interface was studied in [23] for quasistatic driving and it was found that the global motion (i.e., the motion of the center of mass of the interface) is unchanged by the elastic interaction. An instanton equation for the other Fourier modes was derived, but solving it remained a challenge.

Here we extend these results to arbitrary driving velocity. We first study the simpler case of only two elastically coupled particles, and present a direct argument to show that the center of mass is not affected by the elastic interaction and is the same as for a single particle, i.e., model (1) in a rescaled disorder. For two particles, the instanton equation is simpler and more amenable to analytic studies, which allows us to see how local properties (such as the velocity distribution of a single particle) are modified. In the last part, we come back to the interface and show a nonrenormalization property of the theory valid for any driving velocity.

A. Two elastically coupled particles in an ABBM-like pinning-force field

The model we analyze in this section is a two-particle version of (1):

$$\begin{aligned}\partial_t u_1(t) &= F_1(u_1(t)) - m^2[u_1(t) - w(t)] \\ &\quad + k[u_2(t) - u_1(t)], \\ \partial_t u_2(t) &= F_2(u_2(t)) - m^2[u_2(t) - w(t)] \\ &\quad + k\,[u_1(t) - u_2(t)]\,. \end{aligned} \tag{55}$$

We assume $F_1(u_1)$, $F_2(u_2)$ to be independent Gaussian processes with correlations as in (2), i.e.,

$$\overline{[F_1(u) - F_1(u')]^2} = \overline{[F_2(u) - F_2(u')]^2} = 2\sigma|u - u'|.$$

1. Center-of-mass motion

From (55), we obtain the equation of motion for the center-of-mass velocity $\dot{s}(t) = \frac{1}{2}[\dot{u}_1(t) + \dot{u}_2(t)]$:

$$\partial_t \dot{s}(t) = \tfrac{1}{2}\partial_t[F_1(u_1(t)) + F_2(u_2(t))] - m^2[\dot{s}(t) - \dot{w}(t)]. \tag{56}$$

To better understand the effective noise term $\partial_t[F_1(u_1(t)) + F_2(u_2(t))]$, let us compute its generating functional,

$$\begin{aligned} G[\lambda] &= \overline{e^{\int_t \lambda(t)\frac{1}{2}\partial_t[F_1(u_1)+F_2(u_2)]}} \\ &= e^{-\int_{t,t'}\lambda(t)\lambda(t')\frac{\sigma}{8}\partial_t\partial_{t'}[|u_1(t)-u_1(t')|+|u_2(t)-u_2(t')|]}. \end{aligned}$$

Using monotonicity [48,49] of the trajectories (9), we obtain

$$G[\lambda] = e^{\int_t \lambda(t)^2\frac{\sigma}{4}[\dot{u}_1(t)+\dot{u}_2(t)]} = e^{\int_t \lambda(t)^2\frac{\sigma}{2}\dot{s}(t)}.$$

Note that this is the same generating function as for a random pinning force $F(s(t))$ with correlations

$$\overline{[F(s) - F(s')]^2} = \sigma|s - s'|. \tag{57}$$

Thus, we can rewrite (56) as

$$\partial_t \dot{s}(t) = \partial_t F(s) - m^2\,[\dot{s}(t) - \dot{w}(t)]\,, \tag{58}$$

with a rescaled disorder amplitude $\sigma' = \frac{\sigma}{2}$, reducing it to the same form as (7).

This argument extends straightforwardly to any number of elastically coupled particles, and to the continuum limit. Thus, we observe that the dynamics of the center of mass of an extended interface in a pinning-force field, which is correlated as a random walk, is *equivalent* to the one-particle ABBM model (1).

2. Single-particle velocity distribution

On the other hand, observables that cannot be described solely in terms of the center of mass are more complicated. In order to obtain the joint distribution of the particle velocities $\dot{u}_1(t),\dot{u}_2(t)$, one may follow the same route as in Sec. II B. We start from

$$\begin{aligned} G[\lambda_1,\lambda_2,w] &= \overline{e^{\int_t \lambda_1(t)\dot{u}_1(t)+\lambda_2(t)\dot{u}_2(t)}} \\ &= e^{m^2\int_t[\tilde{u}_1(t)+\tilde{u}_2(t)]\dot{w}(t)}, \end{aligned} \tag{59}$$

where $\tilde{u}_1, \tilde{u}_2$ are solutions of the coupled nonlinear differential equations

$$\begin{aligned} -\partial_t\tilde{u}_1(t) + m^2\tilde{u}_1(t) + k[\tilde{u}_1(t) - \tilde{u}_2(t)] - \sigma\tilde{u}_1(t)^2 &= \lambda_1(t), \\ -\partial_t\tilde{u}_2(t) + m^2\tilde{u}_2(t) + k[\tilde{u}_2(t) - \tilde{u}_1(t)] - \sigma\tilde{u}_2(t)^2 &= \lambda_2(t). \end{aligned}$$

In contrast to (5), these cannot be solved in a closed form even for simple choices of $\lambda_{1,2}$. However, one can obtain a perturbative solution for small k around $k = 0$. To give a simple example, one obtains for monotonous driving $w(t) = vt$ and one-time velocity measurements $\lambda_{1,2}(t) = \lambda_{1,2}\delta(t)$:

$$\begin{aligned} G(\lambda_1,\lambda_2) &= [(1-\lambda_1)(1-\lambda_2)]^{-v} \\ &\quad\times\left\{1+vk(\lambda_1-\lambda_2)\left[-\frac{\ln(1-\lambda_1)}{\lambda_1(1-\lambda_2)}+\frac{\ln(1-\lambda_2)}{\lambda_2(1-\lambda_1)}\right]\right. \\ &\quad\left. + O(k^2)\right\}, \end{aligned} \tag{60}$$

where we use rescaled units where k denotes k/m^2 in the original units. As one expects from the preceding section, the correction of order k vanishes if one considers the center-of-mass motion, $\lambda_1 = \lambda_2$. If, on the other hand, one considers the one-particle velocity distribution, i.e., takes $\lambda_2 = 0$, one gets

$$G(\lambda_1,0) = (1-\lambda_1)^{-v(1+k)}\left[1 - vk\frac{\lambda_1}{1-\lambda_1} + O(k^2)\right]. \tag{61}$$

The Laplace transform can be inverted, giving

$$\begin{aligned} P(\dot{u}_1) &= \frac{e^{-\dot{u}_1}\dot{u}_1^{-1+v}}{\Gamma(v)} \\ &\quad\times\{1 + k\left[v - \dot{u}_1 + v\ln\dot{u}_1 - v\psi(v)\right] + O(k^2)\}\,, \end{aligned} \tag{62}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. Simulations for small k confirm this result (see Fig. 3). The next order in k can likewise be calculated, however the resulting expressions are complicated and not very enlightening.

A nontrivial consequence of (62) is that the power-law exponent of the distribution $P(\dot{u}_1)$ for small velocities changes from $\dot{u}^{-1+v}$ to $\dot{u}^{-1+v(1+k)}$.

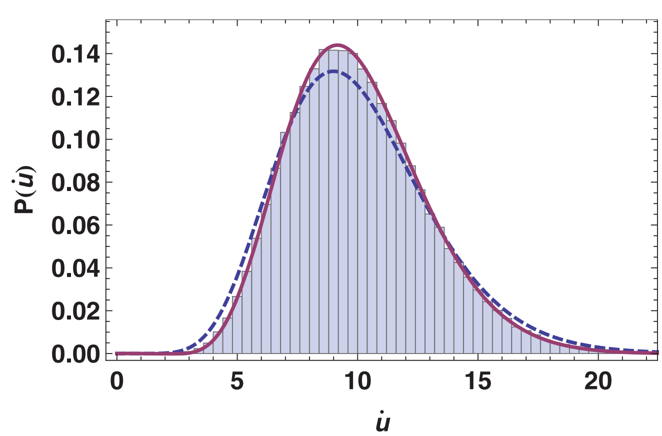

FIG. 3. (Color online) Single-particle velocity distribution $P(\dot{u})$ in the two-particle toy model for weak elasticity. Histogram: Numerical results from simulations for $k = 0.2$. Dashed line: Stationary distribution in the absence of elastic coupling ($k = 0$). Solid (red) line: $O(k)$ result from (62).

B. Continuum limit and nonrenormalization property

Let us now consider a d-dimensional interface in a $(d+1)$-dimensional medium with a generic elastic kernel g_{xy}, such that in Fourier space $g^{-1}_{q=0} = m^2$. Local elasticity corresponds to $g_q^{-1} = q^2 + m^2$. The corresponding generalization of (1) is

$$\partial_t u_{xt} = F(u_{xt},x) - \int_y g^{-1}_{xy} u_{yt} + \tilde{\lambda}_{xt}. \tag{63}$$

For the remainder of this section, we write function arguments as subscripts in order to simplify notations [i.e., $u_{xt} := u(x,t)$]. The source $\tilde{\lambda}_{xt} \geqslant 0$ for the field $\tilde{u}$ is a positive driving, and is related to the velocity of the center of the quadratic well $\dot{w}$ by $\tilde{\lambda}_{xt} = g^{-1}_{xx'}\dot{w}_{x't}$.

The pinning force is chosen Gaussian and uncorrelated in x,

$$\overline{F(u,x)F(u',x')} = \delta^d(x-x')\Delta(u,u'). \tag{64}$$

In the u direction, analogously to (2), we assume Brownian correlations, i.e., uncorrelated increments: $\partial_u\partial_{u'}\Delta(u,u') = \delta(u-u')$. This does not fix F uniquely, with, e.g., two possible explicit choices in (74) and (75) below. However, differences only arise for the position u but not for the velocity $\dot{u}$, as will be discussed below.

Let us write the MSR partition sum in the presence of sources,

$$G[\lambda,\tilde{\lambda}] = \int \mathcal{D}[\dot{u}]\mathcal{D}[\tilde{u}] e^{-S[\dot{u},\tilde{u}] + \int_{xt}\lambda_{xt}\dot{u}_{xt} + \int_{xt}\tilde{\lambda}_{xt}\tilde{u}_{xt}}.$$

The generalization of the MSR action (13) to this situation is

$$S[\dot{u},\tilde{u}] = \int_{xt} \tilde{u}_{xt}\left(\partial_t\dot{u}_{xt} + \int_y g^{-1}_{xy}\dot{u}_{yt} - \sigma\dot{u}_{xt}\tilde{u}_{xt}\right). \tag{65}$$

To arrive at (65), we have again assumed forward-only trajectories $\dot{u}_{xt} \geqslant 0$, guaranteed if $\tilde{\lambda}_{xt} \geqslant 0$ and $\dot{u}_{xt_i} \geqslant 0$ at some large negative initial time t_i.

The solution in Sec. II B generalizes straightforwardly to

$$G[\lambda,w] = \overline{e^{\int_{xt}\lambda_{xt}\dot{u}_{xt}}} = e^{\int_{xt}\tilde{u}^{(s)}_{xt}[\lambda]\tilde{\lambda}_{xt}}, \tag{66}$$

where $\tilde{u}^{(s)}[\lambda]$ is defined as the solution of

$$\partial_t\tilde{u}^{(s)}_{xt} - \int_y g^{-1}_{xy}\tilde{u}^{(s)}_{yt} + \sigma\left(\tilde{u}^{(s)}_{xt}\right)^2 = -\lambda_{xt}. \tag{67}$$

In principle, this can be used to compute any observable of the d-dimensional theory. In practice, Eq. (67) for $\tilde{u}$ is hard to solve analytically for most cases.

In the remainder of this section, instead of discussing specific examples, we show a conceptual consequence of (66): *The action* (65) *does not renormalize*. The effective action Γ is equal to the microscopic action S in any dimension d.

According to (66), the generating functional for connected graphs $W[\lambda,\tilde{\lambda}]$ evaluates to

$$W[\lambda,\tilde{\lambda}] = \ln G[\lambda,\tilde{\lambda}] = \int_{xt}\tilde{u}^{(s)}_{xt}[\lambda]\tilde{\lambda}_{xt}.$$

To perform the Legendre transform from W to the effective action Γ [50], we introduce new fields $\dot{u}_{xt}[\lambda,\tilde{\lambda}]$, and $\tilde{u}_{xt}[\lambda,\tilde{\lambda}]$, defined by

$$\tilde{u}_{xt} = \frac{\delta W[\lambda,\tilde{\lambda}]}{\delta\tilde{\lambda}_{xt}} = \tilde{u}^{(s)}_{xt}[\lambda], \tag{68}$$

$$\dot{u}_{xt} = \frac{\delta W[\lambda,\tilde{\lambda}]}{\delta\lambda_{xt}} = \int_{x't'}\frac{\delta\tilde{u}^{(s)}_{x't'}[\lambda]}{\delta\lambda_{xt}}\tilde{\lambda}_{x't'}. \tag{69}$$

Here and below we drop the functional dependence on the sources when no ambiguity arises. Equation (68) shows that $\tilde{u}^{(s)}_{xt}[\lambda]$ is really the field $\tilde{u}_{xt}$ appearing in the effective action, hence (67) allows to express the field λ_{xt} (on which W depends) in terms of $\tilde{u}_{xt}$ (on which Γ depends).

We can now write down the effective action $\Gamma[u,\tilde{u}]$:

$$\begin{aligned}\Gamma[\dot{u},\tilde{u}] &= \int_{xt}\dot{u}_{xt}\lambda_{xt} + \int_{xt}\tilde{u}_{xt}\tilde{\lambda}_{xt} - W\\ &= \int_{xt}\dot{u}_{xt}\lambda_{xt} \quad \text{since} \quad W = \int_{xt}\tilde{u}_{xt}\tilde{\lambda}_{xt}\\ &= -\int_{xt}\dot{u}_{xt}\left(\partial_t\tilde{u}_{xt} - \int_y g^{-1}_{xy}\tilde{u}_{yt} + \sigma\tilde{u}^2_{xt}\right)\\ &= \int_{xt}\tilde{u}_{xt}\left(\partial_t\dot{u}_{xt} + \int_y g^{-1}_{xy}\dot{u}_{yt} - \sigma\dot{u}_{xt}\tilde{u}_{xt}\right)\\ &= S[\dot{u},\tilde{u}].\end{aligned} \tag{70}$$

This is exactly the same as the bare action S in (65). This nonrenormalization of the action for the particle velocity in ABBM-like disorder is also consistent with a one-loop calculation using functional RG methods (see Appendix B). It is a very nontrivial statement, and shows that, *in some sense*, the MSR field theory for monotonous motion in ABBM-like disorder is exactly solvable in any dimension. The monotonicity assumption implies that the derivatives arising in the formulas above must be performed in the neighborhood of a strictly positive driving source $\tilde{\lambda}_{xt} > 0$. Using the relationship,

$$\dot{u}_{xt}[\lambda,\tilde{\lambda}] = \frac{\overline{\dot{u}_{xt}\exp[\int_{x't'}\lambda_{x't'}\dot{u}_{x't'}]}}{\overline{\exp[\int_{x't'}\lambda_{x't'}\dot{u}_{x't'}]}} \tag{71}$$

(where the average is performed in the presence of $\dot{w} = \tilde{\lambda}$), one sees that (69) maps positive $\tilde{\lambda}$ onto positive $\dot{u}$. On the other hand, the condition $\tilde{\lambda} \geqslant 0$ can be expressed using $\tilde{\lambda}_{xt} = \frac{\delta\Gamma}{\delta\tilde{u}_{xt}}$ as

$$\tilde{u}_{xt} \leqslant \frac{\partial_t\dot{u}_{xt} + \int_y g^{-1}_{xy}\dot{u}_{yt}}{2\sigma\dot{u}_{xt}}. \tag{72}$$

We conclude that the effective action $\Gamma[\dot{u},\tilde{u}]$ is given by the bare action S in the sector of the theory where $\dot{u} \geqslant 0$ and (72) holds as a necessary condition. This in no way implies that $\Gamma = S$ for values of the fields where this monotonicity assumption does not hold. The case of nonmonotonous motion and/or nonmonotonous driving is highly nontrivial and will be studied elsewhere.

In the following section, we shall see how this result generalizes to the field theory of the position $u(t)$, where the relationship between S and Γ is slightly more complicated.

V. FIELD THEORY FOR THE POSITION VARIABLE

So far, we have considered observables that can be expressed in terms of the ABBM velocity $\dot{u}(t)$, or in the case of a manifold $\dot{u}(x,t)$. Here we consider the position $u(x,t)$ itself. One can then formulate the MSR path integral in terms of u and $\hat{u}$, analogous to (12). This was done for a d-dimensional interface in short-ranged disorder in [24], as a starting point for a $d_c - d$ expansion. Here we focus on the simpler and solvable case of the ABBM model, where the MSR path integral reads

$$G[\lambda,w] = \overline{e^{\int_t \lambda(t)u(t)}} = \int \mathcal{D}[u,\hat{u}]e^{-S[u,\hat{u}]+\int_t \lambda(t)u(t)},$$
$$S[u,\hat{u}] = \int_t \hat{u}(t)\{\partial_t u(t) + m^2[u(t) - w(t)]\} - \frac{1}{2}\int_{t,t'} \Delta(u(t),u(t'))\hat{u}(t)\hat{u}(t'). \quad (73)$$

Here, $\Delta(u,u') = \overline{F(u)F(u')}$ is the disorder correlation function. One mathematically simple choice is to assume the random force $F(u)$ to be a one-sided Brownian motion and restrict to $u > 0$:

$$\Delta(u,u') = 2\sigma \min(u,u') = \sigma(u + u' - |u - u'|). \quad (74)$$

Another common choice is the two-sided version, i.e., a Brownian motion on the full real u axis pinned at $F(u = 0) = 0$. With either choice, however, the random force is nonstationary and one loses statistical translation invariance. This is unnatural for certain applications, for example approximating extended elastic interfaces above the critical dimension. In this context, one chooses a stationary variant of (74),

$$\Delta(u,u') = \Delta(u - u') = \Delta(0) - \sigma|u - u'|. \quad (75)$$

Since a stochastic process $F(u)$ can only satisfy (75) for all u in some limit, we always assume (75) to be regularized at large $|u - u'|$.

For observables that can be expressed in terms of the velocity $\dot{u}$, only $\partial_t\partial_{t'}\Delta(u(t),u(t'))$ enters the MSR action (cf. Sec. II B). Hence, choosing (74) or (75) yields the same result (13). However, the choice does matter if one is interested in observables depending on the position, like the mean pinning force $f_p := m^2\overline{[u(t) - w(t)]}$.

In contrast to the velocity theory discussed in the preceding sections, fixing a distribution of positions $u(t_i)$ as the initial condition is problematic. Indeed, in general one cannot exclude that this initial condition leads to backward motion $\dot{u}(t_i) < 0$ for some realizations of the disorder. Hence for the stationary Brownian landscape (75) we will choose $t_i = -\infty$ and assume that the driving $\dot{w}(t) \geqslant 0$ is such that at fixed times the initial condition is forgotten, as discussed in Sec. III C 1. We claim that then

$$G[\lambda,w] = \overline{e^{\int_t \lambda(t)u(t)}} = e^{m^2\int_t \hat{u}(t)w(t) + \frac{\Delta(0)}{2m^4}[\int_t \lambda(t)]^2}\left[1 - \frac{\sigma}{m^4}\int_t \lambda(t)\right], \quad (76)$$

where all time integrals are over $]-\infty,\infty[$. The function $\hat{u}(t) = -\partial_t\tilde{u}(t)$, where $\tilde{u}(t)$ is a solution of

$$\partial_t\tilde{u}(t) - m^2\tilde{u}(t) + \sigma\tilde{u}(t)^2 - \sigma\tilde{u}(t)\tilde{u}(-\infty) = -\int_{t'>t}\lambda(t'). \quad (77)$$

In the particular case of the one-sided Brownian landscape (74), we only consider the initial condition $u(t_i) = 0$. Since $F(0) = 0$ in that case, for $w(t_i) \geqslant 0$ and $\dot{w}(t) \geqslant 0$ the motion will be forward. Then the generating function $G[\lambda,w]$ in (73) takes a form analogous to (6),

$$G[\lambda,w] = \overline{e^{\int_{t>t_i}\lambda(t)u(t)}} = e^{m^2\int_{t>t_i}\hat{u}(t)w(t)}, \quad (78)$$

where $\hat{u}(t) = -\partial_t\tilde{u}(t)$ and $\tilde{u}(t)$ is a solution of (5). In the remainder of this section, we shall prove the above statements and then apply these formulas to determine the distribution of the single-time particle position $u(t)$.

A. Generating functional for stationary Brownian potential

Using the assumption of monotonous motion, the disorder term in the action (73) can be rewritten as

$$\frac{1}{2}\int_{t,t'} \Delta(u(t),u(t'))\hat{u}(t)\hat{u}(t') = \frac{\Delta(0)}{2}\left[\int_t \hat{u}(t)\right]^2 - \sigma\int_{t,t'} u(t)\hat{u}(t)\hat{u}(t')\text{sgn}(t - t').$$

Following the same approach as in Sec. II B, evaluating the path integral over $u(t)$ in (73) yields

$$\int \mathcal{D}[\hat{u}]e^{m^2\int_t \hat{u}(t)w(t) + \frac{\Delta(0)}{2}[\int_t \hat{u}(t)]^2}\delta\bigg(\partial_t\hat{u}(t) - m^2\hat{u}(t) + \sigma\hat{u}(t)\int_{t'}\hat{u}(t')\text{sgn}(t'-t) + \lambda(t)\bigg). \quad (79)$$

Thus

$$G[\lambda,w] = \mathcal{N}e^{m^2\int_t \hat{u}(t)w(t) + \frac{\Delta(0)}{2}[\int_t \hat{u}(t)]^2}, \quad (80)$$

where $\hat{u}(t)$ is a solution to the equation

$$\partial_t\hat{u}(t) - m^2\hat{u}(t) + \sigma\hat{u}(t)\int_{t'}\hat{u}(t')\text{sgn}(t' - t) = -\lambda(t). \quad (81)$$

Substituting $\tilde{u}(t) := \int_t^\infty \hat{u}(t)\text{d}t$, one recovers (77). $\tilde{u}(-\infty)$ is obtained from

$$-m^2\int_{-\infty}^{\infty}\hat{u}(t) = -m^2\tilde{u}(-\infty) = -\int_{-\infty}^{\infty}\lambda(t')\,\text{d}t'. \quad (82)$$

Note that $\tilde{u}(-\infty)$ vanishes for λ such that $\int_{t'}\lambda(t') = 0$. These are exactly those observables which can be expressed in terms of the velocity (or, equivalently, position differences).

As in Sec. II, $\mathcal{N}$ in (80) is the normalization of the path integral and the Jacobian of the operator inside the δ functional in (79). It is independent of $w(t)$, but we cannot fix its value at $w(t) = \text{const}$ as we did for the velocity theory in Sec. II: Even if one keeps $w = \text{const}$ for a long time, the distribution of u will remain nontrivial [unlike the distribution of $\dot{u}$, which will become $\delta(\dot{u})$]. Here, to fix $\mathcal{N}$ we compare to the disorder-free solution ($\sigma = 0$) for which the trajectory $u(t)$ is deterministic and satisfies (80) with $\mathcal{N} = 1$. Hence, we can write $\mathcal{N}$ as a ratio of functional determinants arising from the δ-functional,

$$\mathcal{N}^{-1} = \frac{\det(\partial_t - m^2 - \Sigma^T)}{\det(\partial_t - m^2)} = \det(1 + R\Sigma). \quad (83)$$

Here, R is the disorder-free propagator

$$R := (\partial_t + m^2)^{-1} \Rightarrow R_{t_1,t_2} = \theta(t_1 - t_2)e^{-m^2(t_1-t_2)}, \tag{84}$$

and Σ is the disorder "interaction" term, or "self-energy"

$$\Sigma^T_{t_2,t_1} = \Sigma_{t_1,t_2} = \sigma\delta(t_1 - t_2)\int_{t'} \hat{u}(t')\mathrm{sgn}(t_1 - t') + \sigma\hat{u}(t_2)\mathrm{sgn}(t_2 - t_1). \tag{85}$$

By explicit computation (see Appendix B), one verifies that

$$\mathrm{tr}\,(R\Sigma)^n = -\left[-\frac{\sigma}{m^2}\int_t \hat{u}(t)\right]^n,$$

and hence

$$\det(1 + R\Sigma) = \exp \mathrm{tr} \ln(1 + R\Sigma) = \left(1 - \frac{\sigma}{m^2}\int_t \hat{u}(t)\right)^{-1}.$$

From (81), one further knows that $\int_t \hat{u}(t) = \frac{1}{m^2}\int_t \lambda(t)$.

In total, this proves the expression (76) for the stationary case,

$$G[\lambda,w] = e^{m^2\int_t \hat{u}(t)w(t) + \frac{\Delta(0)}{2m^4}[\int_t \lambda(t)]^2}\left[1 - \frac{\sigma}{m^4}\int_t \lambda(t)\right]. \tag{86}$$

One sees again that for observables expressed in terms of the velocity, where $\int_t \lambda(t) = 0$, the simpler expression (6) is recovered.

In the language of perturbative field theory, the nontrivial functional determinant signifies nonvanishing one-loop diagrams [51]. This is in contrast to the theory for the velocity (Sec. II B), where all observables were given by tree-level diagrams. These loop corrections mean that the nonrenormalization property discussed in Sec. IV B has to be amended when considering the particle position in a stationary potential. After renaming the driving w to $\hat{\lambda} = m^2 w$, the source for the field $\hat{u}$, the generating functional for connected correlation functions becomes

$$W[\lambda,\hat{\lambda}] = \int_t \hat{u}_t[\lambda]\hat{\lambda}_t + \frac{\Delta(0)}{2m^4}\left(\int_t \lambda_t\right)^2 + \ln\left(1 - \frac{\sigma}{m^4}\int_t \lambda_t\right),$$

where $\hat{u}_t[\lambda]$ is a solution of (81). Following the same procedure as in Sec. IV B, one obtains the effective action

$$\begin{aligned}\Gamma[u,\hat{u}] &= \int_t u_t\left[-\partial_t\hat{u}(t) + m^2\hat{u}(t) - \sigma\hat{u}(t)\int_{t'}\hat{u}(t')\mathrm{sgn}(t' - t)\right] \\ &\quad - \frac{\Delta(0)}{2}\left[\int_t \hat{u}(t)\right]^2 - \ln\left[1 - \frac{\sigma}{m^2}\int_t \hat{u}(t)\right] \\ &= S[u,\hat{u}] - \ln\left[1 - \frac{\sigma}{m^2}\int_t \hat{u}(t)\right].\end{aligned} \tag{87}$$

We thus see that the property $\Gamma = S$ seen for the velocity theory is only changed by a simple contribution from the one-loop corrections. The equal-time part of the $\hat{u}^n$ term of these loop corrections coincides with a previous result in [52].

In fact, this calculation can be extended to the d-dimensional interface with elastic kernel g_q of Sec. IV B. There too it ensures that for the position theory, and monotonous driving, Γ differs from S only via the logarithm of a (one-loop) functional determinant. Thus, two- and higher-loop corrections to correlation functions and the effective action vanish. Its expression is particularly simple in the case of a uniform $\lambda_{xt} = \lambda(t)$ leading to a uniform saddle point $\hat{u}_{xt} = \hat{u}(t)$:

$$\begin{aligned}W_{\text{one-loop}} &= L^d\int\frac{\mathrm{d}^d q}{(2\pi)^d}\ln\left[1 - \frac{\sigma g_q}{m^2}\int_t \lambda(t)\right], \\ \Gamma - S|_{\text{uniform }\hat{u}} &= L^d\int\frac{\mathrm{d}^d q}{(2\pi)^d}\ln\left[1 - \sigma g_q\int_t \hat{u}(t)\right].\end{aligned} \tag{88}$$

L^d is the volume of the system. Details and a more general discussion are given in Appendix B, Appendix C, and [24].

B. One-sided Brownian potential

It is instructive to give for comparison the solution for the simpler case of the correlator (74). Using the assumption of monotonous motion, the disorder term in the action (73) can be rewritten as

$$\sigma\int_{t,t'}\min(u(t),u(t'))\hat{u}(t)\hat{u}(t') = 2\sigma\int_t u(t)\hat{u}(t)\int_{t'>t}\hat{u}(t'). \tag{89}$$

Following the same approach as in Sec. II B, evaluating the path integral over $u(t)$ with initial condition $u(t_i) = 0$ in (73) yields Eq. (78), where $\hat{u}(t)$ is a solution to the equation

$$\partial_t\hat{u}(t) - m^2\hat{u}(t) + 2\sigma\hat{u}(t)\int_{t'>t}\hat{u}(t') = -\lambda(t). \tag{90}$$

Note that as in Sec. II B, the initial condition $u(t_i) = 0$ ensures that $G[\lambda, w = 0] = 1$. Hence the functional determinant analogous to (83) is equal to 1 in this case. This is also checked by a direct calculation in Appendix B. For $\lambda(t)$ nonvanishing only around $t \gg t_i$ and $w(t) \gg w(t_i)$, we expect that the influence of the initial condition is negligible. In this particular limit, (78) should hold independently of the initial condition.

Introducing $\tilde{u}(t) := \int_{t'>t}\hat{u}(t')$, (90) gives the following equation for $\tilde{u}(t)$:

$$\partial_t\tilde{u}(t) - m^2\tilde{u}(t) + \sigma\tilde{u}(t)^2 = -\int_{t'>t}\lambda(t'), \tag{91}$$

where we used that $\tilde{u}(t) \to 0$ for $t \to +\infty$ [we recall that $\hat{u}(t)$ must vanish at both $\pm\infty$].

C. Example: Single-time position distribution

To give a simple application of (76), we compute the distribution of the position $u(t)$ at a single time. To do this, set $\lambda(t) = \lambda\delta(t - t_0)$ in (73). For the Brownian case, one obtains

$$\hat{u}(t) = \frac{\lambda(1 - 4\lambda)\theta(t_0 - t)}{\left\{\sinh\left[\frac{\sqrt{1-4\lambda}(t-t_0)}{2}\right] - \sqrt{1 - 4\lambda}\cosh\left[\frac{\sqrt{1-4\lambda}(t-t_0)}{2}\right]\right\}^2}.$$

For the stationary case (77), $\hat{u}(t)$ reads

$$\hat{u}(t) = \frac{\lambda\,(1 - \lambda)^2\,e^{-(t-t_0)(1-\lambda)}\theta(t_0 - t)}{[e^{-(t-t_0)(1-\lambda)} - \lambda]^2}.$$

In both cases, the θ functions come from causality, since the driving $w(t)$ for $t > t_0$ cannot influence the measured position $u(t_0)$. Hence both $\tilde{u}(t)$ and $\hat{u}(t) = -\partial_t\tilde{u}(t)$ must both be identically zero for $t > t_0$.

Let us assume a constant driving velocity, and write $w(t) = v(t - t_i) + w_i$. Then, for the one-sided Brownian with $u(t_i) = 0$ and $w_i \geqslant 0$, we have

$$G(\lambda) = \overline{e^{\lambda u_{t_0}}} = e^{\int_{t_i}^{t_0} dt' \hat{u}(t') v t'}.$$

This leads to a complicated formula that simplifies in the limit $t_i \to -\infty$ at fixed $w(t_0)$,

$$G(\lambda) = \left(\frac{-2\lambda}{1 - 4\lambda - \sqrt{1 - 4\lambda}} \right)^{-v} e^{\frac{w(t_0)}{2}(1-\sqrt{1-4\lambda})}. \tag{92}$$

For the stationary case (restoring units), this is

$$\begin{aligned} G(\lambda) &= e^{m^2 \int_{t'} \hat{u}(t') v t' \, dt' + \frac{\Delta(0)}{2m^4}\lambda^2} \left(1 - \frac{\sigma}{m^4}\lambda\right) \\ &= \left(1 - \frac{\sigma}{m^4}\lambda\right)^{\frac{m^2 v}{\sigma}+1} e^{\lambda v t_0 + \frac{\Delta(0)}{2m^4}\lambda^2}. \end{aligned}$$

Inverting gives a valid distribution only for $|u_{t_0} - v t_0| \ll \sigma/\Delta(0)$, which coincides with the cutoff, which should be used to regularize the stationary Brownian landscape (75).

VI. GENERALIZATIONS

In light of the interesting results obtained for (1), it is natural to ask whether our approach can be extended. In particular, one might want to replace the response function in (1) by a more general response kernel. For example, in order to model eddy currents which change the avalanche shape in real magnets [3,53], one may want to include second-order derivatives in time.

For this, it is useful to view the calculation in Sec. II B from another perspective. Equation (5) for $\tilde{u}$ is identical to the saddle-point equation obtained from the action (13) in the presence of the source λ by taking a functional derivative with respect to $\dot{u}(t)$. The result (6) is then the value of Z at the saddle point obtained by solving (5) for the given choice of λ. The other "coordinate" of the saddle point (which happens not to influence the value of Z in this case, however) is the field $\dot{u}(t)$, fixed by the equation obtained by a functional derivative of (13) with respect to $\tilde{u}(t)$,

$$\partial_t \dot{u}(t) + m^2 [\dot{u}(t) - \dot{w}(t)] - 2\sigma \dot{u}(t)\tilde{u}(t) = 0. \tag{93}$$

This is the trajectory giving the dominant contribution to Z for a given choice of λ. For example, for $\lambda(t) = \lambda\delta(t - t_0)$, $\tilde{u}(t)$ is given by (14); for $w(t) = vt$, the solution of (93) converging to v at infinity then reads

$$\dot{u}(t) = v\left(1 + \frac{\lambda}{1-\lambda} e^{-|t-t_0|}\right).$$

Note that it can also be obtained from the two-time generating function (18), e.g., for $t > t_0$ as $\dot{u}(t) = \partial_{\lambda_2} \ln G(\lambda_1 = \lambda, \lambda_2)|_{\lambda_2=0, t_2=t, t_1=t_0}$. Indeed, since $S = \Gamma$ for monotonous motion, the solution of (93) identifies with (71), i.e., the saddle-point approximation is exact. We thus see, as expected, that if we concentrate on small velocities ($\lambda \to -\infty$), the velocity on the dominant trajectory $\dot{u}(t)$ gets closer and closer to 0 at t_0, but never becomes negative.

Now, the action S generalizing (13) with an arbitrary response kernel $R_{tt'}$ is

$$S[\dot{u},\tilde{u}] = \int_t \left\{ \tilde{u}(t) \left[\int_{t'} R^{-1}_{tt'} \dot{u}(t') - m^2 \dot{w}(t) \right] - \sigma \dot{u}(t)\tilde{u}(t)^2 \right\}. \tag{94}$$

The saddle-point equations read

$$\begin{aligned} \int_{t'} R^{-1,T}_{tt'} \tilde{u}(t') - \sigma \tilde{u}(t)^2 - \lambda(t) = 0, \\ \int_{t'} R^{-1}_{tt'} \dot{u}(t') - 2\sigma \dot{u}(t)\tilde{u}(t) - m^2 \dot{w}(t) = 0. \end{aligned} \tag{95}$$

For a general (bare) response function R, the last term in the action (94) is not exact, since we cannot assume monotonicity of each individual trajectory. However, as long as the saddle-point trajectory defined by (95) for some choice of λ is monotonous [i.e., satisfies $\dot{u}(t) \geqslant 0$ for all t], it gives a well-defined approximation to the value of Z for this particular λ. Investigating the quality of this approximation is an interesting avenue for further research.

VII. SUMMARY AND OUTLOOK

In this paper, we have considered the ABBM model with a monotonous, but nonstationary driving force. Using the Martin-Siggia-Rose formalism, we obtained the generating functional for the velocity from a field theory that can be solved exactly. This was illustrated on several paradigmatic examples (e.g., a quench in the driving velocity). Using our formalism, we also succinctly recovered previous results on the stationary case.

An interesting direction for further research is trying to generalize these results to nonstationary dynamics of models which are not mean-field in nature, like d-dimensional elastic interfaces. Although some work has been done in that direction [54–57], many questions remain open. Another complication arises when adding nonlinear terms to the equation of motion (1) or (63). The effects of the KPZ term $[\nabla u(x)]^2$ have been discussed in [58–60]. An analogous term but with a time instead of a space derivative, i.e., a term $\dot{u}^2$, is related to dissipation of energy [61] and yields a toy model with velocity-dependent friction. This is important as a step toward realistic earthquake models, where it is known that instead of a constant friction coefficient, one has a complicated rate-and-state friction law [12–14]. For the hysteresis loop in the ABBM model, it would be interesting to extend our results to the case of nonmonotonous driving. Unfortunately, this is not an easy task: We crucially used both the monotonicity of the particle velocity, $\dot{u}(t) \geqslant 0$, and the one of the driving, $\dot{w}(t) \geqslant 0$, to simplify the action and compute the path integral in Sec. II B. Without this assumption, neither the result (6) nor the nonrenormalization property in Sec. IV B holds. Assuming the nonrenormalization property, the mean velocity $\overline{\dot{u}(t)}$ would be equal to its value in the system without disorder at all times. This can be seen, e.g., by taking ∂_λ at $\lambda = 0$ in formula (3) and using (6) and (14). However, in numerical simulations one observes that this property breaks down as soon as the driving is nonmonotonous, hence at least the term proportional to $\tilde{u}$ in the effective action is renormalized. We thus leave questions in this direction for future studies.

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APPENDIX A: DERIVATION OF THE NONSTATIONARY SOLUTION IN DISCRETIZED TIME

The path-integral derivation of (6) in Sec. II is, to some extent, formal and neglects subtleties like convergence issues and boundary conditions. To complement it, we provide here a rigorous first-principles derivation of (6) by discretizing the time axis. For a small time step δt, we write (7) as follows:

$$\frac{\dot{u}_{j+1}-\dot{u}_j}{\delta t} = \frac{F(u_j+\delta t\dot{u}_{j+1})-F(u_j)}{\delta t} + m^2(\dot{w}_{j+1}-\dot{u}_{j+1})$$
$$\Rightarrow \dot{u}_{j+1} = X(\dot{u}_{j+1}) + km^2\delta t\dot{w}_{j+1} + k\dot{u}_j, \tag{A1}$$

with $k^{-1} := 1+m^2\delta t$.

$X(\dot{u}_{j+1}) := k[F(u_j+\delta t\dot{u}_{j+1})-F(u_j)]$ is, by the Markov property of Brownian motion, a new Brownian motion with $X(0)=0$ and variance $\overline{X(\dot{u})X(\dot{u}')} = 2\sigma k^2\delta t\min(\dot{u},\dot{u}')$. Equation (A1) is an implicit equation for $\dot{u}_{j+1}$, which has, in general, several solutions $\dot{u}_{j+1}>0$. In fact, its solutions are the intersections of the Brownian motion $X(\dot{u}_{j+1})$ with the line $km^2\delta t\dot{w}_{j+1}+k\dot{u}_j-\dot{u}_{j+1}$. The true $\dot{u}_{j+1}$ describing the motion of the particle is the smallest of these solutions.

Hence, the conditional probability distribution for $\dot{u}_{j+1}$ given $\dot{u}_j$ is the first-passage distribution of Brownian motion, given by

$$P(\dot{u}_{j+1}|\dot{u}_j) = \frac{km^2\delta t\dot{w}_{j+1}+k\dot{u}_j}{\sqrt{4\pi\sigma k^2\delta t}\,\dot{u}_{j+1}^{\frac{3}{2}}}e^{-\frac{(\dot{u}_{j+1}-km^2\delta t\dot{w}_{j+1}-k\dot{u}_j)^2}{4\sigma k^2\delta t\dot{u}_{j+1}}}. \tag{A2}$$

The Laplace transform of this expression, which is the conditional expectation value for $e^{\tilde{u}\dot{u}_{j+1}}$, is given by

$$E(e^{\tilde{u}\dot{u}_{j+1}}|\dot{u}_j) := \int_0^\infty e^{\tilde{u}\dot{u}_{j+1}}P(\dot{u}_{j+1}|\dot{u}_j)\mathrm{d}\dot{u}_{j+1}$$
$$= e^{\frac{\dot{w}_{j+1}m^2\delta t+\dot{u}_j}{2\sigma k\delta t}(1-\sqrt{1-4\tilde{u}\sigma k^2\delta t})}. \tag{A3}$$

This can be rewritten as

$$E(e^{\tilde{u}\dot{u}_{j+1}}|\dot{u}_j) = e^{m^2\tilde{u}'\dot{w}_{j+1}\delta t}e^{\tilde{u}'\dot{u}_j}, \tag{A4}$$

with $\tilde{u}' = \frac{1}{2\sigma k\delta t}(1-\sqrt{1-4\tilde{u}\sigma k^2\delta t})$. Hence, iterating (A4), one obtains

$$\overline{e^{\sum_{j=1}^N \lambda_j\dot{u}_j\delta t}} = e^{m^2\sum_{j=1}^N\tilde{u}_j\dot{w}_j\delta t}\overline{e^{\tilde{u}_1\dot{u}_0}}, \tag{A5}$$

where $\tilde{u}_j$ is defined via the (backward) recursion

$$\tilde{u}_{N+1} = 0,$$
$$\tilde{u}_j = \frac{1-\sqrt{1-4(\tilde{u}_{j+1}\delta t+\lambda_j\delta t^2)\sigma k^2}}{2\sigma k\delta t}, \quad 0<j\leqslant N. \tag{A6}$$

This is the exact solution for the discrete problem with $\delta t>0$. In the continuum limit, we can take the leading order as $\delta t\to 0$. (A5) then reduces to the form (4). The recursion for $\tilde{u}$ becomes

$$\frac{\tilde{u}_j-\tilde{u}_{j+1}}{\delta t} = -m^2\tilde{u}_{j+1}+\lambda_j+\sigma\tilde{u}_{j+1}^2+O(\delta t), \tag{A7}$$

which is the discrete version of (5).

Let us now show the connection with the MSR path integral discussed in Sec. II B. We discretize the action (13) with time step δt using the Itô prescription. Keeping $\dot{u}_j$ fixed, the path-integral formula (12) for the generating function (3) gives us the generating function for $\dot{u}_{j+1}$ as

$$E(e^{\lambda\dot{u}_{j+1}}|\dot{u}_j) = \int_{-\infty}^\infty \mathrm{d}\dot{u}_{j+1}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}\tilde{u}_{j+1}}{2\pi}$$
$$\times e^{-\tilde{u}_{j+1}[\frac{\dot{u}_{j+1}-\dot{u}_j}{\delta t}+m^2(\dot{u}_j-\dot{w}_j)]\delta t+\tilde{u}_{j+1}^2\sigma\dot{u}_j\delta t+\lambda\dot{u}_{j+1}}. \tag{A8}$$

The integrals over $\tilde{u}_{j+1}$ and $\dot{u}_{j+1}$ can be performed explicitly, and yield (taking into account $\delta t>0,\sigma>0,\dot{w}\geqslant 0$, and $\dot{u}_j>0$)

$$E(e^{\lambda\dot{u}_{j+1}}|\dot{u}_j) = \exp[(\lambda-m^2\delta t+\sigma\lambda^2\delta t)\dot{u}_j+\lambda m^2\delta t\dot{w}_j]. \tag{A9}$$

To leading order for $\delta t\to 0$ and substituting $\lambda\to\tilde{u}$, this becomes identical to the generating function (A3). Note that while the first-passage prescription used to obtain (A3) assumed $\dot{u}_{j+1}\geqslant 0$, in (A8) we formally allow the velocity $\dot{u}_{j+1}$ to take any value between $-\infty$ and ∞. Surprisingly, this yields the same result to leading order in δt. It would be interesting to understand how a more rigorous MSR approach could be developed directly on the discrete version for finite δt using first-passage times.

Analogously, one can derive a discretized path integral for the position variable u for the one-sided Brownian potential discussed in Sec. V B.

APPENDIX B: FUNCTIONAL DETERMINANTS AND ONE-LOOP DIAGRAMS

Here we compute $\mathrm{tr}(R\Sigma)^n$, where R is given in (84) and Σ in (85). For simplicity, we set $\sigma=m=1$. Let us recall that in Ito discretization, $\theta(0)=0$. First, note that

$$(R^T\Sigma^T)_{t_1,t_2} = \int_{t'}\hat{u}(t')\mathrm{sgn}(t'-t_2)$$
$$\times[\theta(t'-t_1)e^{-(t'-t_1)}-\theta(t_2-t_1)e^{-(t_2-t_1)}].$$

Applying this to $\mathrm{tr}(R\Sigma)^n = \mathrm{tr}(R^T\Sigma^T)^n$, one gets

$$\mathrm{tr}(R\Sigma)^n = \int_{t_1'\cdots t_n'}\hat{u}(t_1')\cdots\hat{u}(t_n')\int_{t_1\cdots t_n}\prod_{j=1}^n\mathrm{sgn}(t_j'-t_j)$$
$$\times[\theta(t_{j+1}'-t_j)e^{-(t_{j+1}'-t_j)}-\theta(t_{j+1}-t_j)e^{-(t_{j+1}-t_j)}].$$

The convention is that $t_{n+1}=t_1$ and $t_{n+1}'=t_1'$. Now, we conjecture that for any $t_1'\cdots t_n'$,

$$\int_{t_1\cdots t_n}\prod_{j=1}^n\mathrm{sgn}(t_j'-t_j)[\theta(t_{j+1}'-t_j)e^{-(t_{j+1}'-t_j)}$$
$$-\theta(t_{j+1}-t_j)e^{-(t_{j+1}-t_j)}] = (-1)^{n+1}. \tag{B1}$$

We were unable to find an analytic proof, but verified this conjecture for $n \leqslant 5$. Assuming it for any n, one obtains as claimed

$$\mathrm{tr}(R\Sigma)^n = -\left[-\int_t \hat{u}(t)\right]^n.$$

For the one-sided Brownian correlator (74), we find the self-energy analogous to (85) as

$$\Sigma_{t_1,t_2} = -2\delta(t_1-t_2)\int_{t'} \hat{u}(t')\theta(t'-t_2) - 2\hat{u}(t_2)\theta(t_1-t_2). \tag{B2}$$

This implies

$$\begin{aligned}(R^T\Sigma^T)_{t_1,t_2} = -2\int_{t'} \hat{u}(t')[\theta(t_2-t')\theta(t'-t_1)e^{-(t'-t_1)} \\ +\theta(t'-t_2)\theta(t_2-t_1)e^{-(t_2-t_1)}].\end{aligned}$$

One then finds $\mathrm{tr}(R\Sigma)^n = 0$ for $n \geqslant 1$, hence a unit functional determinant as claimed in the text.

This can be generalized to the d-dimensional interface. We need to compute the functional determinant $\det(1+R\Sigma)$ with

$$R^{-1}_{x_1t_1,x_2t_2} = \delta_{t_1t_2}(\partial_{t_2}\delta_{x_1x_2} + g_{x_1x_2}), \tag{B3}$$

$$\Sigma_{x_1t_1,x_2t_2} = \delta_{x_1x_2}\left[\sigma\delta_{t_1t_2}\int_{t'} \hat{u}_{xt'}\mathrm{sgn}(t_1-t') + \sigma\hat{u}_{xt_2}\mathrm{sgn}(t_2-t_1)\right]. \tag{B4}$$

We conjecture that this yields

$$\begin{aligned}\ln\det(1+R\Sigma) &= \mathrm{tr}\ln\left(\delta_{xx'} - \sigma g_{xx'}\int_t \hat{u}_{x't}\right) \\ &= \mathrm{tr}\ln\left[\delta_{xx'} - \sigma g_{xx'}\int_y g_{x'y}\int_t \lambda_{yt}\right].\end{aligned} \tag{B5}$$

For the last equality, we used $\int_t \hat{u}_{xt} = g_{xx'}\int_t \lambda_{x't}$. For a uniform source, one recovers the expression in the text of Sec. V A.

APPENDIX C: ONE-LOOP FUNCTIONAL RG AT FINITE VELOCITY

In [62], the one-loop functional RG equations for a d-dimensional elastic interface at nonzero driving velocity $v > 0$ were derived in the Wilson RG scheme. These equations have resisted analytical (or numerical) solution since then. Here, instead of using Wilson RG with a hard cutoff in momentum space, we regularize our model by a parabolic well with curvature m^2. We point out that the stationary ABBM disorder correlator (64) and (75) yields a simple solution of the corresponding functional RG equations. This also provides an independent check of the nonrenormalization property for ABBM disorder discussed in Sec. IV B using a different method.

For a d-dimensional interface driven by a parabolic well of curvature m^2 centered at $w = vt$, one can derive the functional RG flow equation by computing $-m\partial_m\Gamma$ and reexpressing it as a function of Γ. This is done order by order in Δ, which in this appendix denotes the *renormalized* second cumulant of the disorder (the local part of the term $\hat{u}\hat{u}$ in Γ). The resulting functional RG flow of Δ at finite driving velocity v is [63]

$$\begin{aligned}-m\partial_m\tilde{\Delta}(u) \\ = (\epsilon-2\zeta)\tilde{\Delta}(u) + \zeta u\tilde{\Delta}'(u) + \int_0^\infty ds_1\int_0^\infty ds_2 \frac{e^{-(s_1+s_2)}}{s_1+s_2} \\ \times\{\tilde{\Delta}''(u)[\tilde{\Delta}[\tilde{v}(s_2-s_1)] - \tilde{\Delta}[u+\tilde{v}(s_2-s_1)]] \\ - \tilde{\Delta}'(u+\tilde{v}s_1)\tilde{\Delta}'(u-\tilde{v}s_2) + \tilde{\Delta}'[\tilde{v}(s_1+s_2)] \\ \times[\tilde{\Delta}'(u-\tilde{v}s_1) - \tilde{\Delta}'(u+\tilde{v}s_2)]\}.\end{aligned} \tag{C1}$$

Here $\epsilon = 4-d$, the rescaled correlator is defined via $\Delta(u) = A_d m^{\epsilon-2\zeta}\tilde{\Delta}(um^\zeta)$ with $A_d^{-1} = \epsilon\int\frac{d^dk}{(2\pi)^d}(1+k^2)^{-2}$, and $\tilde{v} = \eta_m v/m^{2-\zeta}$ flows as

$$-m\partial_m\ln\tilde{v} = z-\zeta = 2-\zeta-\int_{s>0} e^{-s}\tilde{\Delta}''(s\tilde{v}). \tag{C2}$$

The flow of $\tilde{v}$ arises because the friction is corrected by disorder. In general, this leads to a nontrivial dynamical exponent z defined by the relation above. For $v \to 0$, one recovers the flow at the depinning threshold obtained in [42]. These equations are sufficient [64] for an expansion in ϵ with $\tilde{\Delta} = O(\epsilon)$.

Plugging in the correlator for ABBM-type disorder, $\tilde{\Delta}(u) = \tilde{\Delta}(0) - \tilde{\sigma}|u|$, and $\zeta = \epsilon$ into (C1), one finds

$$-m\partial_m\Delta(u) = -\tilde{\sigma}^2, \tag{C3}$$

$$-m\partial_m\tilde{v} = z-\zeta = 2-\epsilon. \tag{C4}$$

We see that the dynamical exponent z for ABBM-type disorder takes the value $z = 2$ in any dimension d. The ABBM form of the disorder is preserved with $-m\partial_m\tilde{\sigma} = 0$ and only $\tilde{\Delta}(0)$ flowing as $-m\partial_m\tilde{\Delta}(0) = -\tilde{\sigma}^2$. This is consistent (for $d = 0$) with Eq. (76). In addition, as discussed in Sec. V and Appendix B, two- and higher-loop corrections vanish in any d for monotonous motion in ABBM-type disorder. More precisely, $\Gamma - S$ is the logarithm of a functional determinant computed in Sec. V. This shows that for ABBM-type disorder, (C1) is exact to all orders in $\epsilon = 4-d$.

We note that for ABBM disorder the correlator remains nonanalytic for any v [65]. This is, presumably, a peculiarity of ABBM disorder. For short-ranged disorder this may only hold until some scale, the nonanalyticity being rounded at larger scales (small m). However, further studies are needed to clarify the validity of this hypothesis.

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