

Making complex CFTs real: The two-dimensional Potts model for $Q > 4$ and complex Q

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The two-dimensional Q -state Potts model with real couplings has a first-order transition for $Q > 4$. Starting from a triangular-lattice Potts model with two- and three-spin interactions, we study an equivalent loop model in which Q is a continuous parameter. By a combination of analytical and numerical arguments, we show that this loop model allows for the collision of a critical and a tricritical fixed point at $Q = 4$. These then emerge as a pair of complex conformally invariant theories at $Q > 4$, or even complex Q , for suitable complex coupling constants. We conjecture that all conformal data (such as the central charge, critical exponents, and three-point structure constants) can be obtained by analytic continuation of known exact results for the loop model with $Q \leq 4$. This conjecture is checked, both for real $Q > 4$ and for $Q \in \mathbb{C}$, by extensive transfer-matrix computations and comparison to previous studies for $Q = 5$.

I. INTRODUCTION

Second-order phase transitions in statistical-physics models are signaled by the divergence of thermodynamic quantities, such as the free energy or the specific heat, when a control parameter (usually the temperature) is taken to a suitable critical value. Accordingly, the correlation length associated with the two-point function of a local observable, such as the spin or the energy, diverges at the transition point, and the generic exponential decay of correlations changes to a power-law behavior characterized by universal, critical exponents.

From a more mathematical perspective, this is a manifestation of non-analytic behavior of the thermodynamical quantities with respect to the control parameter. In a classical paper, Lee and Yang [1] initiated the fruitful idea of studying second-order phase transitions from the perspective of complex analysis. Specifically, they studied the free energy of the two-dimensional Ising model in a complex magnetic field. Fisher [2] later undertook a similar study at zero field, where the temperature takes complex values. In these works, the complex plane of the relevant parameter is partitioned into distinct regions, or phases. In each of these regions, the dominant contribution to the free energy takes a different analytical expression. Complex analysis then implies that the phase boundaries take the form of loci across which these expressions cannot be continued analytically. The interest of this approach is that knowledge about the physical model can ultimately be inferred by constraining to real, physical values of the relevant parameter.

This general scenario has been delineated for a multitude of different models, thermodynamical quantities, and control parameters. In particular, in a series of studies [3] the phase transitions in the two-dimensional Q -state Potts model were investigated in the complex Q -plane. In the present paper we also study the complex- Q Potts model in two dimensions, but from a different perspective, where the phase transitions are followed by analytic continuation throughout the complex plane. We pay special attention to the role of conformal field theory (CFT) in the description of the second-order phase transitions, and study the properties of these CFTs by analytic

continuation.

Before moving to field theory, let us start by recalling the lattice definition of the Q -state Potts model. For any (complex) value of Q , its partition function can be reformulated as a Fortuin-Kasteleyn (FK) model [4], in which each lattice bond is considered occupied or empty with a probability related to the temperature. The occupied bonds then form connected components, known as FK clusters, each of which occur with a weight Q . An important special case is bond percolation, which arises in the limit of $Q \rightarrow 1$. In contrast to the usual definition in terms of Q -component spins, the FK formulation permits us to study the Potts model for complex Q .

An additional advantage of the FK formulation, which has attracted much interest since the 1980s, stems from the fact that its correlation functions are related to probability measures in random geometry. Indeed, in the continuum limit, the FK clusters themselves, as well as related geometrical objects (hulls, external perimeters, backbones, shortest paths *etc.*), form scale- and conformally invariant fractals at the critical temperature. In probability theory, starting around the year 2000, these fractals are studied in their own right using rigorous methods, complementary to CFT (see [5–7] and references therein).

Many models, including the Ising model referred to above, harbor a single or a finite number of second-order phase transitions (or universality classes), each one described by a CFT with a symmetry extending that of the discrete model ($\mathbb{Z}_2$ for the case of Ising). From that point of view the FK-Potts model is interesting, because its underlying symmetry is the permutation group S_Q , with Q being allowed to take any complex value. This opens the perspective of a continuum limit whose CFT is characterized by the continuous parameter Q . And indeed, it is known that in dimension $d = 2$, and for ferromagnetic interactions, criticality occurs in the range $0 \leq Q \leq Q_c = 4$ [8], whereas the phase transition turns first-order when $Q > 4$. The situation in higher dimensions is similar, but for a d -dependent value of Q_c . For instance, $Q_c = 10/3$ for $d \rightarrow 6$ [9]. For any d in the range between the lower and upper critical dimensions, $2 \leq d \leq 6$, first-order behavior sets in for $Q > Q_c(d)$.

Intriguingly, it has been observed that when $Q > Q_c$, some

kind of approximate conformal invariance continues to hold, at least provided that Q is not too large [10]. In fact, it has been argued a long time ago [11] that such a scenario can happen when nearby a stable (critical) fixed point of the renormalization group annihilates with another unstable (tricritical) fixed point, upon varying a symmetry-related continuous parameter. For the Potts model this parameter is Q , and in $d = 2$ the annihilation happens at $Q_c = 4$. This annihilation scenario epitomizes other similar situations, occurring for instance in deconfined quantum critical points [12], or in models of quantum impurity spins coupled to a bath [13].

Based on the hypothesis that the CFT description valid for real $Q \leq Q_c$ could be analytically continued in Q , Gorbenko, Rychkov and Zan [14] proposed that approximate conformality in the first-order region $Q > Q_c$ could result from the vicinity, in the plane of complex Q , of a pair of complex conjugate fixed points. Furthermore it was argued [9, 14] that the Potts model could become truly conformal also for $Q > Q_c$ upon introducing suitable complex coupling constants into the model.

The field-theoretical argument applies also to the closely related, but easier to handle $O(n)$ model. This model is originally defined as a lattice model of n -component spins. In two dimensions, it can again be extended to complex values of n , by using a diagrammatic reformulation in terms of lattice loops [15] each of which occurs with weight n . This model has, for any $0 \leq n < n_c = 2$, both a critical fixed point (governing the so-called dense phase of loops, corresponding to temperatures $T > T_c$ for the vector spins) and a tricritical point (governing the dilute phase of loops, corresponding to the critical point $T = T_c$ for the spins). The annihilation of fixed points can be realized when $n \rightarrow n_c$. It was in this model that approximate conformality (with real couplings) and true conformality (with suitable complex couplings) were first established in the range $n > n_c$ [16], extending earlier and (by that time) rather puzzling results [17].

The question remained how to achieve something similar for the two-dimensional Potts model with $Q > Q_c$. An obvious candidate for the lattice model would be the usual FK model defined on a square lattice, but with a complex weight (temperature parameter) for an occupied bond. However, this does not work, since this model realizes only the critical fixed point, but not the tricritical one that is required for the collision of fixed points. In our previous Letter [18] we showed how to overcome this obstacle with a more elaborate FK model defined on a triangular lattice. This model, defined below, contains both two- and three-spin interactions, which compete and can induce tricritical behavior. The collision of fixed points can be realized by this lattice model so as to generate conformal behavior for $Q > Q_c$, and more generally for complex values of Q . A convenient formulation of this model is via completely packed loops on the triangular lattice, with specific discrete-rotational invariant interactions at the vertices.

The purpose of this paper is not only to give a more detailed account of our results than was possible in the Letter [9], but also to extend them in several directions. In the meantime a number of related studies have been published. Ref. [19] con-

siders the quantum version of the 5-state Potts model, where the internal degrees of freedom are explicitly realized. In contrast to our approach, this technique does not allow the authors to take Q arbitrary. This is then extended to boundary critical phenomena [20]. A common feature of these related studies is their use of the Q -state non-Hermitian Potts spin chain. Its representation in terms of Q -component spins requires Q to be a positive integer, mostly taken as $Q = 5$ in order to be in the regime $Q > Q_c$, while still keeping manageable the dimension of the linear operators used in the numerical studies. We believe that our model has the decisive advantage of simply treating Q as a parameter, so that any value of Q , including complex ones, can be accessed in a framework where the dimension of the transfer matrix (or the quantum Hamiltonian) is independent of Q . This confers to the lattice model of loops the same flexibility as that of the corresponding field theory.

To expand on this motivation, we round off this Introduction by a brief overview of the present analytical understanding of the continuum limit of the loop-model version of the FK-Potts model. The loop-model representation itself is due to Temperley and Lieb [21]. The first exact results were obtained by Baxter. Using a mapping to the six-vertex model, valid at the critical temperature, he obtained the partition function and some critical exponents for the square-lattice model using integrability techniques [8]. As we have just explained, this model is not sufficient for the purpose of the present investigation. The required triangular-lattice model with competing interactions was introduced by Wu and Lin [22], but only the case of vanishing three-spin interactions was found to be integrable [23].

Strong results about the continuum limit of this family of models arrived with the field-theoretical advances of the early 1980s. The loops, which are the contours of the FK clusters [24], were found to become, in the continuum limit, the level lines of a compactified bosonic field, to which the methods of Coulomb Gas (CG) [25] and Conformal Field Theory (CFT) [26] can be applied. The study of the model on a torus [27] fixes the set of critical exponents, sometimes known as the *spectrum* of the theory. However, the CG approach is only one part of the field-theoretical description of loop models.

In the terminology currently in use [28, 29] operators in loop models are of three types: (1) degenerate operators (local energy densities), (2) diagonal operators (which modify the weights of loops encircling the insertion point), and (3) non-diagonal operators (which insert defects in the form of open loop segments, or paths). From heuristic arguments, there are indications of a relation to Liouville CFT (LCFT) [30], an exactly solvable interacting CFT. However, LCFT has a continuous spectrum and is unitary for central charge $c \in (1, \infty)$, facts which are in tension with CG results for the Potts loop model for generic $Q \in [0, 4]$, since these establish that the corresponding CFT is non-unitary, has $c \leq 1$, and a discrete spectrum [27].

The last 10 years of research on the loop model have aimed at solving this conundrum. In this endeavor the theoretical physicists have been joined by probability theorists, who use methods such as Schramm-Loewner Evolution (SLE) [31–

[33], the Conformal Loop Ensemble (CLE) [6] and the probabilistic construction of the LCFT path integral [34, 35]. One might wonder whether the loop model could be obtained from LCFT by analytic continuation through complex values of Q ; this idea can, however, be quickly dismissed. Indeed, in LCFT the structure constants of three-point correlation functions are given by the so-called DOZZ formula [36, 37], but analytic continuation of the latter to $c \in (-\infty, 1)$ is impossible. On the other hand, there is a variant of LCFT, called LCFT $_{c \leq 1}$ or sometimes imaginary Liouville CFT, for which it has been established [38, 39] that the DOZZ-type formula for LCFT $_{c \leq 1}$ [40] correctly predicts three-point structure constants of diagonal operators in the Potts loop model. Very recently, this result on the three-point functions has been promoted to a theorem by probability theorists Ang, Cai, Sun and Wu [41], while physicists have established a more general result for three-point structure constants involving any of the three types of operators [42] (one special case of the latter result has been proved [41]). The results in [41, 42] go beyond both the CG and LCFT scenarios.

From this perspective, the present paper establishes numerically that the Potts model outside of the interval $Q \in [0, 4]$ can be made conformal in a large portion of the complex Q -plane, and that the corresponding universality class coincides with an analytic continuation of the CFT discussed above. More precisely, we provide precise numerical evidence that the conformal data (the central charge, the spectrum of critical exponents, and the three-point structure constants) of the two complex CFTs that result from the collision of the real critical and tricritical CFTs coincide with the appropriate analytic continuation of the results in [27–29] for the $c \leq 1$ loop models. Based on this link, we are confident in conjecturing that recent results on three- and four-point correlation functions [29, 43–45] and the symmetries of the space of states [28] of the loop models carry over to the complex CFT by the same analytic continuation.

Let us outline the content of this paper, for which a table of contents can be found on page 44: In Section II, the model is defined (II A), together with its transfer matrix formulation (II B) and its duality and symmetry structure (II C). The self-dual manifold is parametrized by a free parameter z . The phase diagram is then analyzed both in the complex z -plane (II D) and as a function of Q (II F), with section II E describing the analytic continuation of the critical theory to complex Q and $Q > 5$. The operator content and conformal dimensions are given in Sec. II G, and the role of the global S_Q symmetry (II H) and the question how to calculate three-point structure constants (III) are clarified.

We then proceed to our numerical results: Section III briefly studies a simpler model with only nearest-neighbor interactions, showing that it cannot access the complex CFT. Returning then to the model of interest, in Section IV, numerical results for central charge c and scaling corrections ω are presented for $Q < 4$, along with the RG flows between tricritical and critical theories. This sets the stage for section V on $Q = 5$, analyzing the central charge in the complex z -plane in section V A, with improvements in precision discussed in section V B, while Section VI extends the analysis to large real

values of Q , and complex Q . Section VII discusses the spectrum, starting again with a case with $Q < 4$ (VII B), before proceeding to $Q = 5$ (VII C).

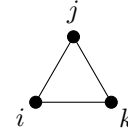
Finally, Section VIII summarizes the results and their implications for complex CFTs in statistical physics.

Appendices A to E provide additional data for large and complex Q values ($Q = 10, 20, 40, 5 + 2i, 8 + i$). We also examine whether the location of a complex CFT can be inferred from data for real couplings only (section F). Finally we provide alternate derivations of some OPE coefficients in the Dotsenko–Fateev framework, complementing the general formula in section III.

II. THEORY

A. Model

We use a variant of the Q -state Potts model introduced by Wu and Lin [22]. It is defined on a triangular lattice $\mathbb{T}_0$, which we orient so that one half of the faces form up-pointing triangles, while the other half are down-pointing. Let Δ_{ijk} denote an up-pointing triangle with vertices i, j, k hosting the three spins $\sigma_i, \sigma_j, \sigma_k$, each of which takes values in $\{1, 2, \dots, Q\}$:



The interactions within Δ_{ijk} are given by the local Hamiltonian $\mathcal{H}_{ijk}$, where

$$\mathcal{H}_{ijk} = -K(\delta_{\sigma_i, \sigma_j} + \delta_{\sigma_j, \sigma_k} + \delta_{\sigma_k, \sigma_i}) + K_3 \delta_{\sigma_i, \sigma_j, \sigma_k}. \quad (1)$$

The Kronecker symbol $\delta_{x,y} = 1$ if $x = y$, and 0 if $x \neq y$; we have set $\delta_{x,y,z} = \delta_{x,y} \delta_{y,z}$ for convenience. The Hamiltonian for the whole lattice is

$$\mathcal{H} = \sum_{\Delta_{ijk}} \mathcal{H}_{ijk}, \quad (2)$$

and the sum runs over all up-pointing triangles. There are no interactions in the down-pointing triangles. Notice that each bond belongs to precisely one up-pointing triangle, so there is a two-spin interaction K along each bond, but a three-spin interaction K_3 only in the up-pointing triangles.

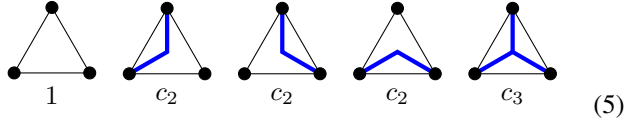
If we absorb $k_B T$ into the coupling constants K and K_3 , the local Boltzmann weights read

$$e^{-\mathcal{H}_{ijk}} = 1 + c_2(\delta_{\sigma_i, \sigma_j} + \delta_{\sigma_j, \sigma_k} + \delta_{\sigma_k, \sigma_i}) + c_3 \delta_{\sigma_i, \sigma_j, \sigma_k}, \quad (3)$$

and the parameters c_2 and c_3 are related to K, K_3 by

$$\begin{aligned} e^K &= 1 + c_2, \\ e^{K_3 + 3K} &= 1 + 3c_2 + c_3. \end{aligned} \quad (4)$$

Each term on the right-hand side of Eq. (3) can be represented graphically as



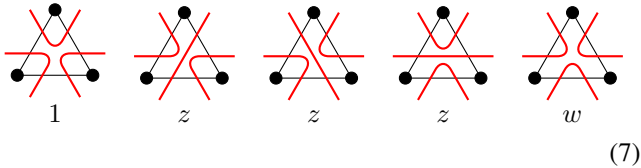
Each connected component of thick blue segments is called a Fortuin-Kasteleyn (FK) cluster. The partition function corresponding to (2) is

$$Z = \sum_{\{\sigma\}} e^{-\mathcal{H}} = \sum_{\Delta} Q^C c_2^{N_2} c_3^{N_3}. \quad (6)$$

Here, the first sum runs over the possible Q states of each spin on the lattice. The second sum is over the five possible configurations (3) of each up-pointing triangle, with N_2 and N_3 denoting the number of diagrams of weight c_2 and c_3 , respectively, and C is the number of FK clusters.

In this last writing Q appears only as a parameter, so we can *define* the FK model for any (complex) Q by the partition function (6), forgetting that Q had to be a positive integer in the original Potts model (2).

In a final step, we trade the clusters for the loops bouncing off their boundaries. To this aim, replace the diagrams (5) by



The red curves go straight across the down-pointing triangles. Applying the Euler relation we have, up to an unimportant global factor,

$$Z = \sum_{\Delta} Q^{\ell/2} z^{N_2} w^{N_3}, \quad (8)$$

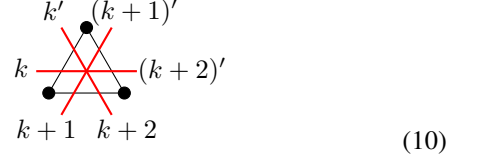
where ℓ denotes the number of closed loops formed by the red curves, while N_2 and N_3 are defined as before. We call this representation the *loop model*. The local weights z and w are related to the previous ones by

$$\begin{aligned} z &= Q^{-1/2} c_2, \\ w &= Q^{-1} c_3. \end{aligned} \quad (9)$$

The diagrams (7) are such that the loops live on another, shifted triangular lattice $\mathbb{T}$. At each vertex of $\mathbb{T}$ (corresponding to the barycenter of a face in the original lattice $\mathbb{T}_0$) the loop segments split so as to avoid crossing one another. We call the loops *completely packed*, since each edge of $\mathbb{T}$ is covered by a loop segment. Each of the five possible splittings (7) at the vertices of $\mathbb{T}$ defines what is sometimes called a non-intersecting transition system. The loop model on $\mathbb{T}$ is thus completely packed, and has configurations given by the sum over all possible non-intersecting transition systems.

B. Transfer matrix

In the numerical part of this paper we will study the model (8) via the transfer matrix T_L which builds a row of L triangles, with periodic boundary conditions. Thus T_L propagates $\sqrt{3}/2$ lattice spacings upwards and $1/2$ to the right. Consider now the operator $\tilde{R}_k$ that propagates through triangle number k of $\mathbb{T}$ within a given row:



The vertex of $\mathbb{T}$ where the red lines cross implies a sum over the five splittings, as in (7), with the corresponding local Boltzmann weight for each term. We have labelled the three lines as respectively $\{k, k+1, k+2\}$ and $\{k', k+1, k+2\}'$ before and after the action by $\tilde{R}_k$; the operator $\tilde{R}_k$ does nothing to the remainder of the system. Note that with this labelling, the imaginary time flows in the North-East direction, as usual in the transfer matrix formalism.

We now explain that the transfer matrix T_L that adds another row of L triangles to the lattice can be expressed as follows:

$$T_L = u^{-1} \text{qTr} \tilde{R}_{2L-2} \cdots \tilde{R}_2 \tilde{R}_0. \quad (11)$$

Before the action by T_L , we suppose that the red lines sticking out of the previously completed row are labelled $1, 2, \dots, 2L-1, 2L$. At the beginning of the new row (i.e., before the action by the first factor $\tilde{R}_0$) the horizontal space (also called ‘auxiliary space’) is inserted at site 0. A further site labelled $2L+1$, initially identified with site 0, is also inserted in order to manage the periodic boundary conditions along the horizontal direction. Acting with the successive factors $\tilde{R}_0, \tilde{R}_2, \dots$ in (11) the auxiliary space is constantly relabelled (as $0 \rightarrow 2 \rightarrow 4 \rightarrow \dots$), and finally comes out with label $2L$ after the action by the last factor $\tilde{R}_{2L-2}$. The operation qTr in (11) denotes the ‘quantum trace’ over the horizontal space, which amounts to the initial insertion of the extra sites 0 and $2L+1$, as well as the final identification of sites $2L$ with $2L+1$ followed by their removal. In this way the periodic boundary conditions have been imposed and a row of the lattice is completed. After the qTr operation, the sites on the top have the new labels $0', 1', 2', \dots, (2L-1)'$, when read from left to right, which we can relabel (working modulo $2L$, and removing the primes) as $2L, 1, 2, \dots, 2L-1$. The final application of the left-shift operator u^{-1} in (11) — where by convention u denotes the operator that shifts the sites cyclically towards the right — brings the labelling back in order as $1, 2, \dots, 2L-1, 2L$, and we are ready for the next application of T_L .

The periodic Temperley-Lieb algebra on $N = 2L$ sites is generated by the identity 1 and the e_i with $i = 1, 2, \dots, N$,

satisfying the algebraic relations

$$e_i^2 = ne_i, \quad (12)$$

$$e_i e_{i\pm 1} e_i = e_i, \quad (13)$$

$$e_i e_j = e_j e_i \quad \text{for } |i - j| > 1, \quad (14)$$

where indices i, j are considered modulo N . The affine Temperley-Lieb algebra is generated by e_i , u and u^{-1} and satisfies the additional relations

$$ue_i u^{-1} = e_{i+1}, \quad (15)$$

$$uu^{-1} = u^{-1}u = 1, \quad (16)$$

$$u^{-2}e_1 = e_{N-1}e_{N-2} \cdots e_2 e_1. \quad (17)$$

These algebras have a faithful diagrammatic representation in which e_i acts non-trivially on sites i and $i + 1$, as

$$e_i = \left| \begin{array}{c} \cdots \\ \text{---} \\ \text{---} \end{array} \right| \cdots \left| \begin{array}{c} i' \quad (i+1)' \\ \text{---} \\ \text{---} \end{array} \right| \cdots \left| \begin{array}{c} \cdots \\ \text{---} \\ \text{---} \end{array} \right|. \quad (18)$$

By inspecting (7) and comparing with (10) we can therefore write

$$\check{R}_k = e_k + z(1 + e_{k+1}e_k + e_k e_{k+1}) + we_{k+1}, \quad (19)$$

where e_i are generators of the periodic Temperley-Lieb algebra [28] on $N = 2L$ sites. Each $\check{R}_k$ acts like the identity operator on the $2L - 3$ sites different from $\{k, k + 1, k + 2\}$. The formulae (19) and (11) define the transfer matrix as an element of the N -site affine Temperley-Lieb algebra.

Notice, however, that the affine Temperley-Lieb algebra is infinite-dimensional, because non-contractible loops may pile up and lines may spiral around the periodic direction. To obtain a manageable problem involving a finite-dimensional linear operator, we therefore let T_L act in the space of link patterns, sometimes called standard modules $\mathcal{W}_{j,\tilde{z}}$, with $2j$ defect lines (FK cluster boundaries) propagating from bottom to top, and (pseudo) momentum $\tilde{z}$ describing the winding of lines with respect to the periodic boundary condition. We refer to [28] for details.

C. Duality and symmetries

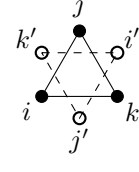
We are interested in placing the model (8) at its critical point, hoping for conformal invariance in the continuum limit. Since this implies continuous rotational invariance, it seems that the least one may ask for in the lattice model is 6-fold rather than 3-fold discrete rotational invariance of the diagrams (7). From this consideration, Wu and Lin [22] conjectured that the critical manifold is given by

$$w = 1. \quad (20)$$

It will be a huge advantage for us that the model with (20) is now defined in terms of only one local weight z . In what follows we shall take $z \in \mathbb{C}$. We sometimes split it into real and imaginary parts,

$$z = x + iy \quad (x, y \in \mathbb{R}). \quad (21)$$

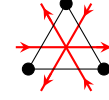
We briefly mention another way of thinking about the constraint (20). In ordinary graph theory, an edge connects precisely two vertices, and the usual FK model (mentioned in the Introduction) consists in summing each edge over an occupied and an empty state. More generally, one may define a hypergraph by allowing for edges to connect more than two vertices. From that point of view, the diagrams (5) correspond to summing each hyperedge in the regular three-uniform hypergraph $\mathbb{T}_0$ over its possible states of occupancy. The dual graph $\mathbb{T}_0^*$ is obtained by replacing each up-pointing triangle with a down-pointing triangle, here shown as a dashed line superimposed on the original triangle:



(22)

The duality transformation amounts to replacing each hyperedge on $\mathbb{T}_0$ with a hyperedge on $\mathbb{T}_0^*$ which is in the opposite state of occupancy: i' is connected to j' iff k is disconnected from i and j , and so on by cyclic permutation of the labels (i, j, k) . In particular, the first and fifth diagrams in (5) or (7) get interchanged (up to rotation). From this perspective, the constraint (20) simply means that we place the model at its *selfdual point* (in the sense of hyperduality).

Let us return to the loop model defined by (8) with configurations (7). Superficially this looks like an $O(n)$ model, with $n = Q^{1/2}$. However, the global symmetry is larger than $O(n)$. Consider endowing the underlying lattice $\mathbb{T}$ with the following *fixed* orientations of its edges:



(23)

Each horizontal line is oriented from left to right, and the other lines are obtained from this by rotations through $\pm \frac{2\pi}{3}$. Then each of the transition systems (7) is such, that an observer moving along any one of its loop segments sees consistently oriented arrows. By extension, each loop is consistently oriented with respect to this fixed orientation of the lattice. This implies that the loop model has in fact the global symmetry $PSU(n)$. The simplest square-lattice loop model with this symmetry was studied extensively in [46]. The symmetry $PSU(n)$ is an example of a Deligne category, which is well-defined for any n (and hence for any $Q \in \mathbb{C}$). The CFT properties of the loop model are enhanced by the properties of this global symmetry, as explained in [46] to which we refer for further details.

D. Phase diagram as a function of z

The ordinary nearest-neighbor Potts model on the triangular lattice is a special case of (1) with $K_3 = 0$. Defining

$c_2 = e^K - 1$ as above, it is known to be integrable along the curve [23]

$$(c_2)^3 + 3(c_2)^2 = Q. \quad (24)$$

From (4) we see that this requires $c_3 = Q$, so from (9) this means $w = 1$ in agreement with (20). In other words, the Wu-Lin model on its critical manifold (20) is a deformation of the integrable nearest-neighbor model, in which the free parameter z determines how K_3 should be adjusted against K to maintain criticality.

A useful parameterisation of the cubic equation (24) is obtained by setting [47]

$$c_2 = -1 + 2 \cos\left(\frac{2\pi(g-1)}{3}\right), \quad Q = 4 \cos^2(\pi g), \quad (25)$$

with $-\frac{1}{2} \leq g \leq 1$. We have then

$$z = \frac{c_2}{\sqrt{Q}} = \frac{-1 + 2 \cos\left(\frac{2\pi(g-1)}{3}\right)}{2 \cos(\pi g)}. \quad (26)$$

Notice that for each $0 < Q < 4$ there are three distinct solutions for g , one in each of the intervals $(-\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Only the latter corresponds to $z > 0$.

It was established numerically [48] that the interval $0 < g \leq 1$ corresponds to a CFT with central charge

$$c = 1 - \frac{6(1-g)^2}{g}. \quad (27)$$

This is easily recognised as the usual Coulomb gas expression. It is this CFT whose analytic continuation shall be our main interest in this paper. Meanwhile, the interval $-\frac{1}{2} < g < 0$ is described by a quite different CFT with central charge

$$c = 2 + 6g \quad (28)$$

and unusual properties, such as a continuous spectrum of critical exponents. It can be identified by the black-hole sigma model, whose square-lattice counterpart has been extensively studied [48–52].

E. Analytic continuation

We henceforth concentrate on the CFT with central charge (27). To discuss its analytic continuation away from the real interval $c \in [-\infty, 1]$, or equivalently away from a Potts model with real $Q \in [0, 4]$, we find it useful to introduce the parameter

$$b = \frac{1}{1-g}. \quad (29)$$

The corresponding CFT, denoted $\mathcal{B}_b$, has

$$Q(b) = 4 \cos\left(\frac{\pi}{b}\right)^2, \quad (30)$$

$$c(b) = 1 - \frac{6}{b(b-1)}, \quad (31)$$

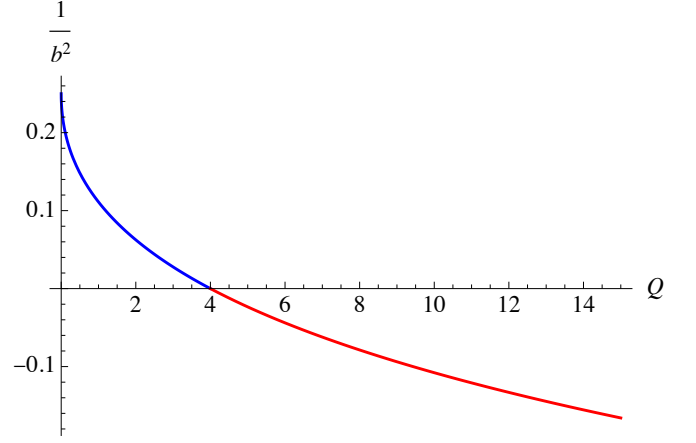


FIG. 1. $1/b^2$ as a function of Q . For $Q \leq 4$, b is real, while for $Q > 4$, b is purely imaginary.

so that

$$b(Q) = \frac{\pi}{\arccos(\frac{\sqrt{Q}}{2})}. \quad (32)$$

The choice of branch is such that $b > 0$ corresponds to $Q < 4$.

Notice that for integer $b = 3, 4, \dots$, (31) coincides formally with the series of unitary minimal models $\mathcal{M}_{m+1,m}$ [53] via the identification

$$\mathcal{M}_{b,b-1} \equiv \mathcal{B}_b. \quad (33)$$

Within these models it is customary to specify the dimension of the holomorphic part of the operator $\phi_{r,s}(z)$ in terms of the Kac table

$$h_{r,s} = \frac{[b(r-s) + s]^2 - 1}{4(b-1)b}. \quad (34)$$

The dimension of the full operator $\Phi(z, \bar{z}) := \phi_{r,s}(z)\bar{\phi}_{\bar{r},\bar{s}}(\bar{z})$ and its spin are given by

$$\Delta_\Phi = h_{r,s} + h_{\bar{r},\bar{s}}, \quad (35)$$

$$s_\Phi = h_{r,s} - h_{\bar{r},\bar{s}}. \quad (36)$$

In this paper we are interested in the continuum limit of the loop model which is neither unitary, nor minimal. Minimality in Eq. (34) would require $1 \leq r \leq b-2$ and $1 \leq s \leq b-1$, but the operator content of loop models is not confined to this range, as will be discussed below. We shall consider (34) a convenient parametrization, and allow $b \in \mathbb{C}$ to take complex values.

While $\mathcal{B}_b$ is a critical model for b integer, its tricritical counterpart identifies with the next minimal model $\mathcal{B}_{b+1}$, which according to the above can also be written as $\mathcal{B}_{-b}$. This means that the collision between critical and tricritical theories occurs when $b \rightarrow \infty$. The desired analytic continuation must be performed around this value, for which $\frac{1}{b(Q)^2}$ has a regular Taylor series around $Q = 4$, see Fig. 1,

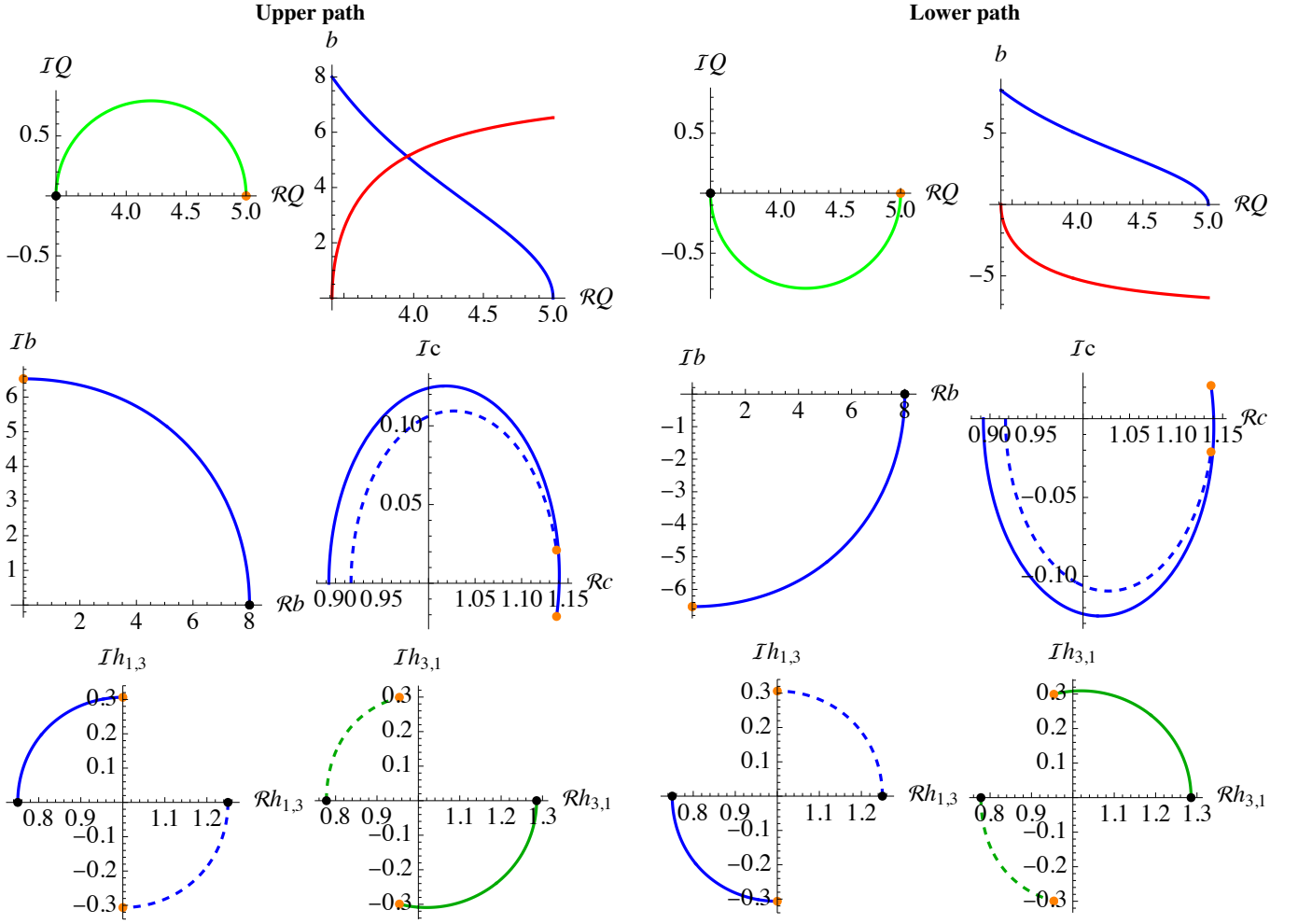


FIG. 2. Upper row: paths taken in the analytic continuation, starting at $Q = 3.41421$ ($b = 8$, black dot) and ending at $Q = 5$ ($b = \pm 6.5285i$, orange dot [gray in B&W]). In the left part of the figure, the path taken is a semicircle in the upper half-plane in Q (“upper path”), while in the right part it is a semicircle in the lower half-plane (“lower path”). The remaining plots show the central charge c , as well as the dimensions $h_{1,3}$ and $h_{3,1}$, for the corresponding continuations of the critical point B_b (solid lines) and the tricritical point B_{-b} (dashed lines). Once we arrive at $Q \rightarrow 5$ (with $b \in i\mathbb{R}$), shown as an orange dot, the two continuations of c , $h_{1,3}$ and $h_{3,1}$, starting from either B_b or B_{-b} , are complex conjugates of each other. The same is true if we compare the two continuations along the upper and lower paths, with the same starting point.

For $Q = 5$ we find¹

$$b(Q = 5) = \mp 6.5285i. \quad (38)$$

The upper sign, corresponding to analytic continuation along the lower path in Fig. 2, is what will be necessary for the critical coupling z_c to belong to the upper half plane. We then use this value for analytic continuation, e.g.

$$c(Q = 5) = 1.13755 \pm 0.0210687i \quad (39)$$

With our choice of $b > 0$ for $0 < Q < 4$, this uniquely defines the analytic continuation in Q . As shown in Fig. 2, there are two distinct paths one can take, passing above or below $Q = 4$ in the complex plane.

¹ There are two more solutions, $b = \pm t$, $t = 0.977075 + 0.149663i$. They have central charge $c = 13. + 38.252i$ (for $t = b$) and $c = -1.98251 + 0.690625i$ for $t = -b$. We do not know what these branches mean. What makes them unlikely to meet our criteria is the fact that they are not complex conjugate of each other, as we expect for real Q .

There are also the complex conjugate solutions $\bar{b}$ and $\bar{c}$. As we show below, for $\Im z_c > 0$ we see the solutions for b and c with the upper sign.

F. Phase diagram as a function of Q

In this section, we determine the stability of the critical and tricritical fixed points in the complex Q -plane by analyzing the subleading scalar operator ε' , which governs the perturbation of the corresponding fixed points, given that we have tuned the amplitude of the leading scalar operator ε to 0 by restricting to the critical manifold (20). Readers familiar with [54] will agree that the most natural candidates for these operators are $\varepsilon = \phi_{2,1} \times \bar{\phi}_{2,1}$ or $\phi_{1,2} \times \bar{\phi}_{1,2}$, and $\varepsilon' = \phi_{3,1} \times \bar{\phi}_{3,1}$ or $\phi_{1,3} \times \bar{\phi}_{1,3}$. Pending the detailed discussion of the operator content of the Potts loop model (section II G 2) and a detailed numerical investigation (section IV C) we shall admit here that the correct identification is

$$\varepsilon = \phi_{2,1} \times \bar{\phi}_{2,1} := V_{(2,1)}^d, \quad (40)$$

$$\varepsilon' = \phi_{3,1} \times \bar{\phi}_{3,1} := V_{(3,1)}^d \quad (41)$$

for both the critical point $\mathcal{B}_b$ and the tricritical point $\mathcal{B}_{-b} \equiv \mathcal{B}_{b+1}$, and indeed also for the complex CFT obtained by making both fixed points collide.

In order to decide whether a fixed point is stable or unstable towards the perturbation by ε' , we thus have to check whether $\Re h_{3,1}$ is smaller or larger than 1. To simplify our considerations, we note the curious relation

$$h_{3,1}(\mathcal{B}_b)h_{3,1}(\mathcal{B}_{-b}) = 1. \quad (42)$$

For $0 \leq Q < 4$, when b and hence $h_{3,1}$ are real, this implies that one of the fixed points is attractive (critical) and the other repulsive (tricritical), in agreement with the discussion above. Having two repulsive fixed points is possible, but only if both $h_{3,1}(\mathcal{B}_b)$ and $h_{3,1}(\mathcal{B}_{-b})$ are complex.

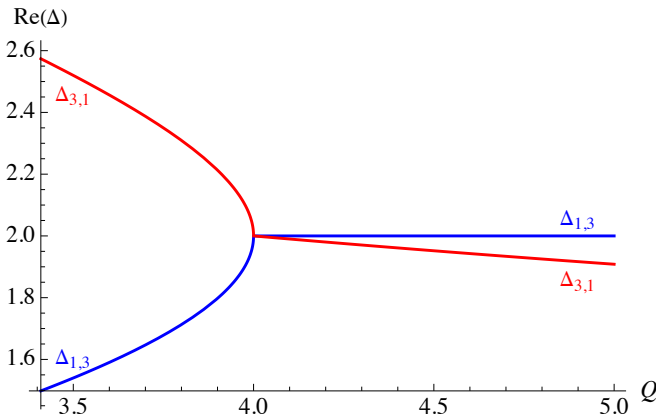


FIG. 3. $\Delta_{1,3}$ (unphysical branch) and $\Delta_{3,1}$ physical branch.

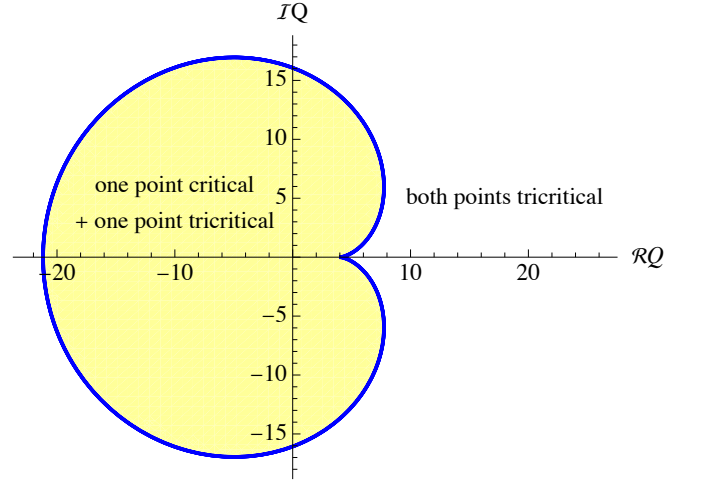


FIG. 4. The critical domain as a function of Q .

For future reference we note the numerical values

$$2h_{1,3}(\mathcal{B}_8) = \frac{3}{2}, \quad (43)$$

$$2h_{3,1}(\mathcal{B}_8) = \frac{18}{7} = 2.57143. \quad (44)$$

$$2h_{1,3}(Q = 5) = 2 - 0.612698i, \quad (45)$$

$$2h_{3,1}(Q = 5) = 1.9083 + 0.598652i. \quad (46)$$

Curiously, for $Q = 5$ the first is marginal, while the second is slightly relevant. This is illustrated in Fig. 3. We will come back to this below.

The separatrix in the complex Q -plane between the two situations is given by the curve $\Re h_{3,1} = 1$. It can be recast in the parametric form [55]

$$Q(\varphi) = 2 + 2 \cos \left(\frac{\pi}{\frac{1}{2} + i\varphi} \right), \quad \varphi \in \left[-\frac{1}{2}, \frac{1}{2} \right] \quad (47)$$

and is plotted in Fig. 4. On the inside of the curve (shown in yellow) we have one critical and one tricritical point, while in its complement there is no stable fixed point (both points are tricritical).

The cusp of the separatrix is at $Q = 4$. The maximum of $\Re Q$ is 7.746 347... and is attained for $\varphi = \pm 0.193 437...$. The maximum of $|\Im Q|$ is 16.949 180... and is attained for $\varphi = \pm 0.336 521...$. Finally, the minimum of $\Re Q$ is $2 - 2 \cosh \pi = -21.183 906...$ and is attained for $\varphi = \pm \frac{1}{2}$.

In Fig. 5 we show the same plot as a function of the conformal weight $h_{3,1}(\mathcal{B}_b)$. The yellow domain with $\Re h_{3,1}(\mathcal{B}_b) > 1$ corresponds to $\mathcal{B}_b$ being attractive (i.e. critical) while $\mathcal{B}_{-b}$ to being repulsive (i.e. tricritical). The other yellow domain corresponds to the opposite situation, where $\mathcal{B}_b$ is tricritical and $\mathcal{B}_{-b}$ is critical. By (42) it is delimited by the separatrix $\frac{1}{1+i\mathbb{R}}$, a fancy rewriting of a circle of radius $1/2$ centered at $1/2$.

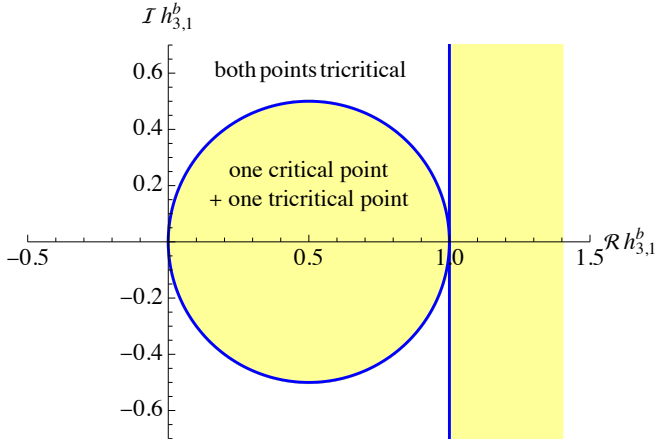


FIG. 5. In yellow we show the domains in which one fixed point is stable while the other is unstable.

G. Operator content and conformal dimensions

The non-unitary CFTs describing the continuum limit of the Q -state Potts-FK model with $0 \leq Q \leq 4$ and the $O(n)$ loop model with $-2 \leq n \leq 2$, both with real central charge $c \leq 1$, were studied in [27] and a recent series of papers including [28, 29, 42–45, 56]. We can formulate the main result of the present paper in the form of a pair of conjectures:

Conjecture 1. For any $Q \in \mathbb{C}$ inside the critical domain, the loop model (8) at its selfdual point (20) and at the critical value z_c of the parameter z (9) has a continuum limit that is a complex CFT.

Conjecture 2. For any $Q \in \mathbb{C}$, this complex CFT is the analytic continuation, as prescribed in Section II E, of the non-unitary CFT studied in the preceding list of references.

We now unroll the consequences of Conjecture 2 in terms of the conformal data for the critical loop models. We wish to make contact with the results and notations contained in the above list of references which treat in a unified way loop models of both $O(n)$ and Potts type. In the latter case, the loop weight $n = Q^{1/2}$. Although the present paper is concerned with the Potts-type loop model (8), we can give results that cover also the $O(n)$ loop model at almost no extra cost. In other words, although Conjectures 1–2 are stated for the Potts loop model, which is our main focus here, the equivalent statements should also hold true for the analytic continuation of the $O(n)$ loop model to $n \in \mathbb{C}$, that was studied in [16].

That being said, there are some notational differences and small subtleties that distinguish the Potts and $O(n)$ models, and which need to be stated clearly. To avoid any undue confusion, we split the remainder of this section into two parts, one for each model.

1. $O(n)$ loop model

We start with the $O(n)$ model, because most of the results in [28, 29, 42–45, 56] are formulated in a way that gives prece-

dence to that model. To make contact with those references, we parametrize the loop weight n by β^2 , as

$$n = -2 \cos(\pi \beta^2), \quad (48)$$

with $\Re \beta^2 > 0$. We also denote the left and right conformal dimension by $(\Delta, \bar{\Delta})$. The dimension and the central charge will be parametrized by

$$c = 1 - 6(\beta - \beta^{-1})^2, \quad (49)$$

$$\Delta_{(r,s)} = \frac{1}{4} (\beta r - \beta^{-1} s)^2 - \frac{1}{4} (\beta - \beta^{-1})^2. \quad (50)$$

It is also useful to rewrite a general conformal dimension Δ and its Kac-table parametrization $\Delta_{(r,s)}$ in terms of momentum variables P and $P_{(r,s)}$ as follows:

$$\Delta = P^2 - P_{(1,1)}^2, \quad (51)$$

$$\Delta_{(r,s)} = P_{(r,s)}^2 - P_{(1,1)}^2, \quad (52)$$

$$P_{(r,s)} = \frac{1}{2} (-\beta r + \beta^{-1} s). \quad (53)$$

It was then shown in [27, 56] that the operator content (or *spectrum*) of the $O(n)$ loop-model CFT is:

Primary fields in the $O(n)$ loop model CFT			
Name	Notation	Parameters	$(\Delta, \bar{\Delta})$
Degenerate	$V_{(r,s)}^d$	$r=1; s \in 2\mathbb{N} + 1$	$(\Delta_{(r,s)}, \Delta_{(r,s)})$
Diagonal	V_P	$P \in \mathbb{C}$	(Δ_P, Δ_P)
Non-diagonal	$V_{(r,s)}$	$r \in \frac{1}{2}\mathbb{N}^*; s \in \frac{1}{r}\mathbb{Z}$	$(\Delta_{(r,s)}, \Delta_{(-r,s)})$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

This spectrum involves three different types of operators. The *degenerate operators* $V_{(r,s)}^d$ encompass the continuum limit of the identity operator, the local energy operator $\varepsilon = V_{(1,3)}^d$, and the whole series of higher energy operators. These operators have one singular vector at level s , so their highest-weight representations are Kac modules. The label $s \in \mathbb{N}^*$ can take any parity for the Potts model, but must be odd for the $O(n)$ model.

The *diagonal operators* V_P correspond to changing the weight of a loop surrounding the insertion point of one such operator from n to the modified value

$$w(P) = 2 \cos(2\pi \beta P), \quad (54)$$

determined by a momentum P , which can be an arbitrary complex number. Such operators are non-degenerate and correspond to a full Verma module.

Finally, the *non-diagonal operators* $V_{(r,s)}$ result from inserting a defect of $2r$ open loop segments, each of which acquires a complex phase $e^{i\pi s}$ upon making a full turn around the insertion point. These open segments are not allowed to contract among themselves, but must end on another non-diagonal operator.² Their conformal spin rs is integer. The

² Open segments can also end on a boundary, if the system has one, and if the boundary conditions are appropriate (non-diagonal). In this paper we consider the bulk CFT, so there are no boundaries.

representations of non-diagonal operators are again Verma modules; when $s > 0$ they are logarithmic with a rank-two Jordan cell at level rs . We are here interested in the spinless case $s = 0$ in which no phase factors appear. These non-diagonal operators correspond to the Temperley-Lieb standard modules $\mathcal{W}_{j,\bar{z}^2}$ discussed in Section II B, setting $j = r$ and $\bar{z} = e^{i\pi s}$ there. We refer to [28] for details.

To establish the link between the parametrizations (48)–(50) and those used above — in particular in the algebraic relation (12) and the parametrizations (30)–(31) — we should set

$$n = 2 \cos\left(\frac{\pi}{b}\right) \implies \beta^2 = g = \frac{b-1}{b}. \quad (55)$$

We then retrieve (30) with $Q(b) = n^2$. Moreover, it is then straightforward to show that the relation to the standard Kac table (34) is

$$\Delta_{(r,s)} = h_{s,r}, \quad [O(n) \text{ model}]. \quad (56)$$

i.e., the indices r, s are *interchanged* on the right-hand side.

2. Potts loop model

The results for the Potts loop model are almost the same, except for this interchange of indices. We have the following operator content:³

Primary fields in the Potts loop model CFT			
Name	Notation	Parameters	$(\Delta, \bar{\Delta})$
Degenerate	$V_{(r,s)}^d$	$s=1; r \in \mathbb{N}^*$	$(\Delta_{(r,s)}, \Delta_{(r,s)})$
Diagonal	V_P	$P \in \mathbb{C}$	(Δ_P, Δ_P)
Order parameter	$V_{(r,0)}$	$r \in \mathbb{N} + \frac{1}{2}$	$(\Delta_{(r,0)}, \Delta_{(r,0)})$
Non-diagonal	$V_{(r,s)}$	$s \in \mathbb{N}^*; r \in \frac{1}{s}\mathbb{Z}$	$(\Delta_{(r,s)}, \Delta_{(-r,s)})$

where now the notation $\Delta_{(r,s)}$ coincides with that of the Kac table (34):

$$\Delta_{(r,s)} = h_{r,s}, \quad [\text{Potts model}]. \quad (57)$$

Accordingly, in the column “Parameters” the labels r, s are reversed with respect to the corresponding table for the $O(n)$ model. For the same reason (52)–(53) are replaced by

$$\Delta_{(r,s)} = P_{(r,s)}^2 - P_{(1,1)}^2, \quad (58)$$

$$P_{(r,s)} = \frac{1}{2} (-\beta s + \beta^{-1} r). \quad (59)$$

³ The modular invariant torus partition function [27] contains the degenerate, order parameter and non-diagonal fields, except non-diagonal fields $V_{(r,s)}$ with $s = 1$. The latter $V_{(r,1)}$, as well as the most general diagonal fields V_P , can be realized in the representation theory of the underlying affine Temperley-Lieb algebra, which is the basis of our transfer-matrix treatment, so they are included in the table.

But there are a few more subtle differences, which we now highlight.

First, in the list of degenerate operators, the non-trivial index r can now take both even and odd values. In particular, the local energy operator is $\varepsilon = V_{(2,1)}^d$.

Second, the non-diagonal operators are now interpreted as inserting a defect of s Potts cluster boundaries. Each such boundary can be thought of as a marked point that inserts a pair of open loop segments, so we have $2s$ open loop segments as before, but now s must be an integer rather than half an integer. Another way to see this is that odd values of $2s$ simply do not make sense in the microscopic formulation of the Potts loop model, because of the completely-packing constraint.

Third, the non-diagonal operator with $s = 1$ actually drops out of the spectrum (it has zero multiplicity in the modular invariant torus partition function). It is replaced by the *order-parameter operators* $V_{(r,0)}$, where $r \in \mathbb{N} + \frac{1}{2}$. The first member of this series is $\sigma = V_{(1/2,0)}$, which can be interpreted in the spin formulation as the local spin operator, and in the cluster formulation as an operator, that inserts one cluster (not a cluster boundary). The order-parameter operators are in fact just special cases of diagonal operators. To see why, notice that for any $r \in \mathbb{N} + \frac{1}{2}$, inserting $P = P_{(r,0)}$ into (54) we find $w(P) = 0$. This means that the two-point function $\langle V_P V_P \rangle$ does not admit any loop that separates one of the operator insertion points from the other. Therefore they will have to belong to the same FK cluster, which is the defining property obeyed by the two-point function $\langle V_{(r,0)} V_{(r,0)} \rangle$ of an order-parameter operator.

We remark in passing that comparing (50) with (52) shows that the dimension $\Delta_{(r,s)}$ of any operator can be evaluated in this way by passing through the equivalent momentum $P = P_{(r,s)}$.

H. Global S_Q symmetry

Instead of using the loop representation (8) of the Potts model, it can also be studied in the spin representation for integer Q . When $Q = 1$ the resulting model is trivial (all Potts spins take the same value). $Q = 2$ and $Q = 3$ give the minimal models with $b = 4$ and $b = 6$, as discussed in Section II E; $Q = 4$ similarly arises as the limit $b \rightarrow \infty$. In those cases the spin representation results from taking a quotient of the underlying Temperley-Lieb algebra. This quotient renders the CFT unitary and, for $Q = 2$ and $Q = 3$ minimal. If the quotient is not taken, the non-local operators make the CFT logarithmic: the same would be true for taking $Q \rightarrow 1$ as a limit in the loop model. Our interest instead is to take *generic* values of the loop weight $n = \sqrt{Q}$ — namely those where $n = q + q^{-1}$, and q is not a root of unity. The diagram algebra of Section II B then has a semisimple representation, corresponding to the non-logarithmic CFT with operator content described in Section II G.

Taking $Q > 4$ integer and using the spin representation is another generic choice. This generic representation allows us to make contact with the works [19, 20] mentioned in the introduction, where the complex Q -state Potts CFT was studied

as the scaling limit of a $Q = 5$ state non-Hermitian spin chain. To clarify this link we need to discuss the global S_Q symmetry of the model, following [28]. As explained there, operators can be classified in terms of S_Q representations in the generic case, where Q may take non-integer or even complex values. Recall that finite-dimensional irreducible representations of S_Q , that we first think of for $Q \in \mathbb{N}^*$, are parametrized by Young diagrams with Q boxes. To extend this to a more general setting, we pick an integer partition $\lambda = [\lambda_1, \lambda_2, \lambda_3 \dots]$ with integer $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ that we can visualize as row lengths of a diagram. Let $|\lambda| = \sum_i \lambda_i$ be the size of this diagram. Then, for $Q \geq |\lambda| + \lambda_1$ integer, $\tilde{\lambda} = [\lambda_0, \lambda]$ is a Young diagram with an extra zeroth row of length $\lambda_0 = Q - |\lambda|$ and size $|\lambda_0| = Q$. It turns out that $\tilde{\lambda}$ is a valid S_Q irreducible representation, in the sense of Deligne categories, even when $Q \in \mathbb{C}$. In particular we can compute its dimension (not necessarily a positive integer) and write tensor products $\tilde{\lambda}_1 \times \tilde{\lambda}_2$ as a sum of irreducibles [28].

For the Potts model, the trivial, one-dimensional (scalar) representation is written $\lambda = []$; a Q -dimensional vector in which the Potts indices are permuted under S_Q decomposes on the irreducibles $[1] + []$; and generally any operator acting on M spins can be decomposed on irreducibles λ with $|\lambda| \leq M$ [57].

The state space $\mathcal{S}$ of the Q -state Potts CFT is a representation of $\mathcal{C}_c \times S_Q$, where $\mathcal{C}_c$ denotes the conformal algebra for the CFT with central charge c , and S_Q is the global symmetry of the symmetric group in the Deligne sense. Decomposing this on irreducibles one finds [28] that

$$\begin{aligned} \mathcal{S} = & \bigoplus_{r \in \mathbb{N}^*} \mathcal{R}_{(r,1)} \otimes [] \oplus \bigoplus_{r \in \mathbb{N} + \frac{1}{2}} \mathcal{W}_{(r,0)} \otimes [1] \\ & \oplus \bigoplus_{s \in \mathbb{N} + 2} \bigoplus_{r \in \frac{1}{s}\mathbb{Z}} \mathcal{W}_{(r,s)} \otimes \Xi_{(r,s)}. \end{aligned} \quad (60)$$

Here the $\mathcal{C}_c$ irreducibles $\mathcal{R}_{(r,1)}$, $\mathcal{W}_{(r,0)}$ and $\mathcal{W}_{(r,s)}$ correspond to the set of degenerate (energy-type) operators $V_{(r,1)}^d$, the order-parameter (spin-type) operators $V_{(r,0)}$, and the non-diagonal (defect-type) operators $V_{(r,s)}$ acting on $2s$ TL lines or s FK clusters. They are exactly the operators that were discussed in Section II G 2. On the right-hand side of the tensor products we find the S_Q representations which are, respectively, for the three types of operators: the trivial representation $[]$, the standard representation $[1]$, and another family of representations $\Xi_{(r,s)}$. This latter family of representations satisfy $\Xi_{(r,s)} = \Xi_{(-r,s)} = \Xi_{(r+1,s)}$, so the representations of $\mathcal{C}_c$ can be regrouped into larger representations of a so-called interchiral algebra, obtained from $\mathcal{C}_c$ by acting with the degenerate energy operator $V_{(2,1)}^d$. This regrouping amounts to constraining the r -index to the interval $0 \leq r \leq 1/2$, keeping the spin rs integer. Finally, $\Xi_{(r,s)}$ can be decomposed into S_Q

irreducibles λ . For the first few cases one finds [28]

$$\Xi_{(0,2)} = [2], \quad (61a)$$

$$\Xi_{(\frac{1}{2},2)} = [1^2], \quad (61b)$$

$$\Xi_{(0,3)} = [3] + [1^3], \quad (61c)$$

$$\Xi_{(\frac{1}{3},3)} = [21], \quad (61d)$$

where $[1^k]$ is a short-hand for $[1, 1, \dots]$ with an index repeated k times.

After this review, we are ready to discuss the link between the operator content of Section II G 2 and that exposed in [19]. Table I of that reference lists 11 low-lying primary operators in the $Q = 5$ state complex Potts CFT. Here is a table where we compare them to the above discussion:

[19]	This work	λ	Dimension	Spin
ε	$V_{(2,1)}^d$	$[]$	$2h_{2,1} = 0.466 + 0.255i$	0
ε'	$V_{(3,1)}^d$	$[]$	$2h_{3,1} = 1.908 + 0.599i$	0
ε''	$V_{(4,1)}^d$	$[]$	$2h_{4,1} = 4.328 + 1.123i$	0
σ	$V_{(\frac{1}{2},0)}$	$[1]$	$2h_{\frac{1}{2},0} = 0.134 + 0.021i$	0
σ'	$V_{(\frac{3}{2},0)}$	$[1]$	$2h_{\frac{3}{2},0} = 1.111 + 0.170i$	0
σ''	$V_{(\frac{5}{2},0)}$	$[1]$	$2h_{\frac{5}{2},0} = 3.065 + 0.469i$	0
Z	$V_{(0,2)}$	$[2]$	$2h_{0,2} = 2.011 - 0.305i$	0
Z'	$V_{(0,3)}$	$[1^3]$	$2h_{0,3} = 4.511 - 0.688i$	0
X	$V_{(\frac{1}{2},2)}$	$[1^2]$	$h_{\frac{1}{2},2} + h_{-\frac{1}{2},2} = 2.134 - 0.286i$	1
Y	$V_{(1,2)}$	$[2]$	$h_{1,2} + h_{-1,2} = 2.500 - 0.230i$	2
Z	$V_{(\frac{3}{2},2)}$	$[1^2]$	$h_{\frac{3}{2},2} + h_{-\frac{3}{2},2} = 3.111 - 0.136i$	3

Our dimensions agree with those of [19] up to complex conjugation⁴. The S_Q irreducibles λ also agree with the Q -box Young diagrams $\tilde{\lambda} = [Q - |\lambda|, \lambda]$ given in [19], once the decompositions (61) of $\Xi_{(r,s)}$ are taken into account. In all but one of the 11 cases this is manifestly simple, because the operator is of the energy or spin type (with $\lambda = []$ or $[1]$ respectively), or because the right-hand side of (61) contains only one term.

The remaining case of the operator called Z' in [19] requires a bit more discussion. In our setup, Z' is contained in the representation $\Xi_{(0,3)}$ which by (61) decomposes into the S_Q irreducibles $\tilde{\lambda} = [Q - 3, 3]$ and $[Q - 3, 1^3]$. The first of those does not exist for $Q = 5$ (since $[2, 3]$ is not a valid Young diagram), but the other one $[2, 1^3]$ does. In Table I of [19], Z' is however identified with the S_5 irreducible $[4, 1]$. Notice that $[4, 1]$ and $[2, 1^3]$ are conjugate tableaux, related by interchanging rows and columns, so they have in particular the same dimension. It is possible that, in the context of the methods employed in [19] and the issues being addressed there, it is permissible to use one or the other. From our point of view the two diagrams do not correspond to the same symmetry.

⁴ For $Q > 4$ and real there is a pair of complex conjugate theories, transforming into each other under complex conjugation.

In summary, we are in agreement with the operator content and (up to the minor point just mentioned) with its S_Q symmetry classification reported in [19] for $Q = 5$; our construction is more general, since it makes sense for all $Q > 4$, even non-integer, as well as for generic $Q \in \mathbb{C}$.

I. Three-point structure constants

The identification of the Q -state Potts CFT by analytic continuation of the results of [28, 29, 42–45, 56] can be taken beyond the operator content (Section II G) and the global symmetry (Section II H). Indeed, by Conjecture 2 it applies to all features of the CFT; we now discuss the structure constants entering into the definition of three-point correlation functions.⁵

A recent piece of work [42] (inspired by [29]) proposed a general formula for the structure constants in the loop model that covers all types of operators discussed in Section II G. It contains as special cases the structure constants of degenerate operators which are covered by the work of Dotsenko and Fateev [58] or can be found by analytic continuation thereof (see Appendix G). It also contains the result for diagonal operators, first found in [39] and then established rigorously [41].

As mentioned in Section II G, the notations used in [42] apply to the $O(n)$ model, but we report below the results for the Potts model — the object of interest in this paper — obtained after swapping the r, s labels with respect to [42], as discussed in Section II G 2. Using the parameter β of (48) and setting

$$Q = \beta + \beta^{-1}, \quad (62)$$

the Barnes double Gamma function $\Gamma_\beta(w)$ is defined from its integral representation, convergent for $\Re w > 0$,

$$\ln \Gamma_\beta(w) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-wt} - e^{-\frac{Q}{2}t}}{(1 - e^{-\beta t})(1 - e^{-\beta^{-1}t})} - \frac{\left(\frac{Q}{2} - w\right)^2}{2} e^{-t} - \frac{\frac{Q}{2} - w}{t} \right]. \quad (63)$$

It satisfies the shift relations

$$\Gamma_\beta(w + \beta) = \sqrt{2\pi} \frac{\beta^{\beta w - \frac{1}{2}}}{\Gamma(\beta w)} \Gamma_\beta(w), \quad (64)$$

$$\Gamma_\beta(w + \beta^{-1}) = \sqrt{2\pi} \frac{\beta^{-\beta^{-1}w + \frac{1}{2}}}{\Gamma(\beta^{-1}w)} \Gamma_\beta(w). \quad (65)$$

⁵ More generally we would expect the identification to hold for any correlation function that can be expressed in terms of the loop model. The cited references contain much progress on the four-point functions on the Riemann sphere, albeit still falling short of full and explicit analytical expressions.

The so-called reference structure constant for three non-diagonal operators, $V_{(r_i, s_i)}$ with $i = 1, 2, 3$, is defined as [42]

$$C_{(r_1, s_1)(r_2, s_2)(r_3, s_3)} = \prod_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm} \Gamma_\beta^{-1} \left(\frac{\beta + \beta^{-1}}{2} + \frac{\beta}{2} |\sum_i \varepsilon_i s_i| + \frac{\beta^{-1}}{2} \sum_i \varepsilon_i r_i \right), \quad (66)$$

where $\Gamma_\beta^{-1} = 1/\Gamma_\beta$. The result for diagonal and degenerate operators is also given by Eq. (66), upon using the indices $(r, s) = (2\beta P, 0)$ with P being the momentum variable defined in (59). For instance, the identity operator has momentum $P = P_{(1,1)}$ and can be represented by using $(r, s) = (1 - \beta^2, 0)$ in (66). The three-point structure constant, normalized by two- and zero-point constants in such a way that the result is invariant under field rescalings $V_{(r,s)} \rightarrow \lambda_{(r,s)} V_{(r,s)}$, is finally given by

$$\omega_{123} = C_{123} \sqrt{\frac{C_{000}}{C_{011} C_{022} C_{033}}}, \quad (67)$$

where subscripts 1, 2, 3 are short-hands for the pair (r_i, s_i) with $i = 1, 2, 3$, whereas 0 refers to the identity field with $(r, s) = (1 - \beta^2, 0)$. This expression can be numerically evaluated using the JULIA package BarnesDoubleGamma.jl [59].

We have checked that the 8 structure constants given in Table II of [19] for the $Q = 5$ state complex Potts CFT agree with (67), up to complex conjugation as noticed above. These structure constants involve the degenerate energy operators $\varepsilon, \varepsilon', \varepsilon''$ and the non-diagonal spin operators $\sigma, \sigma', \sigma''$. For instance we find, for the structure constant denoted $C_{\sigma\sigma'\varepsilon'}$ in [19]

$$\omega_{(\frac{1}{2}, 0)(\frac{5}{2}, 0)(3, 1)} = 0.6710 + 0.1143i, \quad (68)$$

with $\sigma = V_{(\frac{1}{2}, 0)}$, $\sigma'' = V_{(\frac{5}{2}, 0)}$ and $\varepsilon' = V_{(3, 1)}^d$ being identified from the table in Section II G 2.

In the same way we can produce results not covered by Table II of [19]. For instance

$$C_{\sigma\sigma Z} = \omega_{(\frac{1}{2}, 0)(\frac{1}{2}, 0)(0, 2)} = 0.05193 + 0.03344i. \quad (69)$$

The authors of [19] have kindly provided their unpublished numerical result for this structure constant [60], which is $C_{\sigma\sigma Z} = 0.0581 - 0.0352i$ with unspecified error bars. The agreement looks rather convincing, up to the usual complex conjugation.

III. FISHER ZEROES IN THE MODEL WITH NEAREST-NEIGHBOR INTERACTIONS

The bulk of this paper addresses the complex criticality of the model (2) defined on triangular lattice, with local interactions (1) comprising two-spin interactions K between nearest neighbors and three-spin interactions K_3 in up-pointing triangles. In this section we show that the usual Potts model, with only nearest-neighbor interactions, is insufficient to access the complex fixed point.

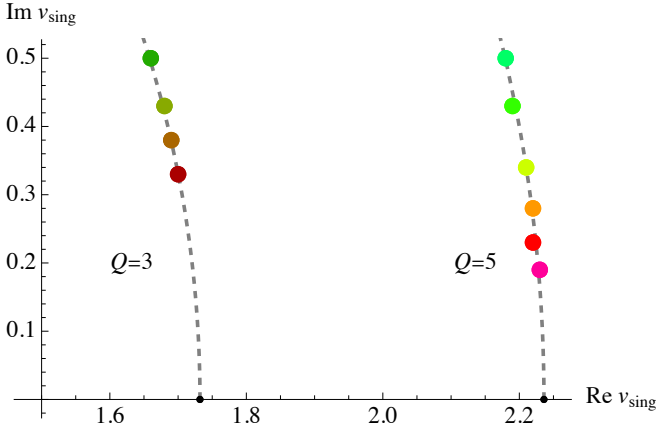


FIG. 6. The location of the nearest Fisher singularity in the complex v_c plane, for $Q = 3$ (left) and $Q = 5$ (right), evaluated on a grid of size $\delta v = 0.01$ for both the real and imaginary part. The different colors indicate different system sizes: for $Q = 3$ (darker) there are $L = 6$ (green, top) to $L = 9$ (red, bottom); for $Q = 5$ (brighter) there are $L = 5$ (blue-green, top) to $L = 9$ (magenta, bottom). The gray dashed lines are the curves $|v_c|^2 = Q$. Small grid artifacts are visible.

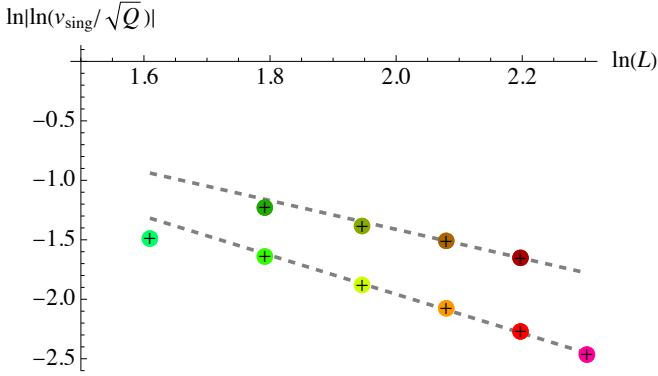


FIG. 7. Scaling of the distance to the critical point $v_c = \sqrt{Q}$ for $Q = 3$ and $Q = 5$. The plotted slopes are $-1.2(2)$ for $Q = 3$ and $-1.6(2)$ for $Q = 5$, fitting to the last points. The crosses use $\text{Im}(\ln v_c/\sqrt{Q})$ instead of $|\ln v_c/\sqrt{Q}|$. We interpret the seemingly increasing slope for large L as a sign for the first-order transition.

Consider therefore the square-lattice Potts model

$$\mathcal{H}_{\text{sq}} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}, \quad (70)$$

where $E = \langle ij \rangle$ denotes the set of nearest neighbors. The corresponding FK cluster model [cf. (6)] takes the simple form

$$Z_{\text{sq}} = \sum_{\{\sigma\}} e^{\mathcal{H}_{\text{sq}}} = \sum_{A \subseteq E} Q^C v^{|A|}, \quad (71)$$

where $v = e^J - 1$, and C is the number of connected components in the subgraph having only the subset of edges A .

To inquire into the analytical structure of (71), we examine its Fisher zeros in the complex v -plane. This is of course a

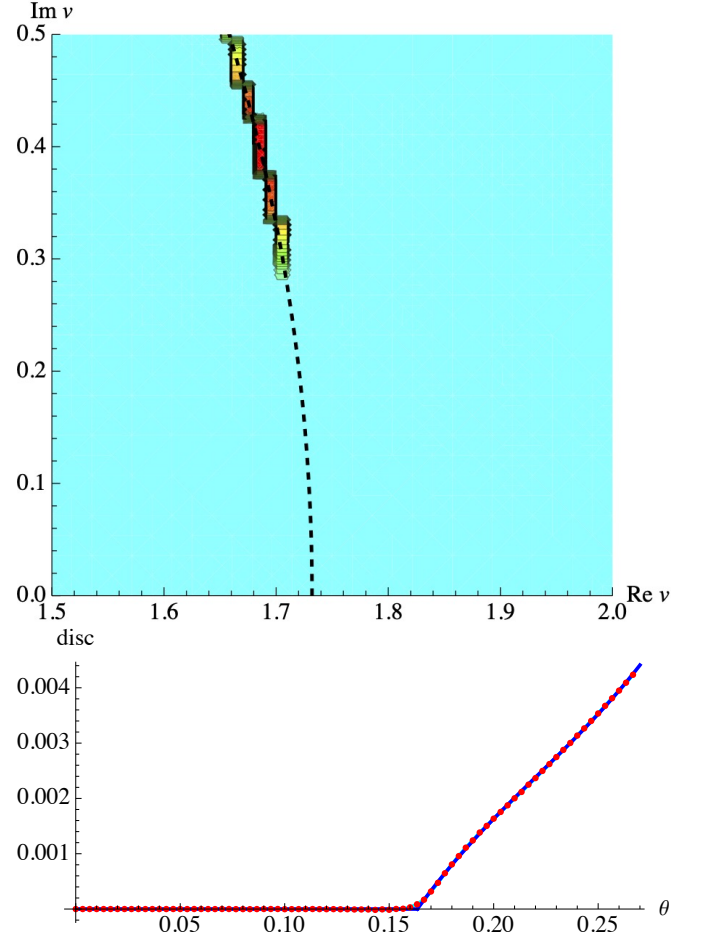


FIG. 8. Top: Singularity of f for $Q = 3$, $L = 10$. Bottom: Discontinuity of $\text{Re}(f)/|f|$ across the cut.

well-studied subject, and since our sole purpose is to elucidate the question stated above, we shall be brief.

Previous studies of Fisher zeros include [61–63] for the Ising model, and [64] for the Potts model. While Refs. [63, 64] consider lattices of size $L \times L$ with free boundary conditions, our study is via the transfer matrix for a cylinder of circumference L , i.e. an $L \times \infty$ lattice. From its dominant eigenvalue Λ_0 (the largest in norm), we obtain $f = -\frac{1}{L} \ln \Lambda_0$, the free energy per unit area.

The critical coupling J_c is given by

$$e^{J_c} - 1 = v_c = \sqrt{Q}. \quad (72)$$

Fig. 6 shows that both for $Q = 3$ and $Q = 5$ there is a singularity close to v_c , at

$$v_{\text{sing}} \approx v_c e^{ia}, \quad a \in \mathbb{R}, \quad (73)$$

i.e. on a circle of radius v_c , slightly off from the real axis. The parameter a decreases with system size L . Numerically we find in Fig. 7

$$a \sim L^{-1.2(2)} \text{ for } Q = 3, \quad (74)$$

$$a \sim L^{-1.6(2)} \text{ for } Q = 5. \quad (75)$$

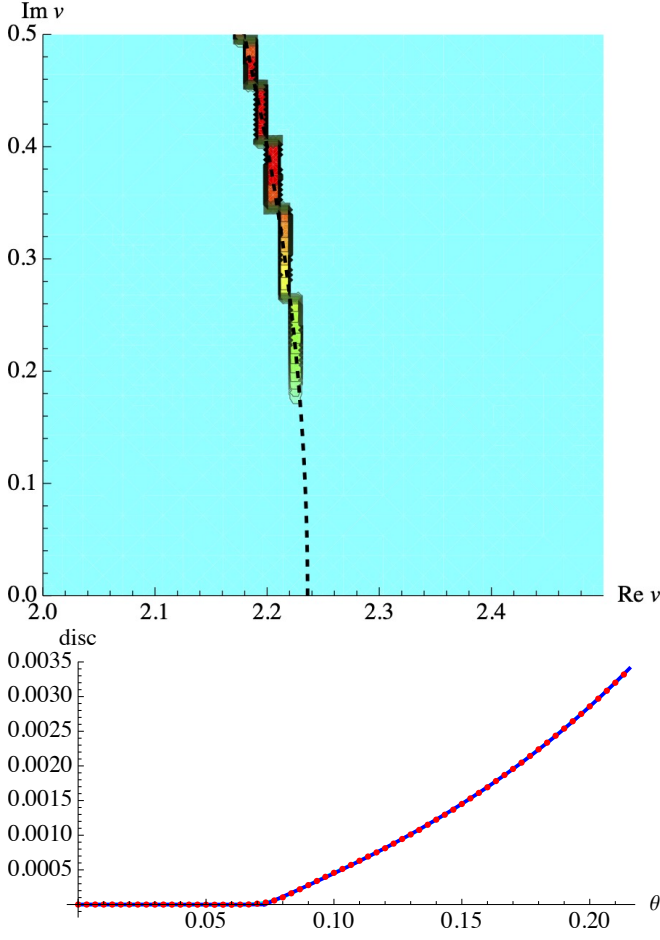


FIG. 9. Top: Singularity in Z for $Q = 5$, $L = 11$. Bottom: Discontinuity of $\text{Re}(f)/|f|$ across the cut. In red numerical points, in blue fit to a polynomial for $\theta > \theta_c$.

This is in reasonable agreement with the theoretical expectation [61]

$$a \sim L^{-\frac{1}{\nu}}. \quad (76)$$

Indeed, the CFT gives the exponent as $\frac{1}{\nu} = 2 - 2h_{2,1}$. For $Q = 3$ and $Q = 5$ this evaluates to

$$\frac{1}{\nu}\bigg|_{Q=3} = 2 - 2h_{2,1}\bigg|_{Q=3} = \frac{6}{5}, \quad (77)$$

$$\frac{1}{\nu}\bigg|_{Q=5} = 2 - 2h_{2,1}\bigg|_{Q=5} = 1.53439 \pm 0.224495i. \quad (78)$$

Eq. (77) is in agreement with Eq. (74) for $Q = 3$. We however know from exact results [8] that the phase transition in the square-lattice model (70) is first order for $Q > 4$. Thus, for $Q = 5$ the argument of the singularity a should not tend to zero as $L \rightarrow \infty$, but rather converge to a very small, but finite value $a_\infty > 0$. So in a log-log plot like Fig. 7 we should eventually see the $Q = 5$ curve level off to a finite value, if only we could access L large enough. The correlation length ξ at v_c can be estimated from its analytical expansion around $Q = 4$ [65]. Evaluated at $Q = 5$ this gives $\xi \approx 10^3$, so

the sizes employed in our study satisfy $L \ll \xi$. Therefore the first-order nature of the transition is not visible to us, and the scaling looks critical, being dominated by the two nearby complex fixed points and a very slow running of the effective coupling constant, a phenomenon known as walking [14, 66]. From the point of view of effective scaling, in the regime $L \ll \xi$ the apparent slope in Fig. 7 should thus be $\Re(\nu^{-1}) \approx 1.53$, which is in reasonable agreement with Eq. (75).

The point is now that even though the model (70) can detect the proximity of the complex fixed point, its parameter space is not large enough to enter its basin of attraction. In particular, we have checked that there is no value of $v \in \mathbb{C}$ for which the finite-size scaling of f reveals the central charge of the complex fixed point.

Figures 8 (for $Q = 3$) and 9 (for $Q = 5$) show that there is a branch cut of the free energy f per site, visible in the color plot at the top; this branch cut sits on a circle of radius $\sqrt{Q}$, i.e. at $v = \sqrt{Q}e^{i\theta}$ (marked by a dashed line). The bottom plots explore the discontinuity across the cut as a function of the angle θ . The discontinuity grows approximately linearly from the cut's endpoint.

IV. NUMERICS FOR c AND ω ON THE REAL LINE ($Q < 4$, $b \in \mathbb{R}$)

A. Outline

Let us return to the model with nearest-neighbor and plaquet interactions defined in section II A. In this section, we extract the central charge c and the correction-to-scaling exponent ω from transfer-matrix calculations of the Potts model on strips of width L . Before addressing the more challenging regime of complex couplings and $Q > 4$, we first study the case of real b (corresponding to $Q < 4$). In this regime the critical behavior is well understood from conformal field theory. This allows us to understand the role of extremas of the effective central charge in locating the critical point. Once these tools are validated, we apply them to the more intricate situations discussed later.

B. Extrapolation on the lattice for $L \rightarrow \infty$

Transfer-matrix calculations give us the free energy density f_L of a system of size L . Following [67–69], we use this to extract the effective central charge c ,

$$f_L = f_\infty - \frac{\pi c}{6L^2} + \dots \quad (79)$$

Solving for two consecutive sizes, we find

$$c_{\text{eff}}(L) \equiv \frac{6L^2(L+1)^2(f_{L+1} - f_L)}{\pi(2L+1)}. \quad (80)$$

This is the effective central charge which we plot as a function of x , or $z = x + iy$. Note that while $c_{\text{eff}}(L)$ is indexed by L , we need system size $L+1$ to evaluate it. The same holds later

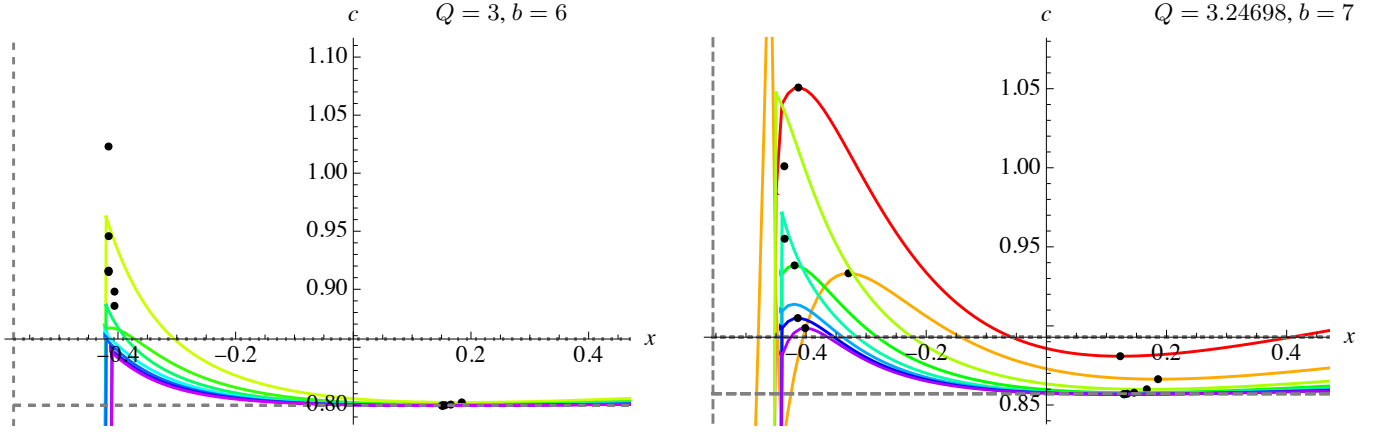


FIG. 10. Left: $c(x)$ for $\mathcal{B}_6$, i.e. $Q = 3$, extracted from Eq. (80) from $L = 2$ and 3 (red) over green, blue to violet for $L = 10$ and 11. The dots are at the maximum and minimum of $c(z)$. The dashed horizontal line represents the central charge for $\mathcal{B}_6$, the dotted line (on the axis) the central charge for $\mathcal{B}_7 \equiv \mathcal{B}_{-6}$. The left dashed vertical line marks the location of the singularity at $x = -1/\sqrt{Q}$. Right: The same for $\mathcal{B}_7$, i.e. $Q = 3.24698$. On both plots, one clearly sees a minimum of the effective central charge c at $x \approx 0.17$, and a maximum at $x \approx -0.4$. For $Q = 6$, the latter is obscured by the singularity at $x = -1/\sqrt{Q}$. It becomes visible at $Q = 7$, but only for large values of L .

for the correction-to-scaling exponent ω , which needs system sizes $L - 1$, L and $L + 1$ to evaluate it.

Before attacking $Q = 5$, we test our transfer-matrix calculations. We start our exploration for $Q < 4$, which corresponds to real b . A special role is played by the minimal models for $b \in \mathbb{N}$, with $b = 6$ for the 3-state Potts model. The limit of $b \rightarrow \infty$ corresponds to $Q = 4$.

C. $\mathcal{B}_6, \mathcal{B}_7$ and $\mathcal{B}_8$

We start our exploration with the models $\mathcal{B}_6$ and $\mathcal{B}_7$, see figure 10. On both plots, one clearly sees a minimum of the effective central charge c at $x \approx 0.17$, and a maximum at $x \approx -0.4$. For $b = 6$, the latter is obscured by the singularity at $x = -1/\sqrt{Q}$.⁶ It becomes visible at $b = 7$, but only for large values of L .

Things get clearer when we proceed to $\mathcal{B}_8$, see Fig. 12. We see a clear minimum of the effective central charge $c_L(x)$ for $x \approx 0.12$, and a maximum at $x \approx -0.34$. Fig. 12 shows that upon increasing L , the minimum of $c_L(z)$ becomes shallower, and the maximum more pointed. We associate this with a critical point for the minimum: increasing L , the measured effective central charge $c_L(z)$ converges to a limiting value, independent of the initial value of z , as long as the latter is close enough to the minimum. In contrast, the value at the maximum becomes more and more sensitive to the value of x , and we associate the deviation from the critical x as a relevant perturbation, characteristic of a tricritical point.

In other words, thanks to the self-duality condition $w = 1$ [see (20)] we are already on a critical manifold for any x . Upon adjusting x to a particular value we can reach a tricritical, repulsive fixed point P_t at x_t . The part $x > x_t$ of the $w = 1$ manifold is then governed by the critical, attractive fixed point P_c at x_c . There is an RG flow from P_t to P_c along which the central charge decreases, in agreement with Zamolodchikov's c -theorem [70].

A quantitative analysis is shown in Fig. 11. One finds for the left fixed point

$$\begin{aligned} z_c(L \rightarrow \infty) &= -0.34036, \\ c(L \rightarrow \infty) &= 0.918224, \\ c(\mathcal{B}_{-8}) &= 0.916667. \end{aligned} \quad (81)$$

The same analysis for the right fixed point yields

$$\begin{aligned} z_c(L \rightarrow \infty) &= 0.129947, \\ c(L \rightarrow \infty) &= 0.892867, \\ c(\mathcal{B}_8) &= 0.892857. \end{aligned} \quad (82)$$

We see that for both fixed points, the central charge can be extracted, and agrees rather precisely with the prediction (31) from the minimal model.

Let us next return to our discussion of critical versus tricritical behavior. This can be made quantitative through a finite-size analysis of the curvature $c_L''(z)$. It allows us to extract the correction-to-scaling exponent ω via

$$c_L''(z_c) \sim L^{-2\omega}. \quad (83)$$

Derivation: Since the critical points are stationary points (maxima or minima in z),

$$c_L(z) = c(z_c) + \frac{1}{2} \partial_z^2 c(z)(z - z_c)^2 + \dots \quad (84)$$

Here, $z - z_c$ is the deviation from the critical point in terms of the bare coupling z . In terms of the renormalized coupling

⁶ The limit $z \rightarrow -1/\sqrt{Q}$, or $c_2 \rightarrow -1$ from Eq. (9), corresponds to $K \rightarrow -\infty$ by Eq. (4), hence an infinitely strong antiferromagnetic two-spin interaction. Since $c_3 = Q$ at criticality by Eq. (20), we must have $K_3 \rightarrow \infty$ in Eq. (5) in such a way that keeps $e^{K_3+3K} = Q - 2$ fixed, implying an infinitely strong ferromagnetic three-spin interaction when $Q > 2$.

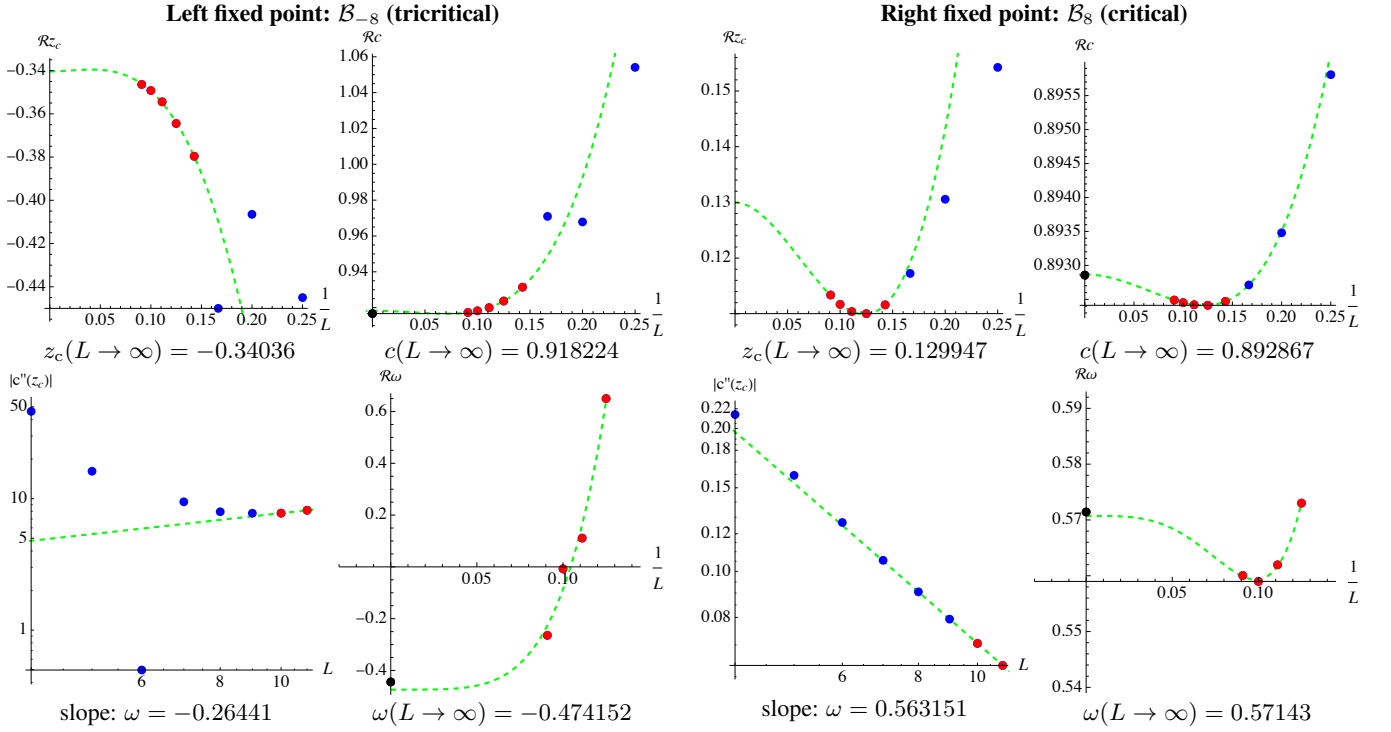


FIG. 11. Left half of plot: Scaling analysis of $c(z)$ (see Fig. 12) for the model B_8 ($Q = 3.41421$), for the left fixed point: The first plot shows the convergence of $x \equiv \Re z$ towards its critical value. (All imaginary parts vanish.) The next plot shows the convergence of $c = \Re c$, followed by an estimate of ω from $c''(z)$ from the two largest system sizes, and finally an extrapolation to $L = \infty$ for ω . Right half of plot: *idem* for the right fixed point.

z^R , this reads

$$(z^R - z_c)L^\omega = z - z_c \quad (85)$$

Inserting Eq. (85) into Eq. (84) yields Eq. (83).

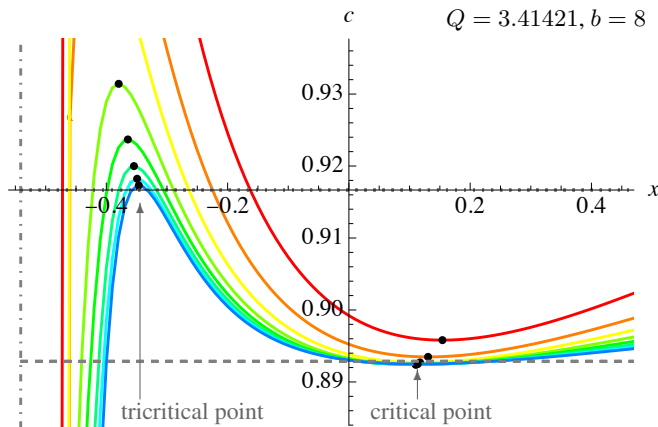


FIG. 12. $c_L(x)$ for $b = 8$, $Q \simeq 3.41421$, using sizes $L = 4$ (red) to 12 (blue). The discontinuity near $x = -1/\sqrt{Q}$ (vertical dot-dashed line) is related to level crossings for $K_2 \rightarrow -\infty$. These prevent us from using this approach for smaller Q . CFT predictions are shown for B_8 (dashed line) and $B_{-8} \equiv B_9$ (axis, and gray dots).

A more precise way to extract ω is to write

$$\frac{c''(L)}{c''(L+1)} = \frac{L^{-2\omega}}{(L+1)^{-2\omega}}. \quad (86)$$

Taking the logarithm yields

$$\ln \frac{c''(L)}{c''(L+1)} = 2\omega \ln \left(1 + \frac{1}{L}\right). \quad (87)$$

We define

$$\omega_L := \frac{1}{2} \frac{\ln \left(\frac{c''(L)}{c''(L+1)} \right)}{\ln \left(1 + \frac{1}{L} \right)}. \quad (88)$$

This is plotted on the lower left plot in Fig. 11. Finally, we can extrapolate these values to $L = \infty$, as shown in the following plot of Fig. 11.

The exponent ω is related to the dimension Δ of the perturbing operators as $\Delta = 2 + \omega$. (The 2 comes from the spatial integral over the perturbing operator.) Using what we found in Fig. 11, we get (with c = critical and t = tricritical)

$$\Delta^c = 2 + \omega^c \approx 2.57, \quad \Delta^t = 2 + \omega^t \approx 1.53, \quad (89)$$

$$2h_{3,1}^{b=8} = \frac{18}{7} = 2.571, \quad 2h_{3,1}^{b=-8} = \frac{14}{9} = 1.556. \quad (90)$$

We find good agreement of measurements (89) with the predictions of CFT (90).

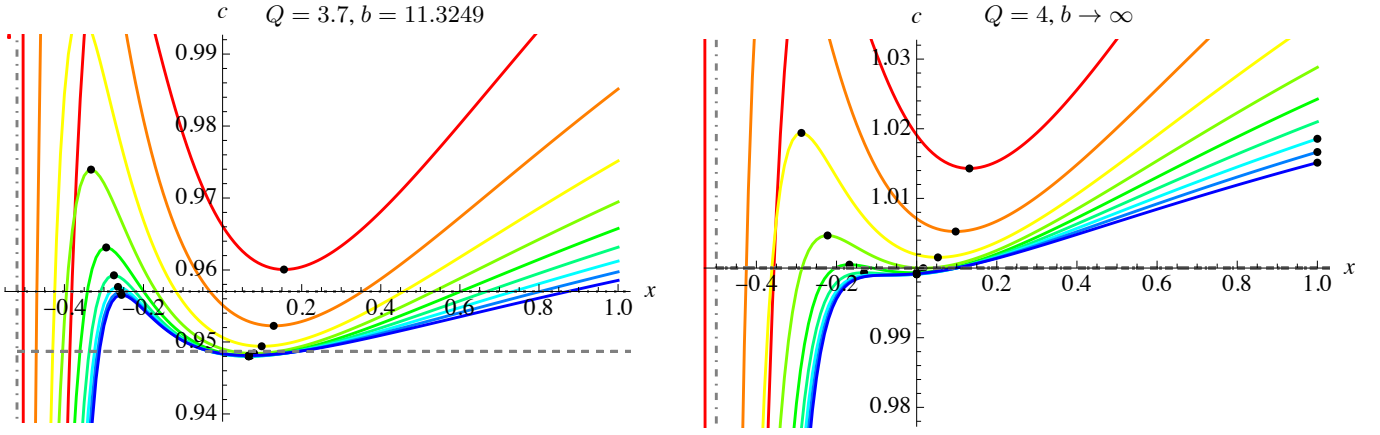


FIG. 13. Left: $c(x)$ for $b = 11.3249$, i.e. $Q = 3.7$, extracted from Eq. (80) from $L = 3$ over green, blue to violet for $L = 11$ and 12; see Fig. 10 for details on what is plotted. Right: *idem* for $Q = 4$ ($b \rightarrow \infty$). We see that for $Q \rightarrow 4$, the two fixed points approach, and merge at $Q = 4$, at $z_c \approx -0.02 \pm 0.02$.

At the critical point P_c , according to section II G 2 the leading energy operator is $\varepsilon = V_{(2,1)}^d$. It governs the perturbation away from the critical manifold $w = 1$ and the latter is obtained, in the continuum limit, by tuning the amplitude of ε to zero. The sub-leading energy operator $\varepsilon' = V_{(3,1)}^d$ is irrelevant and governs the corrections to scaling upon flowing into P_c along the critical manifold. At the tricritical point P_t the leading perturbation is again by $V_{(3,1)}^d$ and it governs the flow towards P_c .

This result was subsumed in section II F and used there to determine the phase diagram as a function of $Q \in \mathbb{C}$. It appears to be in tension with a classical paper by Zamolodchikov [54], in which he establishes a massless flow from the minimal model $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ induced by the perturbation by $\phi_{1,3}$ (not $\phi_{3,1}$). However, there needs not be any contradiction. The models $\mathcal{B}_b$ considered here are loop models, not minimal models, even though we have often taken b integer for practical purposes (e.g. having rational exponents). In particular, the loop models have the spectrum given in section II G 2 which does simply not contain the field $V_{(1,3)}^d$. Indeed, if this field was present in our CFT all along the RG trajectory, it would lead to a contradiction with the flow from P_t to P_c observed in Fig. 12, since according to Eq. (43) the field $V_{(1,3)}^d$ is relevant. So though they look similar, our flow from $\mathcal{B}_{b+1}$ to $\mathcal{B}_b$ is not the same as the one studied by Zamolodchikov [54].

To summarize this section, we are able to identify a critical or tricritical point by $c'(z) = 0$, and use a finite-size analysis on $c_L'(z_c)$ to decide whether it is attractive (critical) or repulsive (tricritical) once T_c is fixed.⁷

D. The limit of $Q \rightarrow 4$

In Fig. 13 we study what happens in the limit of $Q \rightarrow 4$. One sees that maximum and minimum values get closer together, and become equal at $Q = 4$ ($b \rightarrow \infty$). This phenomenology is accounted for by the following toy model inspired by conformal perturbation theory [14, 66]

$$c(z) = 1 + 3z^3 + \frac{3(Q-4)}{(2\pi)^2}(2+3z). \quad (91)$$

The coefficients are chosen s.t. they are analytic in z and Q ; especially there is no singularity at $Q = 4$. The maximum and minimum obtained from $c'(z) = 0$ are at

$$z_{\pm} = \pm \frac{\sqrt{4-Q}}{2\pi} \quad (92)$$

$$c(z_{\pm}) = 1 - \frac{3(4-Q)}{2\pi^2} \pm \frac{3(4-Q)^{3/2}}{4\pi^3} \quad (93)$$

The chosen numerical values in the model (91) ensure that $c(z_+) = c(b(Q)) + \mathcal{O}(4-Q)^2$, and $c(z_-) = c(-b(Q)) + \mathcal{O}(4-Q)^2$. These values are unchanged under smooth deformations, as e.g. $z \rightarrow a(Q)z - z_c(Q)$, thus the model (91) is not unique. A very similar approach was used in [14, 66], which gives Eq. (93), evaluating to

$$c(Q=5) \approx 1.152 \pm 0.0242i. \quad (94)$$

This should be compared to Eq. (39).

V. $Q = 5$

A. Central charge in the complex z -plane

We now proceed to $Q = 5$, for which we run the transfer-matrix calculations on a discretization of the complex $z = x + iy$ plane. We chose the range $-0.5 \leq x \leq 0.5$, and

⁷ Such a prescription is missing in related work on the $O(n)$ model, where the location of the critical point is known analytically [16].

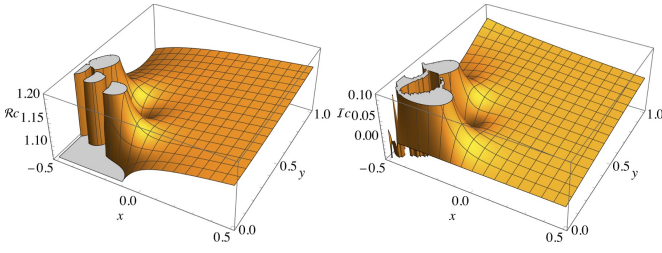


FIG. 14. $Q = 5$, $L = 10$. Data for $\Re c$ (left) and $\Im c$ right, as a function of $z = x + iy$.

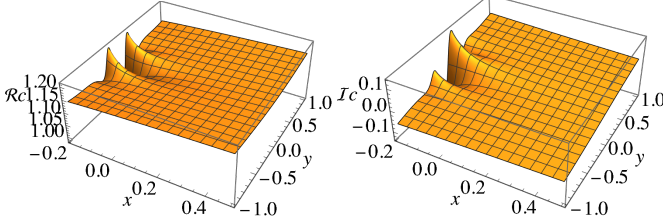


FIG. 15. Data from Fig. 14, after restriction to a smaller range in z , and doubling the data using that $c(\bar{z}) = \overline{c(z)}$. (Note the different scales between Figs. 14 and 15.)

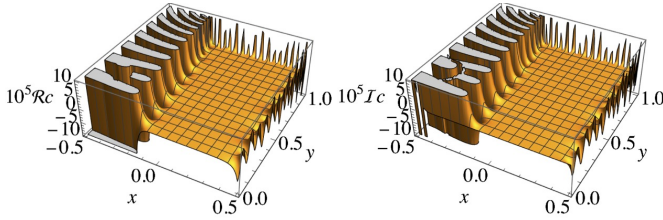


FIG. 16. Error made in the interpolation for the real and imaginary parts of c in Fig. 15, $L = 10$. Note multiplication by 10^5 .

$0 \leq y \leq 1$, with a step size of $\delta z = 0.01$. For future reference we call this numerical scheme “grid”.

In Fig. 14, we show the effective L -dependent central charge as defined by Eq. (80), as a function of z . Since Q is real,

$$c_L(\bar{z}) = \overline{c_L(z)}, \quad (95)$$

thus we know $c_L(z)$ for $-0.5 \leq x \leq 0.5$, and $-1 \leq y \leq 1$. In order to proceed, we wish to fit $c_L(z)$ with a high-order polynomial (order 130). In order that this fit be precise, we exclude the “spiky” region in Fig. 14, and only fit to the data in Fig. 15. The resulting data minus this fit is shown in Fig. 16. Noting the multiplication by 10^5 , we conclude that the fit is excellent inside the domain chosen for it in Fig. 15, with errors apparent on the boundary.

As the fit is a polynomial in z with complex coefficients, we conclude that to the best of our knowledge $c_L(z)$ is an analytic function in this domain. This is non-trivial, as $c_L(z)$ could have been a function of both z and $\bar{z}$.

To locate a critical point, analyticity strongly suggest to replace “maximum” and “minimum” used in section IV B by

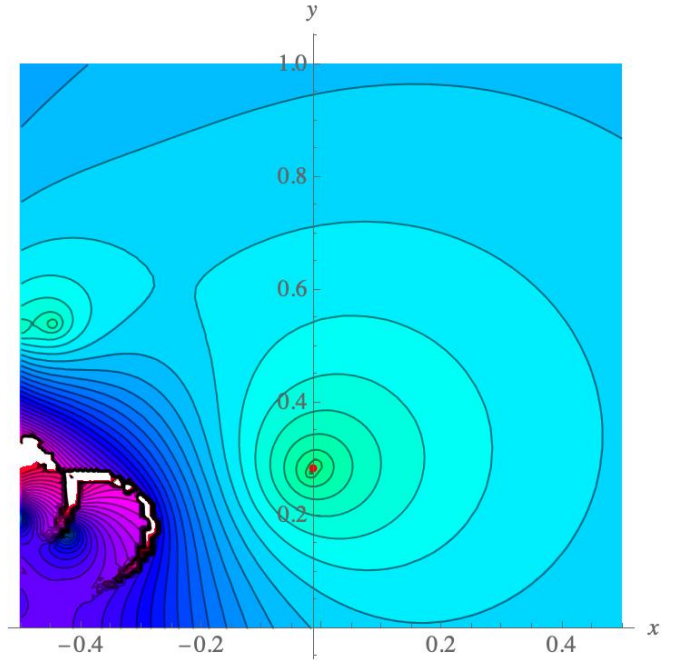


FIG. 17. $|c_L(z) - c_L(z_c)|^{0.25}$ for $Q = 5$, $L = 10$. To the lower left we see singularities at the level-crossing lines.

stationary point,

$$\partial_z c_L(z) \Big|_{z=z_c} = 0. \quad (96)$$

Due to Eq. (95), all coefficients are real. Solving this polynomial equation (of degree 130) yields the zeros, represented as dots in the left of Fig. 18. The domain of data used are shown in yellow. We see zeros close to the boundary, as well as for $|z| \approx 1$; these spurious solutions are marked in black. The two physical solutions (in red) are close to the imaginary axis, for

$$z_c^{L=10} \approx -0.013767 \pm 0.283318i. \quad (97)$$

The central charge at this point is

$$c_{L=10} = 1.13543 \pm 0.0225995i, \quad (98)$$

$$c_{\text{ana}} = 1.13755 \pm 0.0210687i, \quad (99)$$

where on the second line we give the analytical result.

It is instructive to study how the effective $c_L(z)$ behaves close to its saddle point. Recall that our data are well fit by a polynomial in z with complex coefficients. As the leading Taylor coefficient vanishes, see Eq. (96), close to the saddle point the Taylor expansion has the form $c_L(z) = c_L(z_c) + \frac{1}{2}c_L''(z_c)(z - z_c)^2 + \dots$. This is shown in Fig. 17. What we also see are the strong singularities already observed in Fig. 14. We believe that as for real z they can be interpreted as a level crossing between physically different eigenvalues. This is consistent with the observation that they seemingly form curves in the complex plane.

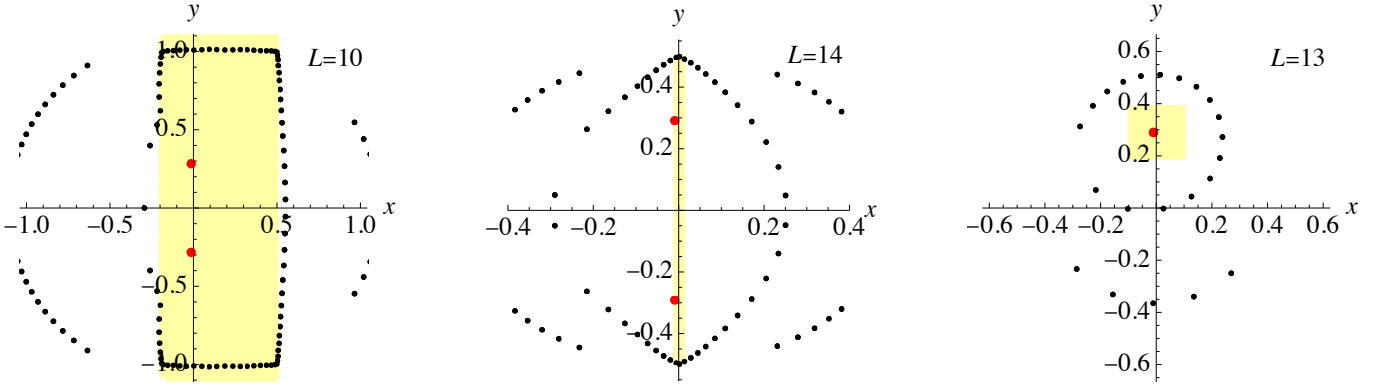


FIG. 18. Zeros of $c'(z)$ using measurements on a rectangular grid of points indicated by the yellow shaded region. Left: data in the plane with $\delta z = 0.01$, for $L = 10$, with 10^4 data points, plus their complex conjugate images. Middle: $L = 14$ on the imaginary line, $\delta z = 0.01$, 50 data points, plus their complex conjugate images. Right $L = 13$ on the mini-grid, $\delta z = 0.05$, 25 data points. We always display the largest available system size.

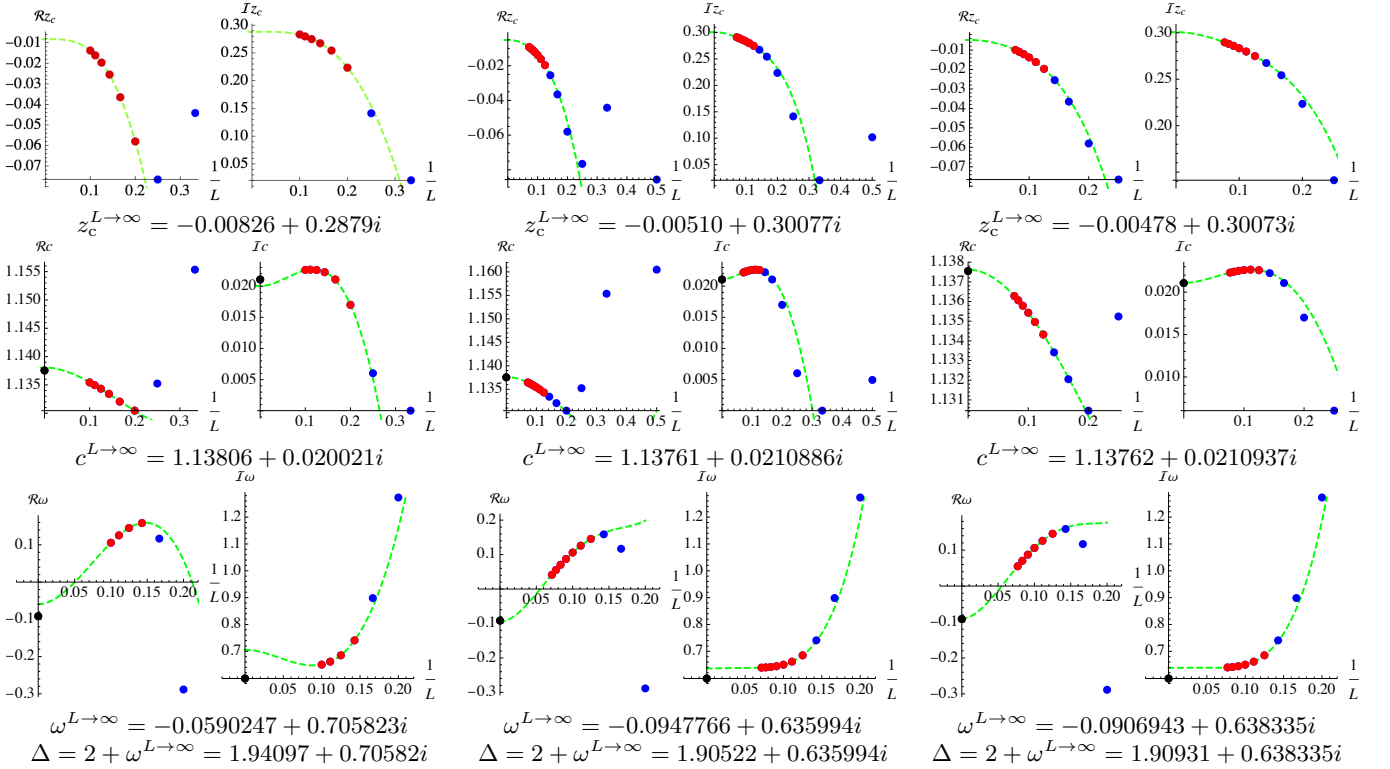


FIG. 19. Values of z_c (top), c (middle) and ω (bottom) for all available system sizes L . The left two columns are from the grid data, the middle two from line data, and the right two from the minigrid; see Fig. 18 for the range covered by these data. The black dots represent the analytically predicted values for c and ω . The fitting polynomial is of the form $a_0 + a_2 L^{-2} + a_3 L^{-3}$ for the grid, and $a_0 + a_2 L^{-2} + a_3 L^{-3} + a_4 L^{-4}$ for the line and minigrid. Only red points are used for the fit, blue points are dropped.

B. Improving the precision

Knowing that $c_L(z)$ is an analytic function in z , and where the fixed point is approximately, there are several strategies to improve the measurements, both using much fewer points. The one we study first, and which we also use for the spectrum later, is to sample on the imaginary axis (we call this numerical scheme “line” for easy reference), with the same step size

of $\delta z = 0.01$. This strategy works since for $Q = 5$, the fixed point is very close to the imaginary axis, see Figs. 17 and 18.

The second strategy is to sample only the immediate vicinity of the fixed point, which was first determined coarsely by the “grid” scheme, followed by an $L \rightarrow \infty$ extrapolation. In this new scheme, nicknamed “minigrid”, we use a 5×5 mesh around the “grid” estimate of the fixed point, using a larger step size of $\delta z = 0.05$. Interpolations are then done via

Q, b	$L_{\max}$	z_c	c_{ana}	$c_{L \rightarrow \infty}$	$\omega_{L \rightarrow \infty}$	$2 + \omega_{L \rightarrow \infty}$	$2h_{3,1}$
$b = 6$	10	0.1711	0.8	0.80001	0.774	2.774	2.8
$b = 7$	10	0.1426	0.85714	0.85715	0.604	2.604	2.667
$b = -7$	10	-0.38	0.89286	0.88812	—	—	1.5
$b = 8$	12	0.1299	0.89286	0.89287	0.570	2.570	2.571
$b = -8$	12	-0.3404	0.91667	0.91822	-0.264	1.735	1.556
5 grid	11	$-0.0083+0.2879i$	$1.13755+0.02107i$	$1.13806+0.02002i$	$-0.059+0.706i$	$1.941+0.706i$	$1.908+0.599i$
5 line	15	$-0.0051+0.3008i$		$1.13761+0.02109i$	$-0.095+0.640i$	$1.905+0.636i$	
5 minigrid	14	$-0.0049+0.3007i$		$1.13762+0.02109i$	$-0.091+0.638i$	$1.909+0.638i$	
10	11	$0.1369+0.6262i$	$1.58411+0.19183i$	$1.58465+0.19083i$	$-0.481+1.999i$	$1.519+1.999i$	$1.611+1.186i$
20	9	$0.2807+0.8234i$	$2.04608+0.48070i$	$2.04599+0.47921i$	$-0.482+2.930i$	$1.518+2.930i$	$1.303+1.518i$
40	8	$0.3686+0.9135i$	$2.50576+0.87158i$	$2.5122+0.89312i$	$-2.138+3.079i$	$-0.138+3.079i$	$0.996+1.734i$
5+2i upper grid	11	$-0.1885+0.4978i$	$1.11224+0.25058i$	$1.11029+0.25394i$	$-0.470+0.525i$	$1.530+0.525i$	$1.502+0.635i$
5+2i upper minigrid	16	$-0.1845+0.4831i$		$1.11201+0.25059i$	$-0.511+0.617i$	$1.489+0.617i$	
5+2i lower grid	11	$0.2298-0.2968i$	$1.24645+0.22938i$	$1.24678+0.2295i$	$0.648-0.994i$	$2.648-0.994i$	$2.259-0.955i$
5+2i lower minigrid	16	$0.2252-0.3330i$		$1.24647+0.22935i$	$0.603-0.955i$	$2.603-0.955i$	
8+i upper grid	11	$0.0546+0.5700i$	$1.40726+0.20408i$	$1.40773+0.20324i$	$0.534+1.007i$	$2.534+1.007i$	$1.617+0.996i$
8+i upper minigrid	16	$0.0489+0.5516i$		$1.40746+0.20395i$	$-0.640+0.945i$	$1.360+0.945i$	
8+i lower grid	11	$0.1530-0.5198i$	$1.47775-0.04199i$	$1.47782-0.04162i$	$-0.321-1.639i$	$1.679-1.639i$	$1.793-1.105i$
8+i lower minigrid	16	$0.1273-0.5027i$		$1.47777-0.04178i$	$-0.334-0.496i$	$1.666-0.496i$	

FIG. 20. All our data. Data for the tricritical fixed point $\mathcal{B}_b$ with $b = 6$ do not exist, and are not shown.

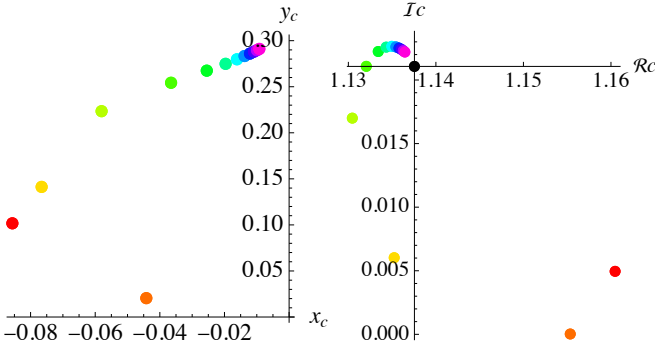


FIG. 21. z_c (left) and c (right), as a function of L , line data for $Q = 5$, sizes $L = 4$ to $L = 14$. A rather regular behavior starts at $L = 7$.

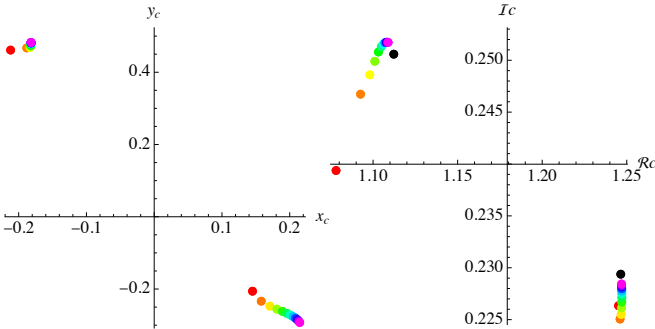


FIG. 22. z_c (left) and c (right), as a function of L (minigrid data) for $Q = 5 + 2i$, sizes $L = 4$ to $L = 14$.

a polynomial of maximal degree (with terms z^0 up to z^{24}).

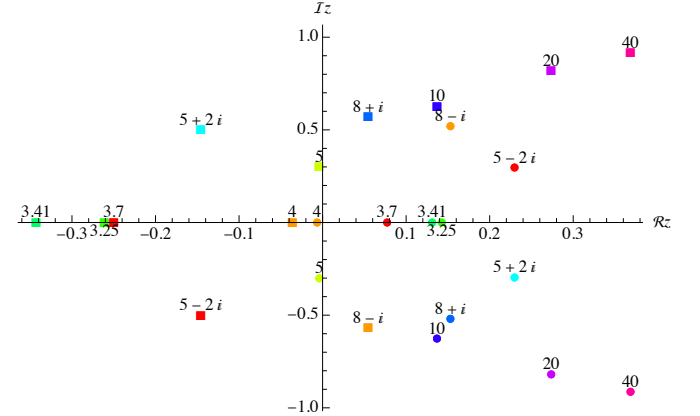


FIG. 23. z_c for all cases considered. Disks are the continuation of the critical FPs, squares the continuation of the tricritical ones. Labels are the values of Q .

This strategy can also be employed for complex Q , namely $Q = 5 + 2i$ (section D), and $Q = 8 + i$ (section E).

The advantage of the line and minigrid strategies is that one needs far fewer points, allowing one to go to larger L : for the large grid we went up to $L = 11$, while on the line we could go up to $L = 15$. This in turn leads to a much higher precision for c , and especially ω , as is shown in Fig. 19.

The data presented in Fig. 19 should be compared to the analytical predictions

$$c(Q = 5) = 1.13755 + 0.0210687i \quad (100)$$

$$\Delta = 2h_{3,1}(Q = 5) = 1.9083 + 0.598652i \quad (101)$$

The agreement is good in all cases, especially for the line and minigrid data, which go to larger L . The central charge comes very close to its analytical prediction, with an error of about 10^{-4} . The correction-to-scaling exponent ω takes longer to stabilize, and we see a clear advantage of having larger systems at our disposal. The largest remaining discrepancy is in the imaginary part of $\Delta \equiv 2 + \omega$, which is of about 0.04. Given that this is extracted from three consecutive sizes, this remains quite satisfactory.

In Fig. 21, we show the movement of the fixed point z_c and the central charge c as a function of L . The data for c follow a spiral, consistent with the complex exponent found for ω . It is hard to make this quantitative, since both z_c^L and $c^L(z)$ change with L : a fit of the form $c(L) = c_{\text{ana}} + aL^{-\omega}$ does not work. We did not pursue this further.

VI. OTHER VALUES OF Q

As discussed in section II, our approach allows us to access any value of Q . To illustrate this, we first enlarge Q up to $Q = 40$. The results are collected in appendices A ($Q = 10$), B ($Q = 20$) and C ($Q = 40$). Figs. 35 ($Q = 10$, $L = 10$), 41 ($Q = 20$, $L = 8$), and 47 ($Q = 40$, $L = 7$) show how the fixed point approaches the level-crossing lines. While our analysis confirms that we can assess all practically relevant Q values, we did not rule out whether there is a critical Q , beyond which even for larger L we cannot locate the fixed point. For the examples given, and within the limits of precision for these smaller sizes, we find agreement between theory and numerics, both for c and ω .

We then proceed to complex values of Q , first $Q = 5 + 2i$ (appendix D) and $Q = 8 + i$ (appendix E). According to section IIF, see Fig. 4, $Q = 5 + 2i$ should have one stable and one unstable fixed point, while $Q = 8 + i$ should have two unstable fixed points and no stable one. Since our ability to assess complex Q , and the agreement between theory and numerics are key claims of this work, we invested much effort in making our claims precise. Let us sketch our approach for $Q = 5 + 2i$; the analysis for $Q = 8 + i$ proceeds in the same way.

Our first step is to sweep the whole lattice in the “grid” scheme (appendix D 1). Key results are the contour plot showing fixed points and level-crossing lines (Fig. 56), and localizing the fixed points (Fig. 57). On the next page, in Figs. 58 to 60 we show the analysis for the fixed point in the upper half plane. Figs. 61 to 63 repeats this analysis for the lower half plane.

Knowing the location of both fixed points approximately, we repeat the analysis on a mini grid around our approximate fixed-point values for the coupling. Section D 2 (Figs. 64 to 68) shows the analysis in the upper half plane, while section D 3 (Figs. 69 to 73) repeats the analysis for the lower half plane. This analysis confirms that theory and transfer matrix give the same value for c with 4-digits accuracy at $L = 15$, see Eqs. (D14) and (D19).

Another key claim of ours is that in contrast to $Q = 5$, the Potts model at complex $Q = 5 + 2i$ has both a stable and an

unstable fixed point. Via transfer matrix we find an *unstable* fixed point in the upper half plane with $\omega = -0.51 + 0.62i$, see Eq. (D15) close to the theoretical prediction of $\omega = -0.49781 + 0.63539i$ in Eq. (D17). The fixed point in the lower half plane has $\omega = 0.44 - 1.48i$, see Eq. (D20), comparable to the analytic prediction of $\omega = 0.25868 - 0.95536i$, see Eq. (D22); it is stable. We attribute the lack of precision to the closeness of further irrelevant couplings; the latter are better separated at the unstable fixed point.

We refer the reader to the results for $Q = 8 + i$ in appendix E, paralleling the analysis for $Q = 5 + 2i$: section E 1 with figures 74 to 79 show the grid data. The fixed points are analyzed in Figs. 80 to 85. The refined analysis is performed separately for the fixed point in the upper and lower half planes. Results for the upper half plane are shown in appendix E 2, Figs. 86 to 90; appendix E 3 with figures 91 to 95 contains the results in the lower half plane.

VII. SPECTRUM

A. Outline

Extracting the spectrum from the transfer matrix is an important check on our claim that the full conformal information can be extracted by analytic continuation of the $c \leq 1$ loop-model CFTs. To sharpen our understanding we first consider $b = 8$ in section VII B for which the spectrum of a corresponding loop model is well established, both analytically and numerically [56]. We then proceed to $Q = 5$ in section VII C.

While we have not evaluated the spectrum for complex values of Q , we are confident it works there as well. This is confirmed by our ability to extract the first subleading scalar $\Delta_{3,1}$ for $Q = 5 + 2i$, see Eqs. (D16)-(D17) for the fixed point in the upper half plane, and Eqs. (D21)-(D22) for the fixed point in the lower half plane. This analysis is repeated for $Q = 8 + i$, see Eqs. (E16)-(E17) for the fixed point in the upper half plane, and Eqs. (E21)-(E22) for the fixed point in the lower half plane.

B. The case $b = 8$ ($Q = 3.41421$)

In Fig. 24 on page 22, we show the spectrum for $b = 8$. We have marked with a red dashed vertical line the location of the minimum ($x_c = 0.1152$) which we previously identified with the minimal model $\mathcal{B}_8$, and with a blue vertical line the location of the maximum ($x_c = -0.3447$) which we previously identified with the minimal model $\mathcal{B}_{-8} \equiv \mathcal{B}_9$. By horizontal lines, we show the expected spectrum for this model. This spectrum contains the spinless primary operators $V_{(r,1)}^d$ with

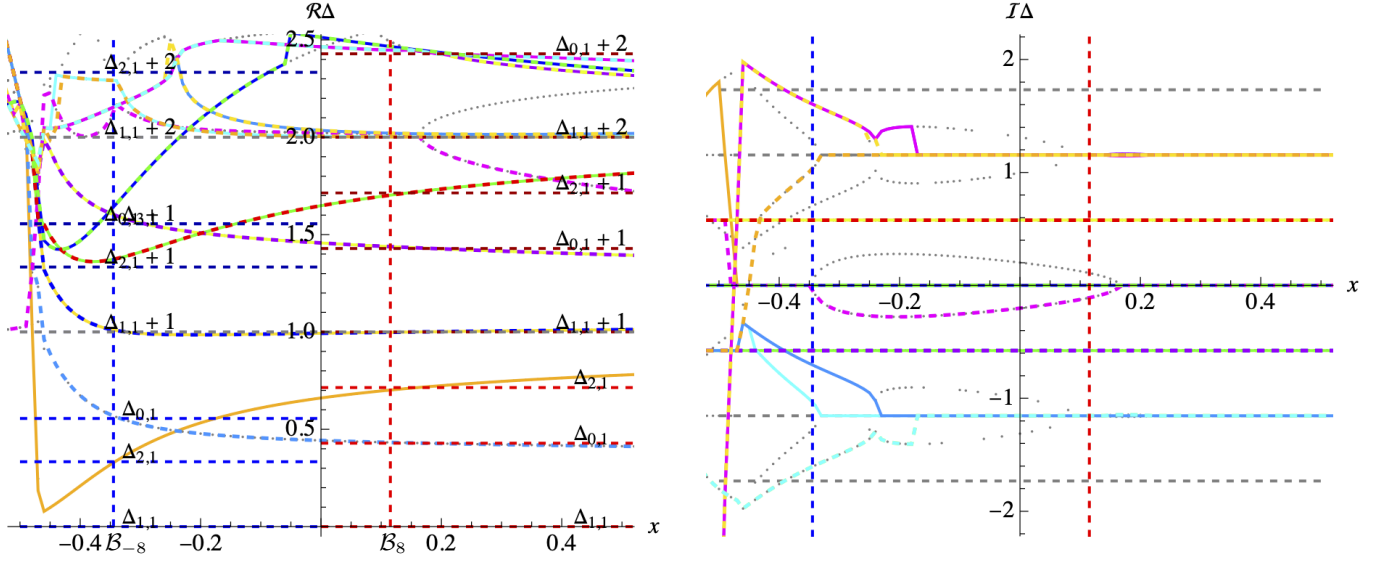


FIG. 24. The spectrum for B_8 at $L = 12$. Left real part, right imaginary part, in the same color code. Not all levels could be reconstructed, due to a clipping of part of the spectrum for $\Re\Delta > 2$. In the imaginary part, many curves lie indistinguishably on top of each other.

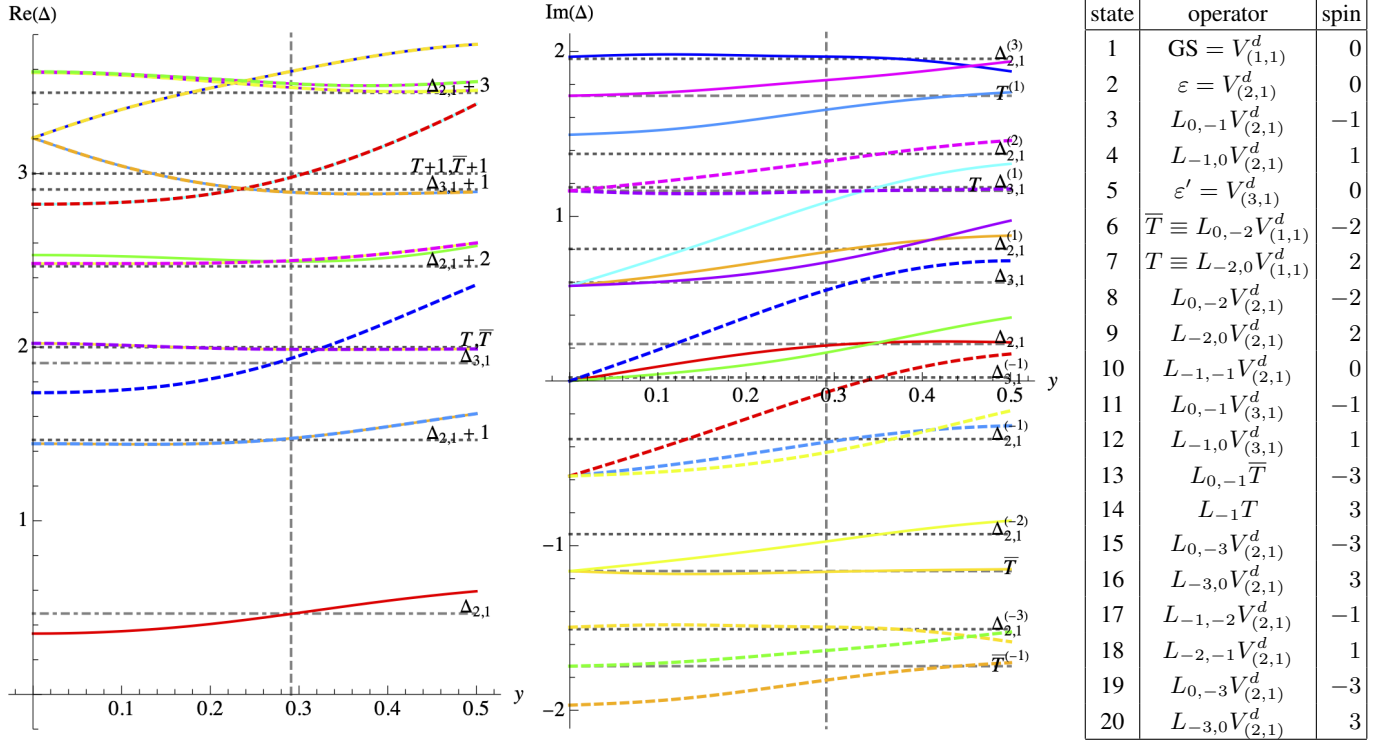


FIG. 25. Real and imaginary part $\Delta_{r,s}^{(S)}$ of spectrum in the ground-state sector for $Q = 5$, $L = 15$, taking only the first 20 EVs. Primaries in red, dashed, descendants in darker red, dotted. The red dashed vertical lines denote the position of the critical point.

exponents⁸

$$\Delta_{r,1} = 2h_{r,1}(b = \mp 8), \quad (102)$$

⁸ We apologize to the reader for the requirement to distinguish between the notation $\Delta_{(r,s)} = h_{r,s}$ of section II G 2 — necessary to make contact with [28, 29, 42–45, 56] — and the notation $\Delta_{r,s} = 2h_{r,s}$ used here and below, to remain consistent with our previous work [9].

starting with $r = 1$ for the vacuum (baseline). For the left FP $b = -8$, while for the right FP $b = 8$. We verify the appearance of $V_{(0,1)}$ and $V_{(2,1)}^d$, with the proper dimensions both for the real and imaginary parts (it vanishes), as well as some of their descendants. Descendants may have a non-vanishing spin S . The dimension of the primary increases by 1 for each level of descendance, and by $iS/\sqrt{3}$ due to the spin.

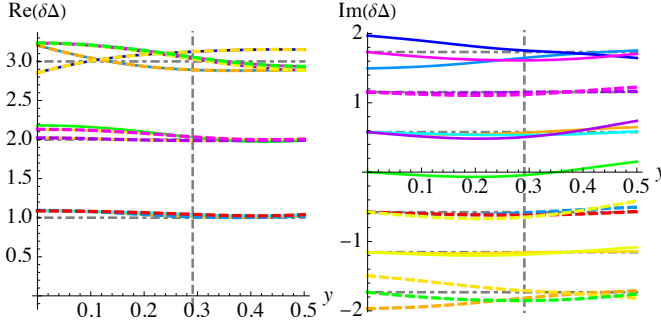


FIG. 26. Difference in real and imaginary part for the descendants minus their primaries in the ground-state sector for $Q = 5$, $L = 15$. Same color-code as in Fig. 25.

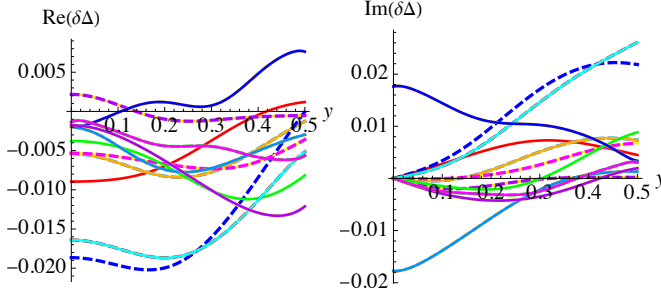


FIG. 27. Difference in real and imaginary part for the spectrum in the ground-state sector for $Q = 5$, $L = 15$, when moving the path from $i\mathbb{R}$ to $i\mathbb{R} - 0.01$. Same color-code as in Fig. 25.

The latter imaginary contribution arises because our transfer matrix T_L defined by (11) is a mixture of a dilation and a rotation in radial quantization. Indeed, T_L propagates by $\frac{\sqrt{3}}{2}$ lattice spacings upwards (the height of an equilateral triangle of side length 1), and by $\frac{1}{2}$ lattice spacings rightwards, as can be seen from (10). The former factor $\frac{\sqrt{3}}{2}$ is taken into account in the scaling analysis when normalizing the free energy per unit area, but the ratio $\sqrt{3}$ of the two factors remain. Using $L_{-N, -\bar{N}} V_{(r,1)}^d$ as a shorthand for any descendent on chiral level N and antichiral level $\bar{N}$, we get

$$\Delta(L_{-N, -\bar{N}} V_{(r,1)}^d) = \Delta_{r,1} + (N + \bar{N}) + \frac{1}{\sqrt{3}}(N - \bar{N})i. \quad (103)$$

The counting of descendants confirms that $V_{(r,1)}^d$ are indeed degenerate and form Kac modules with one null state at level r . The content of primaries and the structure of the modules were predicted from the torus partition function [27] and previously verified numerically for real $Q < 4$ in a square-lattice loop model with only nearest neighbor interactions [56] (see also section II G 2).

C. $Q = 5$

1. Degenerate operators $V_{(r,1)}^d$

We now study the spectrum at $Q = 5$ in the ground-state sector, using the quotient representation $\bar{\mathcal{W}}_{0,q^2}$ of defect-free

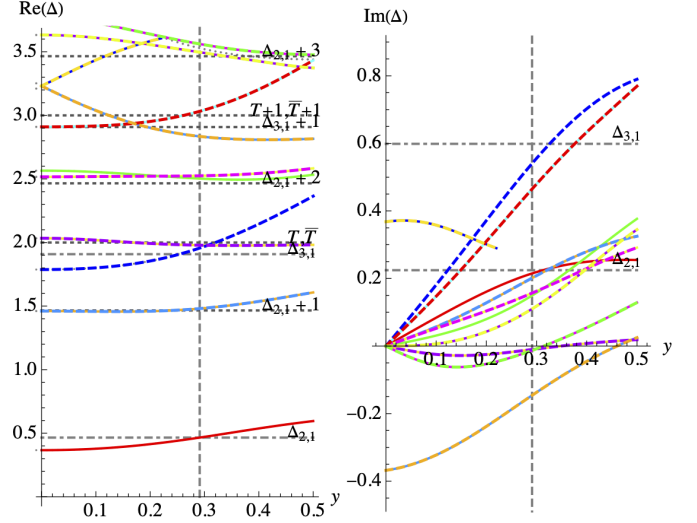


FIG. 28. Real and imaginary part $\Delta_{r,s}^{(S)}$ of spectrum in the ground-state sector for $Q = 5$, $L = 12$, maximally 20 EVs, two consecutive layers. Primaries in dot-dashed, descendants dotted. The dashed vertical lines denote the position of the critical point.

link patterns of dimension $\frac{1}{L+1} \binom{2L}{L}$, where $\sqrt{Q} = q + q^{-1}$; see sections II G 2 and II H as well as [28] for details. Fig. 25 shows the exponents corresponding to the first 20 eigenvalues (EV) in this sector. As the critical points have $x \approx 0$ (see Fig. 18), we take $x = 0$ and plot against $y \geq 0$. The vertical red dashed line indicates the fixed point.

This spectrum contains the spinless primary operators $V_{(r,1)}^d$ with exponents $\Delta_{r,1} = 2h_{r,1}^{-b}$, starting with $r = 1$ for the vacuum (baseline). The general formulas are given in Eqs. (102)-(103) and the paragraph in between. We verify the appearance of $\varepsilon = V_{(2,1)}^d$ and $\varepsilon' = V_{(3,1)}^d$, with the proper dimensions both for the real and imaginary parts. We see in Fig. 25 that all of this is perfectly respected at $Q = 5$ for the 20 lowest-lying states: we confirm that the results for the spectrum carry over to the complex CFT by analytic continuation.

Descendants: As discussed above, descendants may have a non-vanishing spin S : as a result, the dimension of the primary increases by 1 for each level, and by $iS/\sqrt{3}$ due to the spin. This is verified on Fig. 26. Apart from errors which we associate to system size $L = 15$ being too small, the spacing of the real part is indeed integer, and for the imaginary part multiples of $1/\sqrt{3}$.

Double-layer transfer matrix: It is of course possible to study the spectrum using a pure dilation operator. To this end, we performed transfer-matrix calculations on two consecutive layers, such that the first step is up right, and the second up left, with no resulting translation. (In the notation of section II B this means diagonalizing the operator $u^{-1}T_L^2$.) We then expect the last term $(N - \bar{N})/\sqrt{3}$ in Eq. (103) to be absent. This can be checked in Fig. 28. In particular, for real z , i.e. $y = 0$, the imaginary part of the spectrum vanishes. This is indeed the case for all levels, except two. The latter belong to levels degenerate at $z = 0$, with $\Delta = 3.231 \pm 0.368i$. We

believe that this feature is a finite-size effect. From a CFT point of view, values outside the fixed point are not universal, so we should focus on what happens at the fixed point: there seem to be both level-3 descendants of the identity, or level-1 descendants of the stress tensor.

Insensitivity to the chosen path: In Fig. 27 we show the difference in real and imaginary part for the spectrum in the ground-state sector for $Q = 5$, when moving the path from $i\mathbb{R}$ to $i\mathbb{R} - 0.01$. The latter is the approximate uncertainty in the determination of the critical point z_c . We conclude that an uncertainty of about 0.01 in the location of z_c translate into a similar uncertainty in the spectrum.

2. Non-diagonal operators $V_{(r,s)}$

We now study the non-diagonal operators $V_{(r,s)}$ by diagonalisation in the standard modules $\mathcal{W}_{j,\tilde{z}}$ with $j = 2s$ defects. For simplicity we choose the trivial phase factor $\tilde{z} = 1$ corresponding to the spinless non-diagonal operators $V_{(0,s)}$. We shall use the notation $X_j(L)$ for the effective exponent obtained from the finite-size scaling analysis, to be compared to $2h$. Notice that although the operator $V_{(0,1)}$ is absent from the torus partition function, as explained in section II G 2, it can still be realized in the transfer-matrix formalism within the corresponding standard module.

We were able to use

$$X_2(L) = X_2(L+1) \quad (104)$$

to locate the FP. An example is shown in Fig. 29 on the left, with the movement as a function of L on the right. Extrapolation yields

$$z_c \simeq 0.005 + 0.294i. \quad (105)$$

This is comparable to what we got from the central charge, see table 22, even though our largest system size was only $L = 10$.

Fitting the exponents for $j = 2$ and $j = 4$ defects at finite sizes (not shown), and extrapolating to $L \rightarrow \infty$ yields

$$\Delta_{X_2} = 0.512959 - 0.0772247i, \quad (106)$$

$$2h_{0,1} = 0.511462 - 0.0748315i, \quad (107)$$

$$\Delta_{X_4} = 2.01179 - 0.317301i, \quad (108)$$

$$2h_{0,2} = 2.01146 - 0.304593i. \quad (109)$$

This again provides an excellent agreement between the transfer matrix results and the analytic continuation of the CFT.

3. Additional states in the non-quotient spectrum

Rather than diagonalizing T_L in the quotient representation $\overline{\mathcal{W}}_{0,q^2}$, as in section VII C 1, we also examined the non-quotient representation $\mathcal{W}_{0,q^2}$. From the point of link patterns this means that we now work in a larger space that distinguishes whether the link between two sites straddles the periodic boundary condition. In particular, if we changed the twist

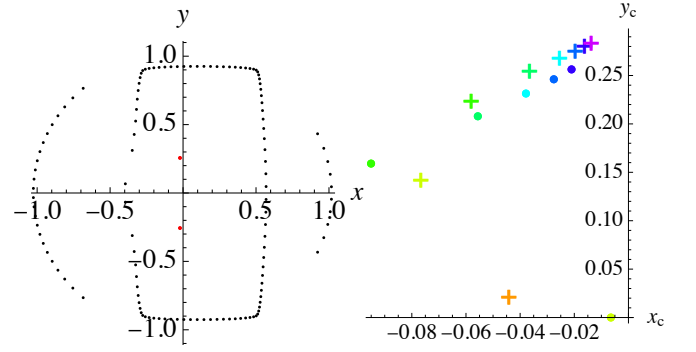


FIG. 29. Left: Loci of $X_2(L) - X_2(L+1) = 0$, at size $L = 9$. Red dots show the estimates of z_c . Right: Movement of z_c as a function of L , from red ($L = 4$) to dark blue ($L = 9$) to magenta ($L = 10$). Crosses come from the analysis of c , dots from X_2 .

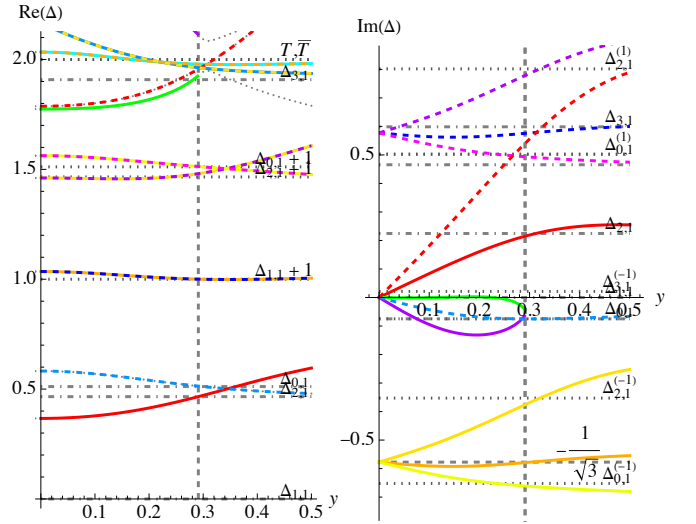


FIG. 30. Full spectrum. Note the additional states as compared to Fig. 25.

parameter in this representation, using $\mathcal{W}_{0,\tilde{q}^2}$, we would be able to give a different weight $\tilde{n} = \tilde{q} + \tilde{q}^{-1}$ to non-contractible loops, while still giving the weight $n = q + q^{-1}$ to contractible loops. But even taking $\tilde{q} = q$, as we did, the algebraic structure changes: the non-quotient representation $\mathcal{W}_{0,q^2}$ decomposes as a direct sum of $\overline{\mathcal{W}}_{0,q^2}$ and the 2-defect standard module $\mathcal{W}_{1,\pm 1}$.

As a result, we obtain more states than in Fig. 25. In Fig. 30 we show the spectrum without taking the quotient, restricted to the lowest 20 eigenvalues. We notice in particular the appearance of the non-diagonal primaries $V_{(0,1)}$ and $V_{(1,1)}$, in agreement with section II G 2 and Conjecture 2.

The field $\bar{J} = V_{(1,1)}$ with conformal weights

$$(\Delta_{(1,1)}, \Delta_{(1,-1)}) = (0, 1) \quad (110)$$

is of particular interest. It was shown in [71] to be one of a pair of non-conserved currents, the other one being $J = V_{(1,-1)}$

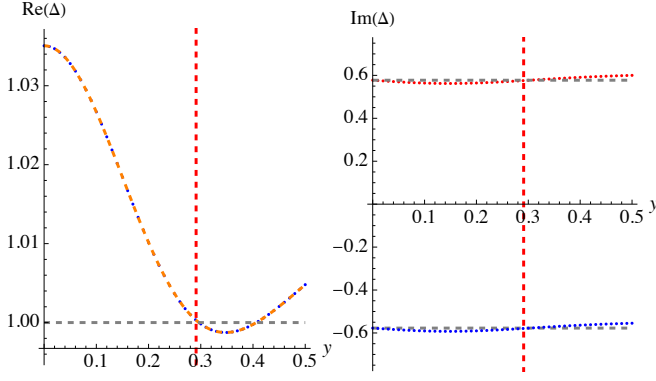


FIG. 31. Δ for EVs 4 and 5 at $L = 12$. At the fixed point this becomes $\Delta = 1 \pm i/\sqrt{3}$.

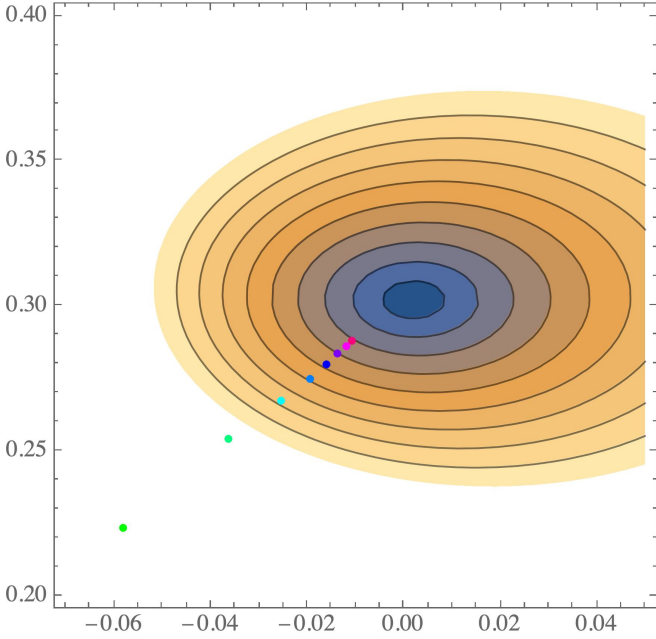


FIG. 32. The location $z_c = x + iy$ where $\Delta = 1 + i/\sqrt{3}$. The contour plot shows $|\Delta - 1 - i/\sqrt{3}|^2$. The dots show the successive approximations from $c'(z) = 0$.

with conformal weights $(1, 0)$, that satisfy

$$\bar{\partial}J = \partial\bar{J} \neq 0. \quad (111)$$

and which do *not* giving rise to a Kac-Moody algebra.

Using here our numerical approach at $Q = 5$, we find at the critical point, indicated with a red dashed line for $L = 12$ in Fig. 31

$$\Delta = 1.00047 + 0.575189i \approx 1 + \frac{i}{\sqrt{3}}, \quad (112)$$

in fine agreement with $(h, \bar{h}) = (1, 0)$ for the field J [cf. Eq. (103)].

As before, we fit the spectrum as a function of y by a polynomial. We can then use this high-quality fit to get the location

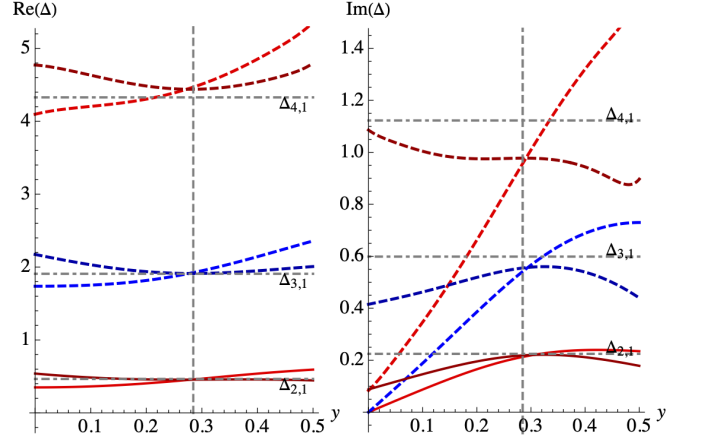


FIG. 33. Left: real part of the spectrum for $L = 15$ with the first three primaries as a function of y : $\Delta_{2,1}$ (red, solid, bottom), $\Delta_{3,1}$ (blue dashed, middle) and $\Delta_{4,1}$ (top, red dashed). We use the same colors as in Fig. 25. Note that $\Delta_{4,1}$ is not shown there, and we changed the auxiliary lines to grey to improve readability. The same levels corrected around z_c for the influence of $V_{(3,1)}^d$ are shown in the same darkened color. Right: *idem* for the imaginary part. The darker lines are the improvements from Eq. (114) with parameters as given in Eqs. (115)-(116).

where $J = 1 + i/\sqrt{3}$, as shown in the contour plot of Fig. 32. We also show the convergence of z_c obtained via $c'(z) = 0$. The location is seemingly right on the imaginary axes. We can do this more precisely, by fitting a polynomial, and searching for $\Delta = 1 + i/\sqrt{3}$. The result is

$$z_c = 0.00182 \pm 0.30206i. \quad (113)$$

This should be compared to our best estimate from $c'(z) = 0$, namely $z_c = 0.0049 + 0.3007i$, see Fig. 20.

We finally note that we found another element in the spectrum, with $\Delta \approx 2 \times (1 + i/\sqrt{3})$. It can be identified with the stress tensor T with conformal weights $(h, \bar{h}) = (2, 0)$.

4. OPE coefficients: Quotient spectrum corrected by ε'

Up to now we read off exponents as $\Delta = \Delta(z_c)$. When one is not exactly at the fixed point, there are corrections due to subdominant operators. The leading one is $\varepsilon' = V_{(3,1)}^d$, integrated over all of space. Denoting by g its effective coupling constant, we expect to see a corrected value $\Delta_{i,1}^{\text{corr}}(z)$ [72]

$$\Delta_{i,1}^{\text{corr}}(z) = \Delta_{i,1}(z) - g(z - z_c)C_{(i,1),(i,1),(3,1)}. \quad (114)$$

Here $C_{(i,1),(i,1),(3,1)}$ denotes the OPE coefficient (3-point structure constant) of twice the primary $V_{(i,1)}^d$ and once $V_{(3,1)}^d$, which is evaluated in Eq. (67) and in appendix G. In Fig. 33 we used

$$z_c = -0.00852 + 0.28466i, \quad (115)$$

$$g = 0.2 - 0.8i. \quad (116)$$

In principle the effective coupling can be gotten as

$$g \approx g_2 := \frac{\partial_z \Delta_{2,1}(z)}{C_{(2,1),(2,1),(3,1)}} = 0.349 - 0.885i \quad (117)$$

$$g \approx g_3 := \frac{\partial_z \Delta_{3,1}(z)}{C_{(3,1),(3,1),(3,1)}} = 0.297 - 0.874i \quad (118)$$

$$g \approx g_4 := \frac{\partial_z \Delta_{4,1}(z)}{C_{(4,1),(4,1),(3,1)}} = 0.210 - 0.829i \quad (119)$$

The value used is a compromise, to get the spectrum (dark lines in Fig. 33) as flat as possible around y_c . Since $\Delta_{4,1}$ has uncorrected the largest slope, this value is close to the coupling constant extracted from the measurement of this OPE coefficient. This procedure can be improved by adding more operators. Another possibility is to directly measure a 3-point function. This was persued in [19, 60], and yields much more precise results.

VIII. CONCLUSIONS

In this paper we have significantly substantiated our claims, first summarized in [18], that complex CFTs can be realized as the continuum limit of lattice models, both for $Q > 4$ and complex Q . The corresponding lattice realization (8) is a loop model that stems from a triangular-lattice Potts model with competing two- and three-spin interactions. Along its self-dual manifold, parametrized by z for $w = 1$ [see Eq. (20)], it possesses a pair of complex fixed points z_c for well-chosen values of $z \in \mathbb{C}$. Their respective continuum limits are complex CFTs (see Conjecture 1).

Our numerical study is based on the diagonalization of the transfer matrix T_L in the basis of standard modules of the affine Temperley-Lieb algebra. We have first established a robust and versatile methodology for determining the critical points z_c for $Q \in \mathbb{C}$. Using this, we have determined their conformal data: central charge, critical exponents and three-point structure constants (in the latter case comparing to published and unpublished work by [19]). We have established that all of this is in agreement with Conjecture 2, which identifies the complex CFT with the analytic continuation of the non-unitary CFT studied in a body of recent papers [28, 29, 42–45, 56]. This result is accompanied by a careful discussion of how the analytical continuation should be taken.

Concerning the operator content of the complex CFT, we have checked in detail the presence of degenerate and non-diagonal operators predicted in the table of section II G 2. The diagonal operators, including the special case of order-parameter operators, could be similarly identified by diagonalizing T_L in the representation $\mathcal{W}_{0,\tilde{q}^2}$, for appropriate values of the twist parameter $\tilde{q}$.

The ongoing work on the non-unitary CFT of loop models already contains, or is about to soon produce, more detailed results, including higher-order correlation functions, families of boundary conditions parametrized by discrete or continuous parameters, and results in geometries with non-trivial topology. We would expect this to carry over to the complex CFT setting as well. A few examples for boundary data have already appeared in [20].

We finally point out that while we have framed this work in the context of the Q -state Potts model, a very similar sce-

nario plays out in the complex $O(n)$ model. The relevant predictions gathered in section II G 1, can again be obtained by analytic continuation.

The present work raises the question of its possible experimental realization. Proposals for non-Hermitian quantum mechanics are widely discussed in the literature, see [73, 74] for a review. The most direct way seems to be via post-selection in monitored quantum systems [75, 76], which has successfully been implemented in experiments on trapped ions [77, 78] or superconducting Q-bits [79]. In contrast, realizing a complex CFT has to our knowledge not been achieved, even though walking behavior, i.e. the presence of a close-by complex CFT is widely discussed, see e.g. [80]. There are attempts to get closer to a complex CFT in quantum Monte Carlo simulations [81]. A recent proposal to realize the $O(3)$ complex CFT is given in [82], using a no-clicks measurement protocol which can be mapped to a Lindblad master equation.

To realize a complex CFT in an experiment, one can use the equivalence of a $d + 1$ dimensional classical system to a d -dimensional quantum system. There the spectrum of a CFT maps onto energy eigenvalues of a quantum Hamiltonian via $\Delta = v_F(E + i/\tau)$, where v_F is the Fermi velocity, and E the energy of the state. The wave function in time of a mode Δ evolves as $|\psi_\Delta(t)\rangle \sim e^{iv_F t(E + i/\tau)}$. This is a *quasiparticle* interpretation, with a positive imaginary part of Δ indicating decay, and a negative imaginary part growth. The most easily realizable CFT arises when the imaginary parts all have the same sign. As complex CFTs come in pairs, let us choose the branch where the operators of interest have Δ with positive imaginary part, and let us further assume that the imaginary part grows with the energy. This means that higher excitations decay faster, which naturally select the lowest-lying states over time, ensuring converges to the complex CFT. We have verified that both the family of degenerate operators $V_{(i,1)}$ which contains ε , and the family of order-parameter operators, $V_{(r,0)}$ with $r \in \mathbb{N} + \frac{1}{2}$, which contain σ , in our choice of branch have $\Im \Delta > 0$. Moreover, $\Im \Delta \geq \frac{1}{|b|} \Re \Delta$ for any $Q > 4$ (e.g., $\Im \Delta \geq 0.153 \Re \Delta$ for $Q = 5$). Thus higher excited states would indeed be decaying faster. On the other hand, defect operators have $\Im \Delta < 0$. While it is not a contradiction to have both types of states, the latter ones would grow exponentially. As they do not result from the fusion of spin or energy operators, they may simply be absent from an experiment.

We hope that efforts to realize the complex CFT within an experiment will succeed, and make the present study more broadly relevant.

ACKNOWLEDGMENTS

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Appendix A: $Q = 10$

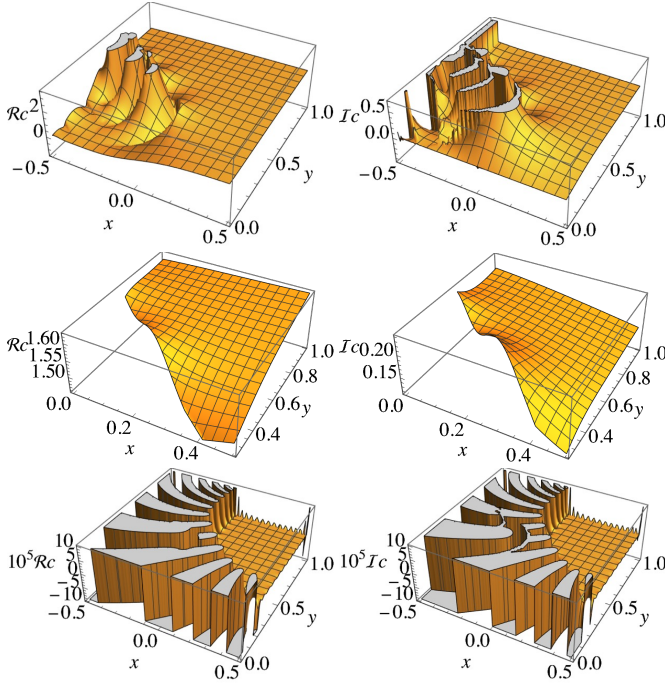


FIG. 34. $c(z)$ as in Figs. 14-16. Top: full data. Middle: selected data. Bottom: fitting error.

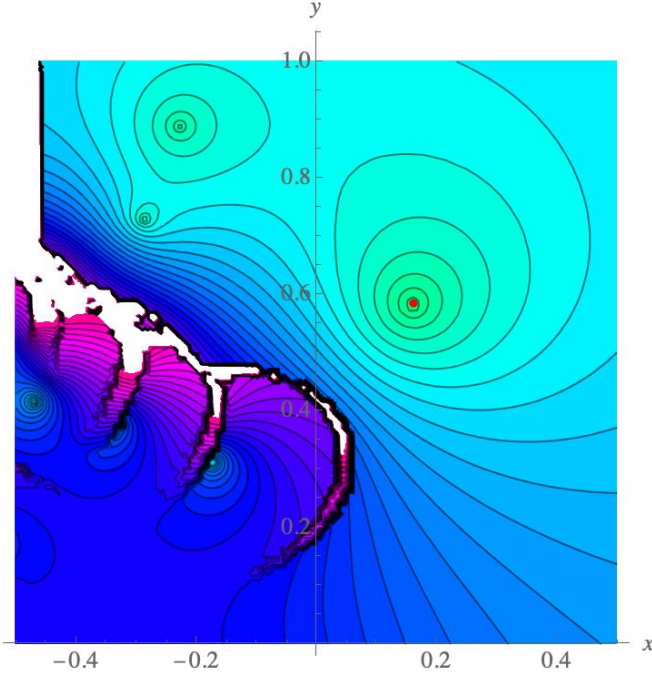


FIG. 35. Contour plot for $|c(z) - c_{\text{theory}}|^{0.3}$ for $Q = 10$, $L = 10$. Saddle point marked by red dot.

Here we repeat for $Q = 10$ the data analysis performed in section V for $Q = 5$. Fig. 34 shows our data on the grid; Fig. 35 the effective central charge $c_L(z)$. Fig. 36 show the location of $c'(z) = 0$, Fig. 37 extrapolations for z_c to $L = \infty$, Figs. 38 and 39 extrapolations for c and ω . The fitting polynomial consists of $\{1, x^2, x^3\}$, with $x := 1/L$.

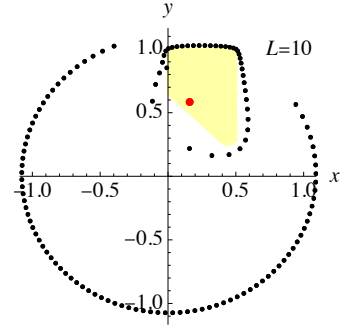


FIG. 36. Zeros of $c'(z)$ at $L = 10$.

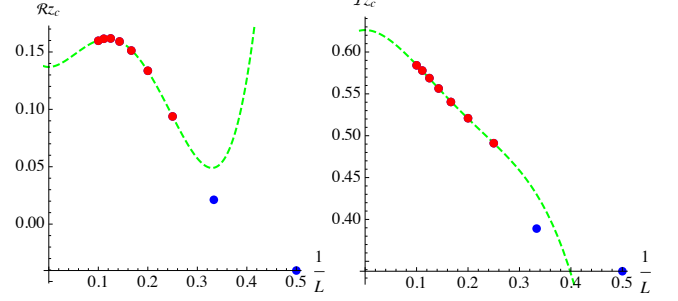


FIG. 37. Extrapolation of z_c for $Q = 10$. This gives $z_{c,L \rightarrow \infty}^{\text{upper}} = 0.136932 + 0.62623i$.

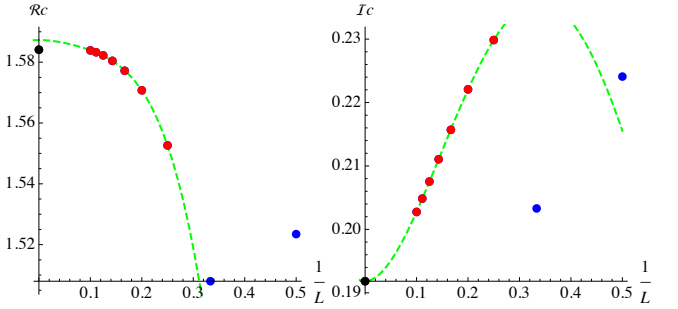


FIG. 38. Extrapolation of c for $Q = 10$ yields $c_{L \rightarrow \infty}^{\text{upper}} = 1.58465 + 0.190826i$. Compare to $c_{\text{ana}}^{\text{upper}} = 1.58411 + 0.191825i$.

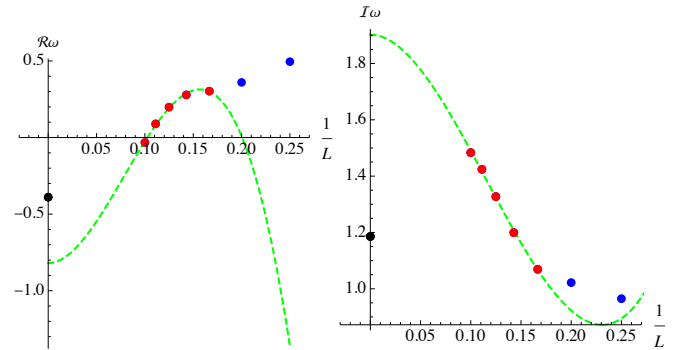


FIG. 39. Extrapolation for ω . $\omega^{\text{upper}}(L \rightarrow \infty) = -0.480646 + 1.99852i$, $\Delta^{\text{upper}}(L \rightarrow \infty) = 2 + \omega^{\text{upper}}(L \rightarrow \infty) = 1.51935 + 1.99852i$, $2h_{3,1} = 1.6106 + 1.18574i$.

Numerical values for c and ω are compared to theoretical predictions in Figs. 38 and 39. The agreement for c is good, while larger sizes would be needed for a better agreement of ω .

Appendix B: $Q = 20$

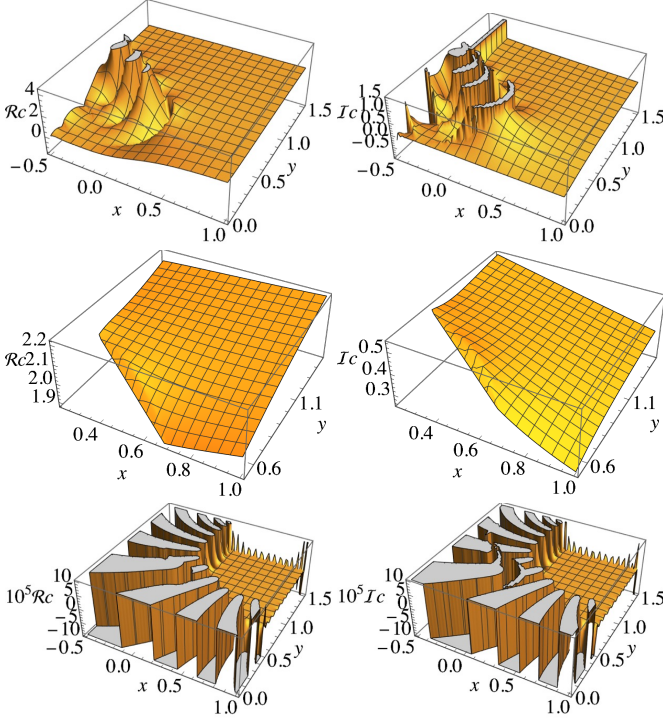


FIG. 40. $c(z)$ as in Figs. 14-16. Top: full data. Middle: selected data. Bottom: fitting error.

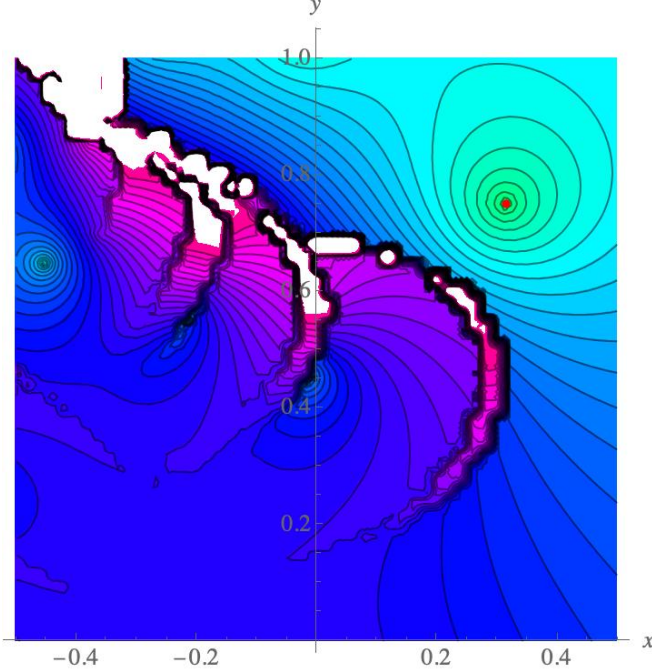


FIG. 41. Contour plot for $|c(z) - c_{\text{theory}}|^{0.3}$ for $Q = 20$, $L = 8$. Saddle point marked by red dot.

Here we repeat for $Q = 20$ the data analysis performed in section V for $Q = 5$. Fig. 40 shows the data on the grid, Fig. 41 the effective central charge $c_L(z)$, and Figs. 43-45 the extrapolations for z_c , c , and ω . The fitting polynomial in

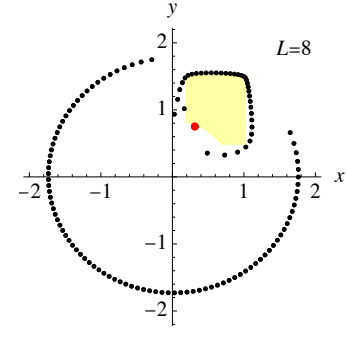


FIG. 42. Zeros of $c'(z)$ at $L = 8$.

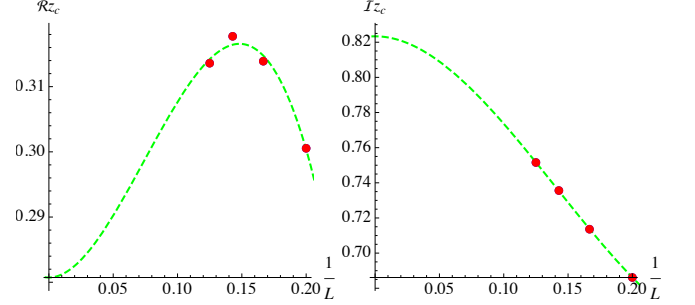


FIG. 43. Extrapolation of z_c for $Q = 20$. This gives $z_{c,L \rightarrow \infty}^{\text{upper}} = 0.136932 + 0.62623i$.

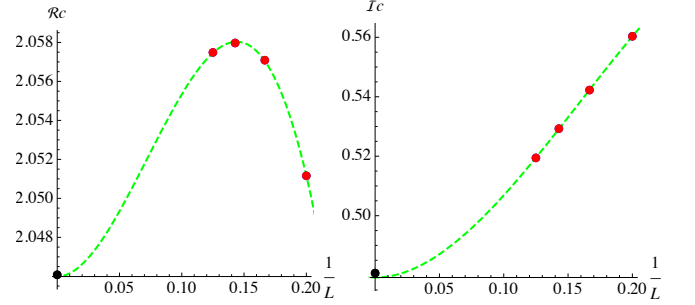


FIG. 44. Extrapolation of c for $Q = 20$ yields $c_{L \rightarrow \infty}^{\text{upper}} = 1.58465 + 0.190826i$. Compare to $c_{\text{ana}}^{\text{upper}} = 1.58411 + 0.191825i$.

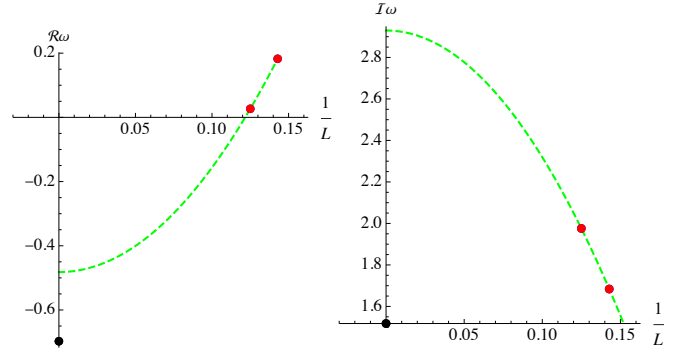


FIG. 45. Extrapolation for ω : $\omega_{L \rightarrow \infty} = -0.482 + 2.930i$, $2 + \omega = 1.518 + 2.930i$, $2h_{3,1} = 1.303 + 1.518i$.

$x := 1/L$ consists of $\{1, x^2, x^3\}$. Extrapolations for ω are shaky due to the small system size, and the closeness to the singularity visible in Fig. 41.

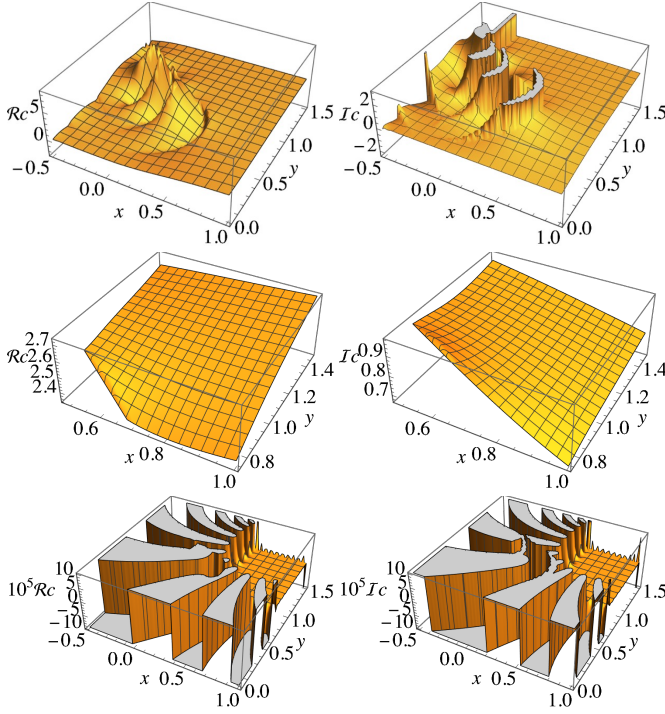
Appendix C: $Q = 40$ 

FIG. 46. $c(z)$ as in Figs. 14-16. Top: full data. Middle: selected data. Bottom: error.

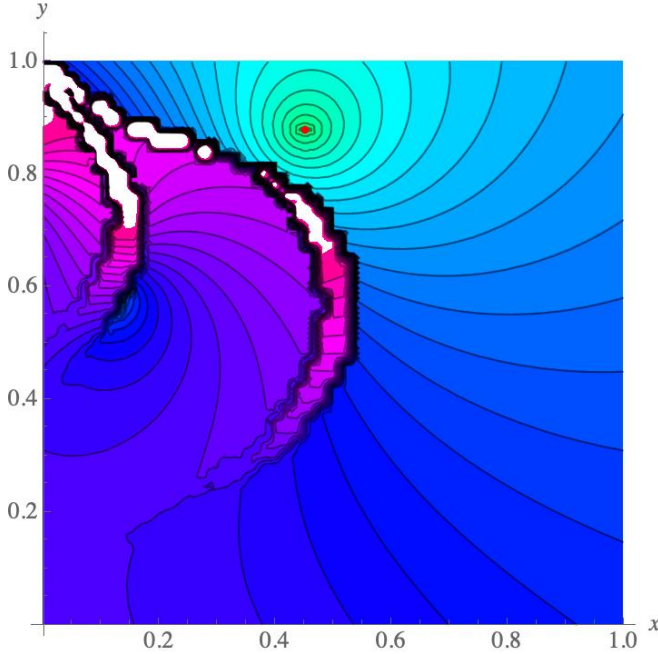


FIG. 47. Contour plot for $|c(z) - c_{\text{theory}}|^{0.3}$ for $Q = 40$, $L = 7$. Saddle point marked by red dot.

Here we repeat for $Q = 40$ the data analysis performed in section V for $Q = 5$. Fig. 46 shows the data on the grid, Fig. 47 the effective central charge $c_L(z)$, and Figs. 49–51 the extrapolations for z_c , c , and ω . The fitting polynomial contains $\{1, x^2, x^3\}$, with $x := 1/L$. Extrapolations for ω are

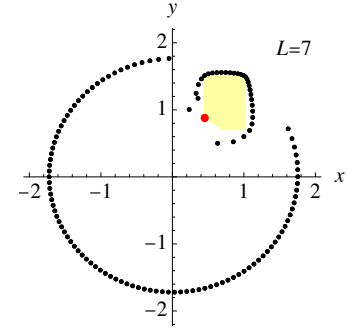


FIG. 48. Zeros of $c'(z)$ at $L = 7$.

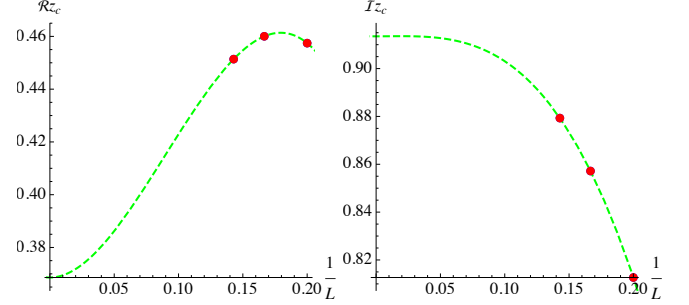


FIG. 49. Extrapolation of z_c for $Q = 40$. This gives $z_{c,L \rightarrow \infty}^{\text{upper}} = 0.136932 + 0.62623i$.

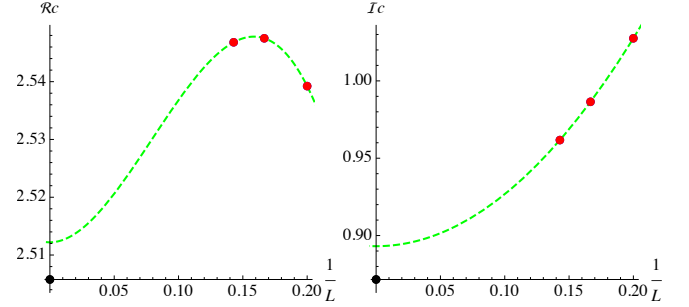


FIG. 50. Extrapolation of c for $Q = 40$ yields $c_{L \rightarrow \infty}^{\text{upper}} = 2.5122 + 0.89312i$. Compare to $c_{\text{ana}}^{\text{upper}} = 2.50576 + 0.871581i$.

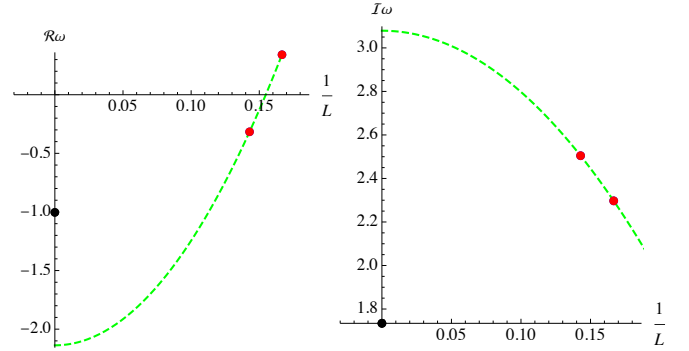


FIG. 51. Extrapolation for ω : $\omega_{L \rightarrow \infty} = -2.138 + 3.079i$, $2 + \omega_{L \rightarrow \infty} = -0.138 + 3.079i$, $\Delta_{3,1} = 0.996 + 1.734i$

shaky due to the small system size, and the closeness to the singularity visible in Fig. 47.

Appendix D: $Q = 5 + 2i$

1. Grid

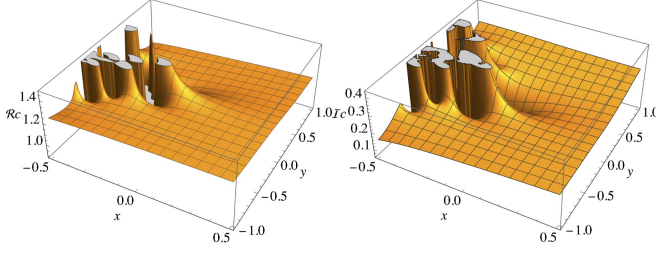


FIG. 52. c , $L = 10$, raw data.

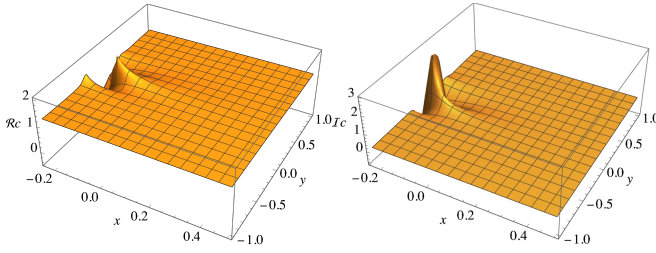


FIG. 53. c , $L = 10$, selected data.

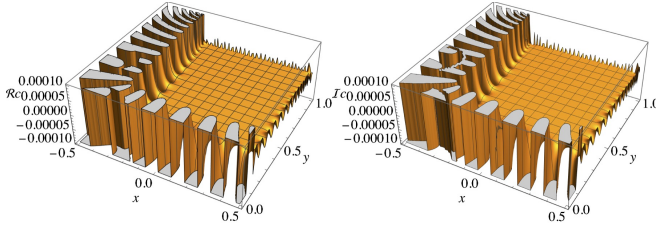


FIG. 54. c , $L = 10$, subtracted data, upper half plane.

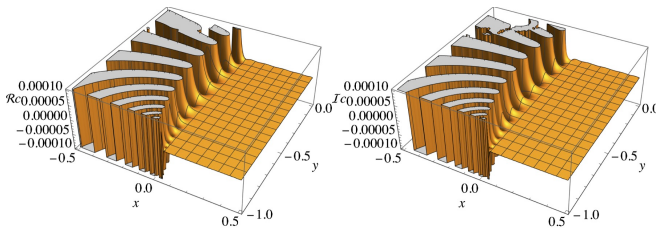


FIG. 55. c , $L = 10$, subtracted data, lower half plane.

The contour plot 56 gives a good idea about the location of the singularities. Note that the two non-trivial fixed points marked by the red dots in Fig. 56 are not complex conjugate of each other. The precise locations of $c'(z) = 0$ are shown in Fig. 57; again the non-trivial ones are in red.

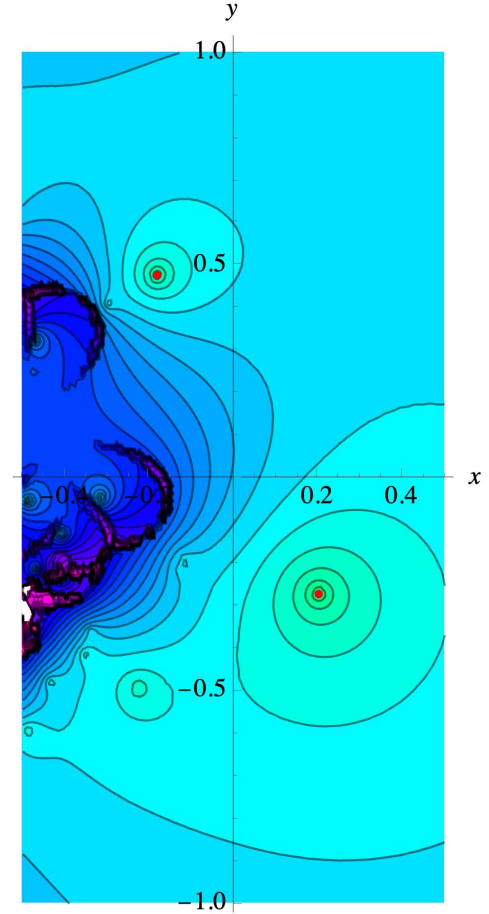


FIG. 56. Contour plot for $|c(z) - c_{\text{theory}}|^{0.3}$ for $Q = 5 + 2i$, $L = 10$. Saddle points $c'(z) = 0$ are marked by a red dot. The level-crossing lines are clearly visible.

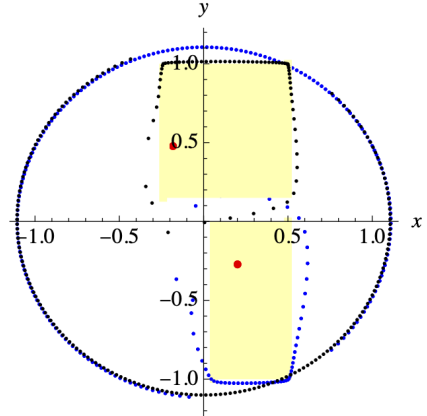


FIG. 57. Zeros of $c'(z)$ at $L = 10$. The yellow areas denote the points used to generate the fitting polynomials for upper and lower half plane.

Extrapolations for z_c , c and ω are given on the next page, separately for the two fixed points.

Fixpoint B_{-b} (upper half plane)

Fits for upper half plane. The fitting polynomial always contains $\{1, x^2, x^3\}$, with $x := 1/L$.

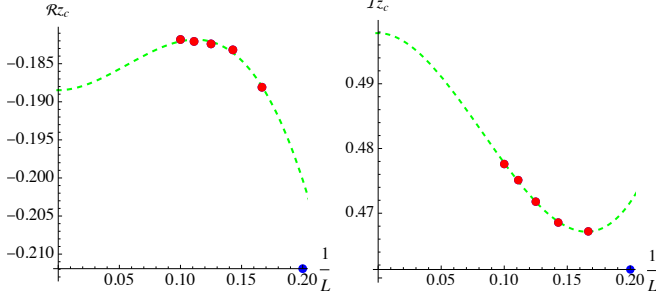


FIG. 58. Extrapolation of z_c .

$$z_c^{\text{upper}}(L \rightarrow \infty) = -0.188476 + 0.497827i. \quad (\text{D1})$$

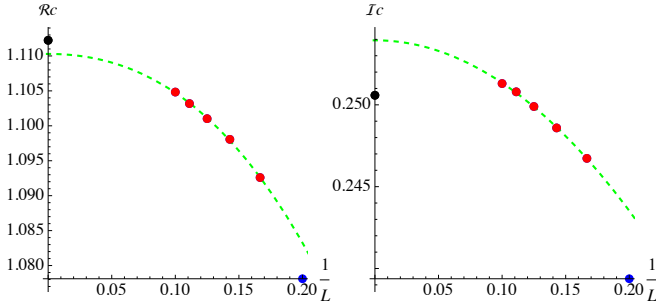


FIG. 59. Extrapolation of c .

$$c_{L \rightarrow \infty}^{\text{upper}} = 1.11029 + 0.253942i \quad (\text{D2})$$

$$c_{\text{ana}}^{\text{upper}} = 1.11224 + 0.250583i. \quad (\text{D3})$$

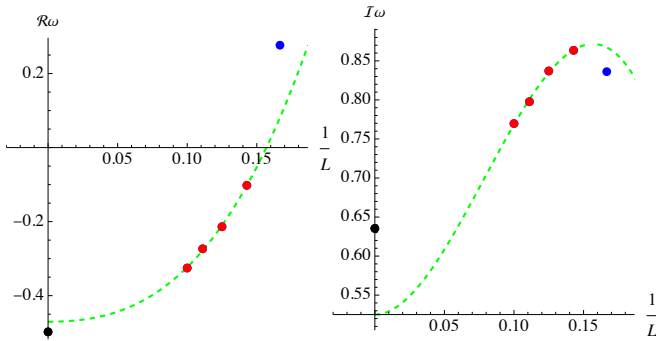


FIG. 60. Extrapolation of ω .

$$\omega_{L \rightarrow \infty}^{\text{upper}} = -0.470446 + 0.525107i, \quad (\text{D4})$$

$$\Delta_{L \rightarrow \infty}^{\text{upper}} = 2 + \omega_{L \rightarrow \infty}^{\text{upper}} = 1.52955 + 0.525107i \quad (\text{D5})$$

$$2h_{3,1} = 1.50219 + 0.635388i. \quad (\text{D6})$$

Fixpoint B_b (lower half plane)

Fits for lower half plane as for upper half plane (left).

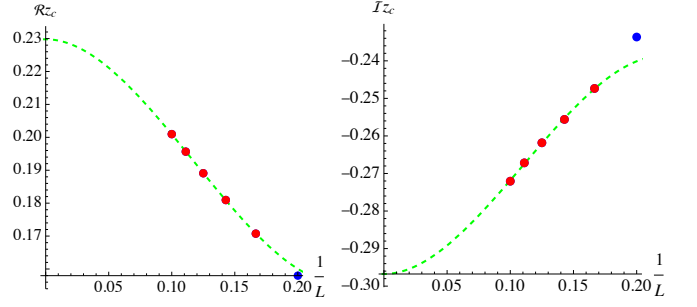


FIG. 61. Extrapolation of z_c .

$$z_c^{\text{lower}}(L \rightarrow \infty) = 0.229789 - 0.296768i. \quad (\text{D7})$$

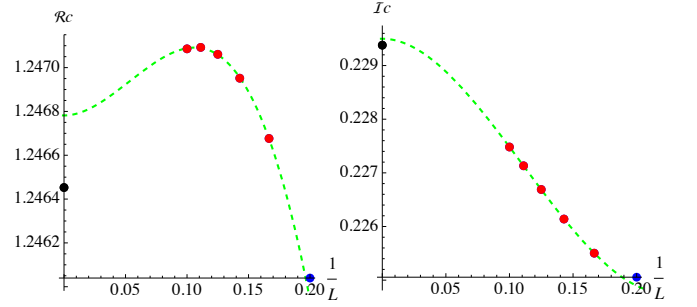


FIG. 62. Extrapolation of c .

$$c_{L \rightarrow \infty}^{\text{lower}} = 1.24678 + 0.2295i. \quad (\text{D8})$$

$$c_{\text{ana}}^{\text{lower}} = 1.24645 + 0.22938i \quad (\text{D9})$$

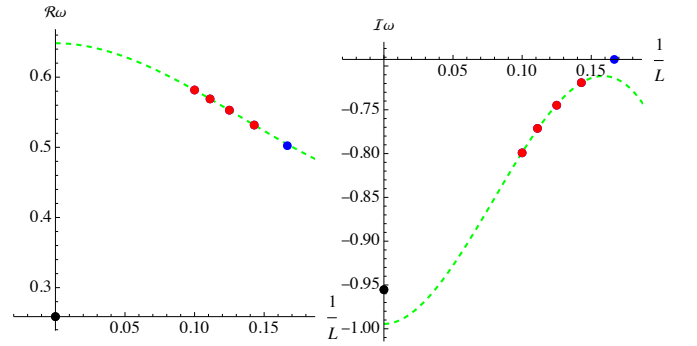


FIG. 63. Extrapolation of ω .

$$\omega_{L \rightarrow \infty}^{\text{lower}} = 0.648385 - 0.994432i \quad (\text{D10})$$

$$\Delta_{L \rightarrow \infty}^{\text{lower}} = 2 + \omega_{L \rightarrow \infty}^{\text{lower}} = 2.64838 - 0.994432i \quad (\text{D11})$$

$$2h_{3,1} = 2.25868 - 0.955363i. \quad (\text{D12})$$

2. Minigrid, upper half plane

Here we use a minigrid of 5×5 points, centered around the non-trivial fixed point identified in the preceding section. In Fig. 64 we show the area covered by it, as well as the zeroes of the fitting polynomial, for our largest system sizes, $L = 15$. The non-trivial solution marked in red is clearly visible.

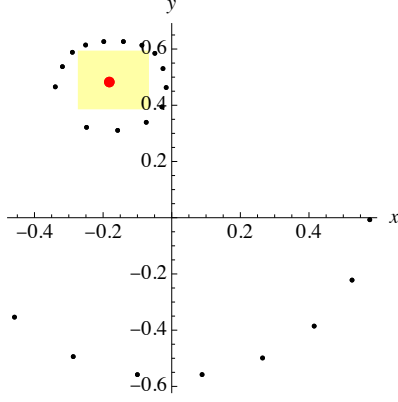


FIG. 64. Zeros of $c'(z)$, $L = 15$.

In Figs. 65 to 68 we show fits using a polynomial with $\{1, x^2, x^3\}$, $x := 1/L$.

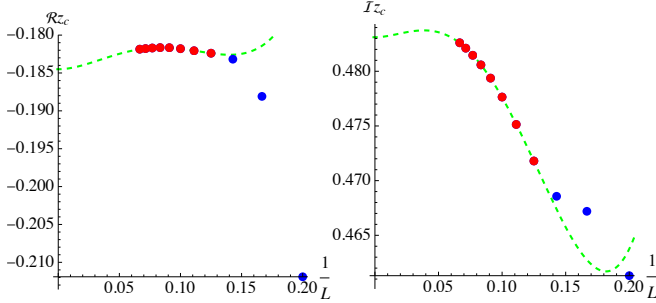


FIG. 65. Extrapolation of z_c .

Extrapolation yields

$$z_c^{\text{upper}}(L \rightarrow \infty) = -0.184504 + 0.483112i. \quad (\text{D13})$$

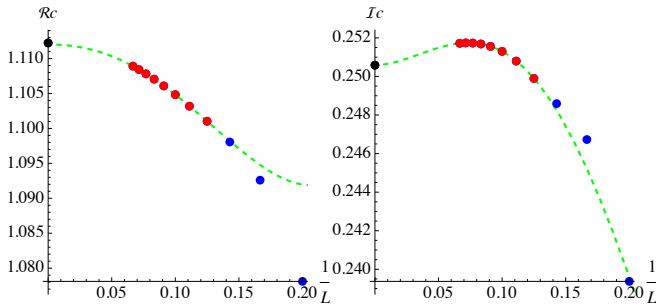


FIG. 66. Extrapolation of c .

For the central charge we find from extrapolation and analytics

$$\begin{aligned} c_{L \rightarrow \infty}^{\text{upper}} &= 1.11201 + 0.250589i \\ c_{\text{ana}}^{\text{upper}} &= 1.11224 + 0.250583i. \end{aligned} \quad (\text{D14})$$

The agreement is excellent.

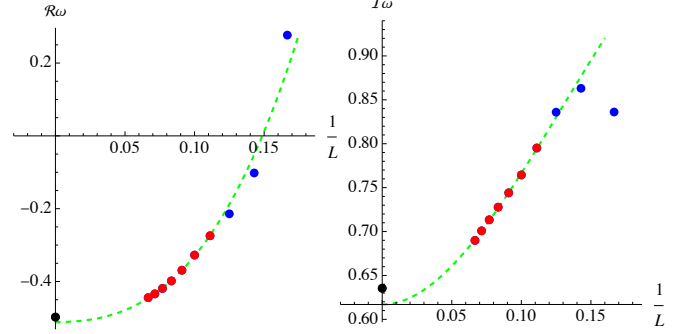


FIG. 67. Extrapolation of ω .

$$\omega_{L \rightarrow \infty}^{\text{upper}} = -0.511328 + 0.616934i, \quad (\text{D15})$$

$$\begin{aligned} \Delta_{L \rightarrow \infty}^{\text{upper}} &= 2 + \omega_{L \rightarrow \infty}^{\text{upper}} \\ &= 1.48867 + 0.61693i \end{aligned} \quad (\text{D16})$$

$$2h_{3,1} = 1.50219 + 0.63539i. \quad (\text{D17})$$

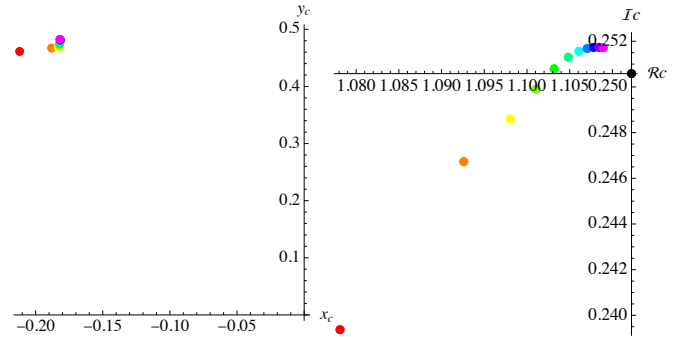


FIG. 68. Movement of z_c (left) and c (right) with L from $L = 4/5$ (red) to $L = 15/16$ (magenta). The black dot is the analytic result.

3. Minigrid, lower half plane

As in subsection D 2, we use a minigrid of 5×5 points, centered around the non-trivial fixed point identified in the preceding section. In Fig. 69 we show the area covered by it, as well as the zeroes of the fitting polynomial, for our largest system sizes $L = 15/16$. The non-trivial solution marked in red is clearly visible.

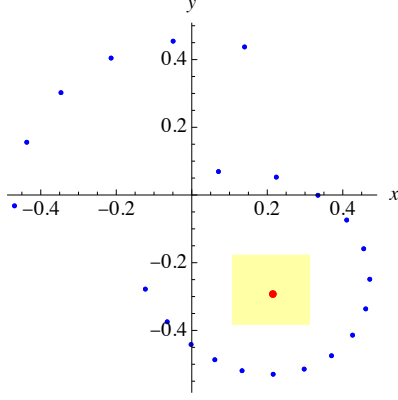


FIG. 69. Zeros of $c'(z)$, $L = 15$.

In Figs. 70 to 73 we show fits using a polynomial in $x := 1/L$ with monomials $\{1, x^2, x^3\}$.

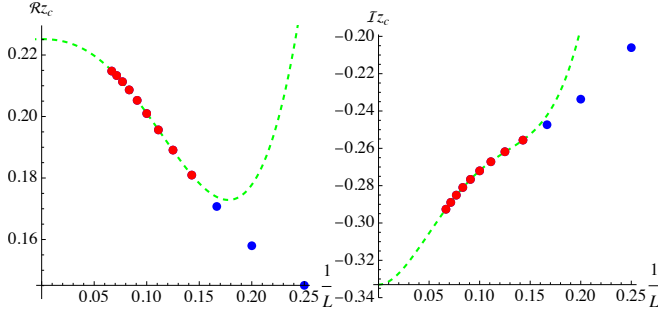


FIG. 70. Extrapolation of z_c .

$$z_c^{\text{lower}}(L \rightarrow \infty) = 0.225171 - 0.333031i. \quad (\text{D18})$$

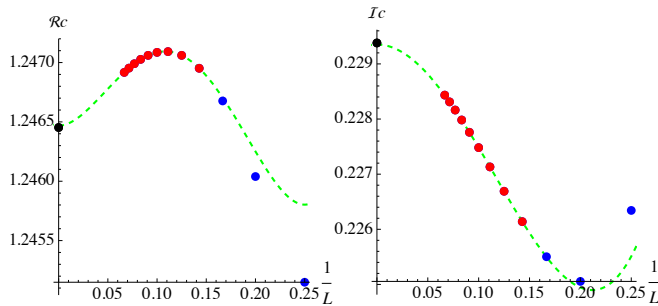


FIG. 71. Extrapolation of c .

For the central charge we find from extrapolation and analytics

$$\begin{aligned} c_{L \rightarrow \infty}^{\text{lower}} &= 1.24647 + 0.22935i \\ c_{\text{ana}}^{\text{lower}} &= 1.24645 + 0.22938i. \end{aligned} \quad (\text{D19})$$

The agreement is excellent.

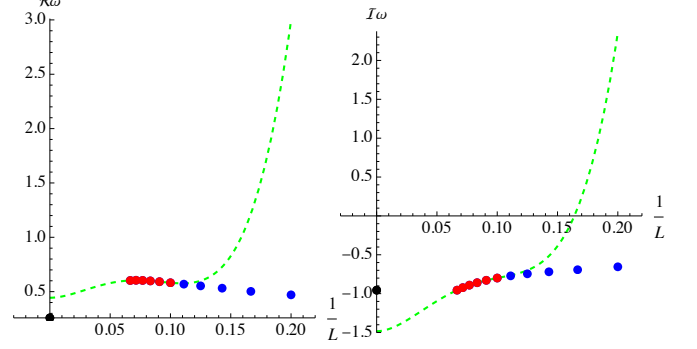


FIG. 72. Extrapolation of ω .

$$\omega_{L \rightarrow \infty}^{\text{lower}} = 0.443876 - 1.48309i, \quad (\text{D20})$$

$$\begin{aligned} \Delta_{L \rightarrow \infty}^{\text{lower}} &= 2 + \omega_{L \rightarrow \infty}^{\text{lower}} \\ &= 2.44388 - 1.48309i \end{aligned} \quad (\text{D21})$$

$$2h_{3,1} = 2.25868 - 0.955363i. \quad (\text{D22})$$

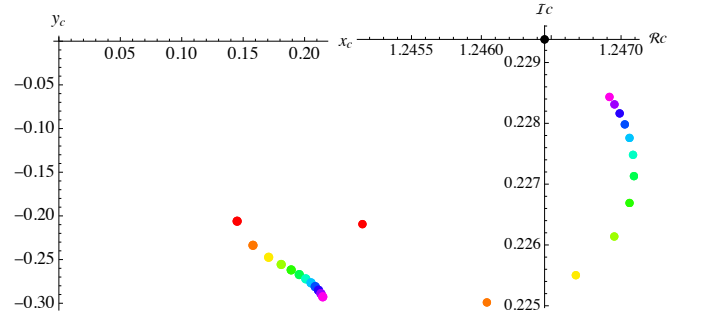


FIG. 73. Movement of z_c (left) and c (right) with L from $L = 4$ (red) to $L = 15$ (magenta). The black dot is the analytic result.

Appendix E: $Q = 8 + i$

1. Grid

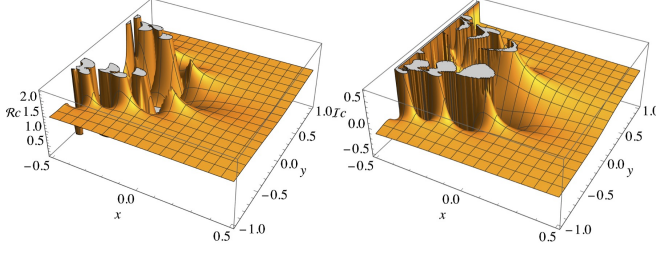


FIG. 74. c at $L = 10$, raw data.

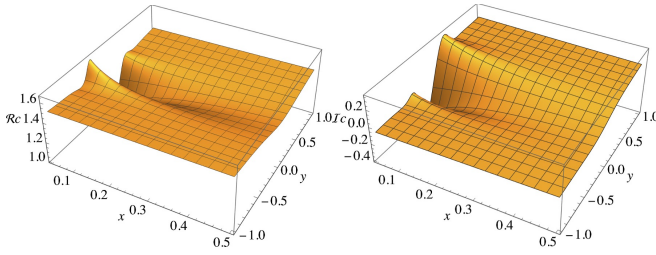


FIG. 75. c at $L = 10$, selected data.

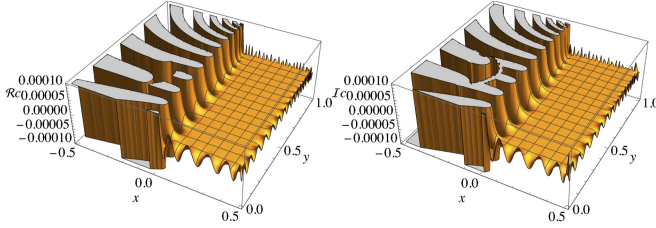


FIG. 76. c , $L = 10$, subtracted, upper half plane.

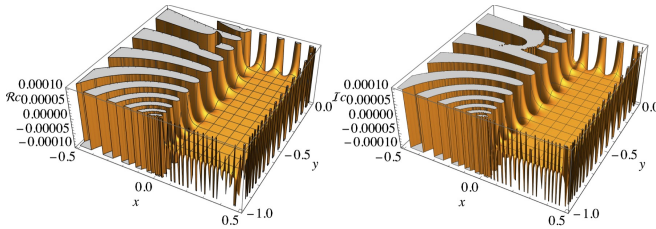


FIG. 77. c , $L = 10$, subtracted, lower half plane.

The contour plot 78 gives a good idea about the location of the singularities. Note that the two non-trivial fixed points marked by the red dots in Fig. 78 are not complex conjugate of each other. The precise locations of $c'(z) = 0$ are shown in Fig. 79; again the non-trivial ones are in red.

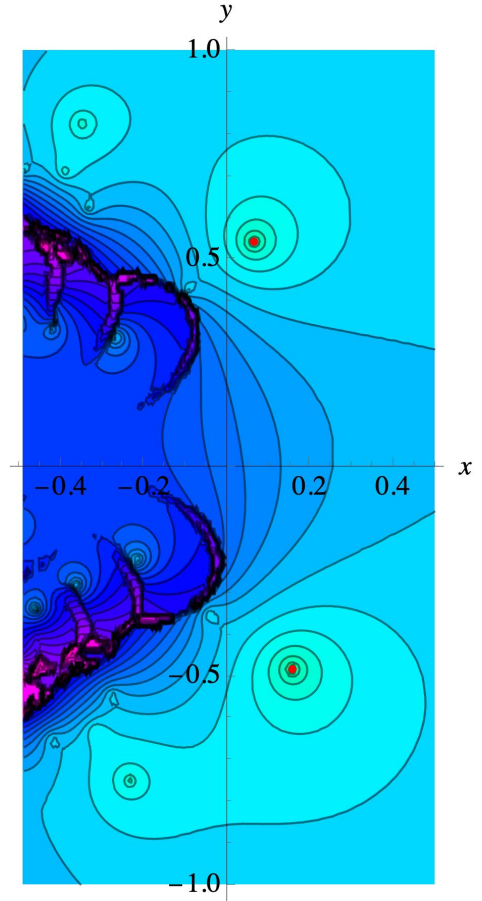


FIG. 78. Contour plot for $|c(z) - c_{\text{theory}}|^{0.3}$ for $Q = 8 + i$, $L = 10$. Saddle points $c'(z) = 0$ are marked by a red dot. The level-crossing lines are clearly visible

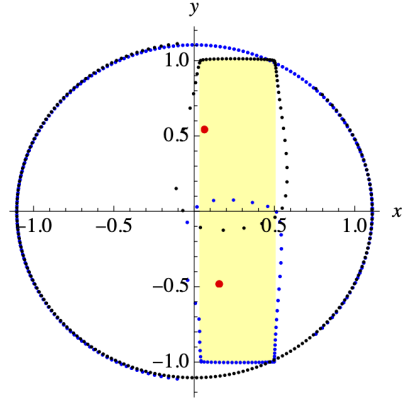


FIG. 79. Zeros of $c'(z)$ at $L = 10$. The yellow areas denote the points used to generate the fitting polynomials, separately for upper and lower half plane.

Extrapolations for z_c , c and ω are shown on the next page, separately for the two fixed points.

Fixpoint B_{-b} (upper half plane)

Fits for upper half plane. The fitting polynomial always contains $\{1, x^2, x^3\}$, with $x := 1/L$.

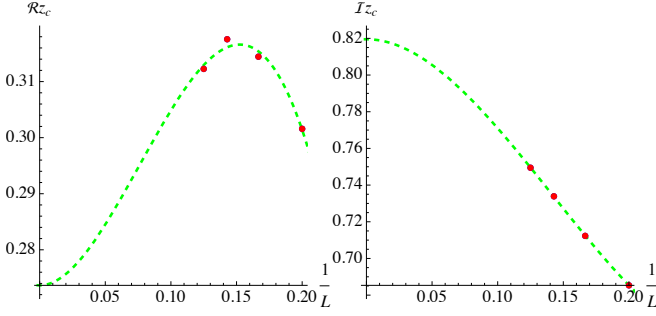


FIG. 80. Extrapolation of z_c in UHP.

$$z_c^{\text{upper}}(L \rightarrow \infty) = 0.0546142 + 0.570012i. \quad (\text{E1})$$

Fixpoint B_b (lower half plane)

Fits for lower half plane, with the same polynomial as for the upper half plane (see left).

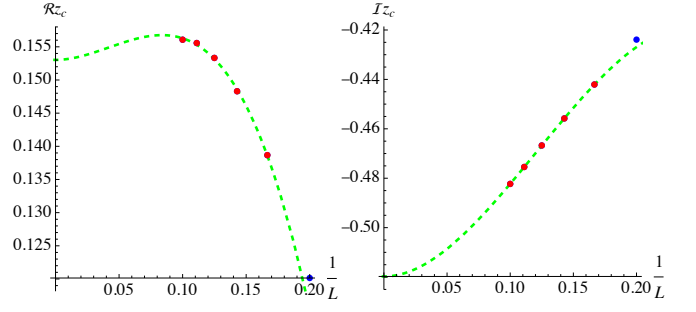


FIG. 83. Extrapolation of z_c in LHP.

$$z_c^{\text{lower}}(L \rightarrow \infty) = 0.153031 - 0.519754i. \quad (\text{E7})$$

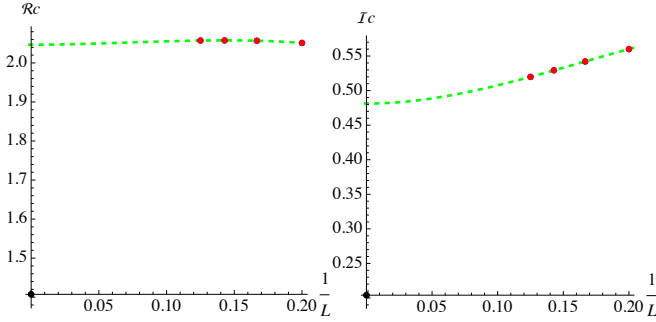


FIG. 81. Extrapolation of c in UHP.

$$c^{\text{upper}}(L \rightarrow \infty) = 1.40773 + 0.203235i, \quad (\text{E2})$$

$$c_{\text{ana}}^{\text{upper}} = 1.4073 + 0.2041i. \quad (\text{E3})$$

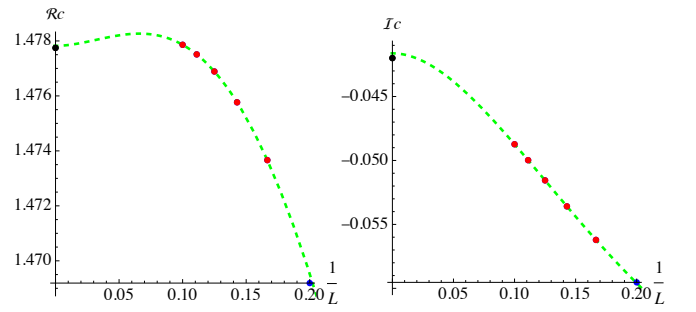


FIG. 84. Extrapolation of c in LHP.

$$c^{\text{lower}}(L \rightarrow \infty) = 1.47782 - 0.0416244i, \quad (\text{E8})$$

$$c_{\text{ana}}^{\text{lower}} = 1.4778 - 0.0420i \quad (\text{E9})$$

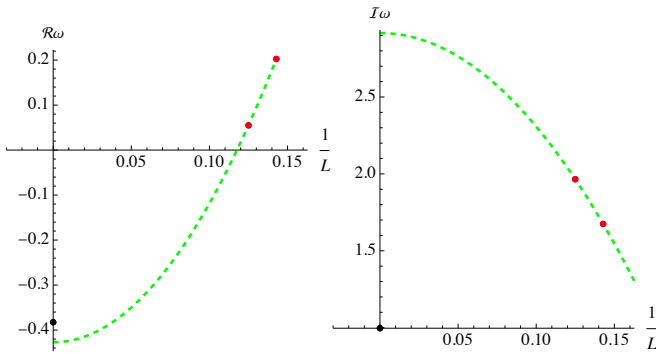


FIG. 82. Extrapolation of ω in UHP.

$$\omega^{\text{upper}}(L \rightarrow \infty) = -0.534022 + 1.00708i, \quad (\text{E4})$$

$$\Delta^{\text{upper}}(L \rightarrow \infty) = 2 + \omega^{\text{upper}}(L \rightarrow \infty) = 1.46598 + 1.00708i \quad (\text{E5})$$

$$2h_{3,1} = 1.61731 + 0.99649i. \quad (\text{E6})$$

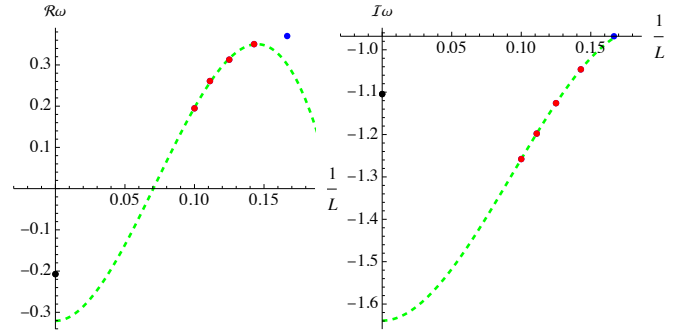


FIG. 85. Extrapolation of ω in LHP.

$$\omega^{\text{lower}}(L \rightarrow \infty) = -0.320701 - 1.63909i \quad (\text{E10})$$

$$\Delta^{\text{lower}}(L \rightarrow \infty) = 2 + \omega^{\text{lower}}(L \rightarrow \infty) = 1.6793 - 1.63909i \quad (\text{E11})$$

$$2h_{3,1} = 1.79269 - 1.10454i. \quad (\text{E12})$$

2. Minigrid, upper half-plane

Again we use a minigrid of 5×5 points. As in subsection D 2, we use a minigrid of 5×5 points, centered around the non-trivial fixed point identified in the preceding section. In Fig. 86 we show the area covered by it, as well as the zeroes of the fitting polynomial, for our largest system sizes $L = 15$. The non-trivial solution marked in red is clearly visible.

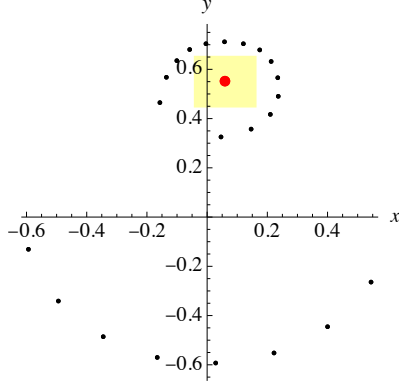


FIG. 86. Zeros of $c'(z)$, $L = 15$, UHP.

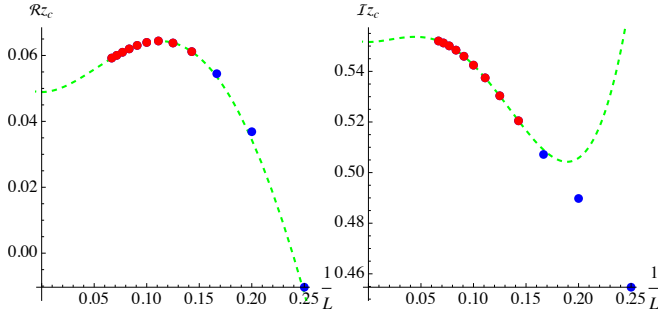


FIG. 87. Extrapolation of z_c , UHP.

$$z_c^{\text{upper}}(L \rightarrow \infty) = 0.0488999 + 0.551616i. \quad (\text{E13})$$

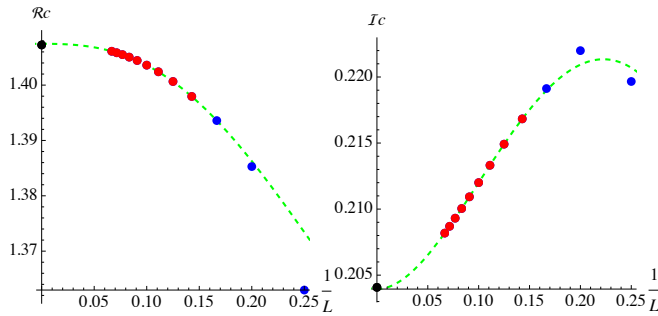


FIG. 88. Extrapolation of c , UHP.

For the central charge, we find from extrapolation and analytics

$$\begin{aligned} c_{L \rightarrow \infty}^{\text{upper}} &= 1.40746 + 0.203946i \\ c_{\text{ana}}^{\text{upper}} &= 1.40726 + 0.204076i. \end{aligned} \quad (\text{E14})$$

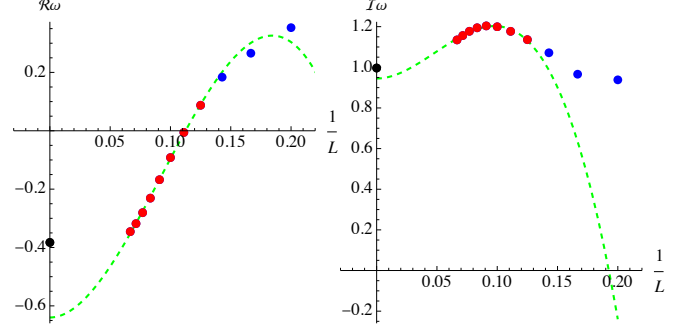


FIG. 89. Extrapolation of ω , UHP.

$$\omega_{L \rightarrow \infty}^{\text{upper}} = -0.639713 + 0.944913i, \quad (\text{E15})$$

$$\begin{aligned} \Delta_{L \rightarrow \infty}^{\text{upper}} &= 2 + \omega_{L \rightarrow \infty}^{\text{upper}} \\ &= 1.36029 + 0.944913i \end{aligned} \quad (\text{E16})$$

$$2h_{3,1} = 1.61731 + 0.99649i. \quad (\text{E17})$$

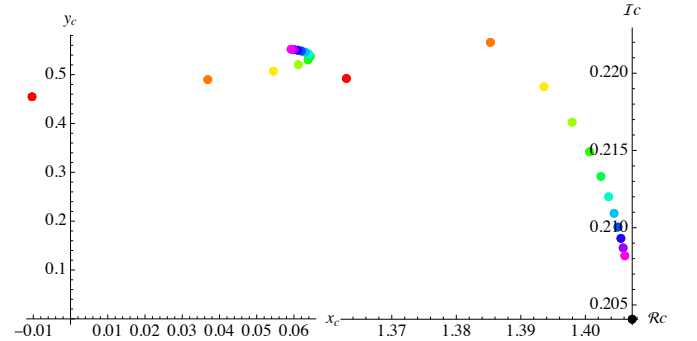


FIG. 90. Movement of z_c (left) and c (right) for $Q = 8 + i$.

3. Minigrid, lower half plane

Here we use a minigrid of 5×5 points.

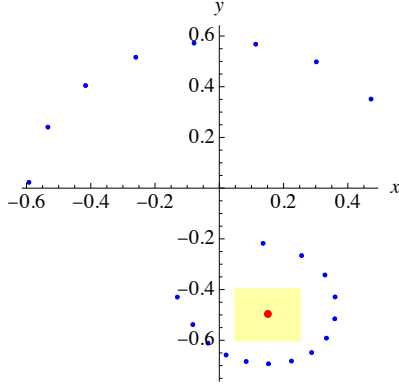


FIG. 91. Zeros of $c'(z)$, $L = 15$, LHP.

Fixpoint B_{-b}

Fits for upper half plane. If not stated otherwise, the fitting polynomial contains $\{1, x^2, x^3\}$, with $x := 1/L$.

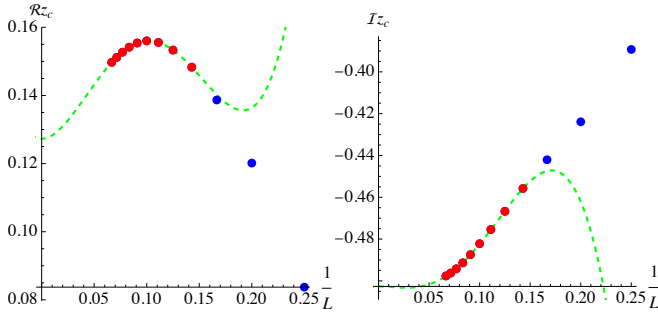


FIG. 92. Extrapolation of z_c , LHP.

$$z_c^{\text{lower}}(L \rightarrow \infty) = 0.127284 - 0.502652i. \quad (\text{E18})$$

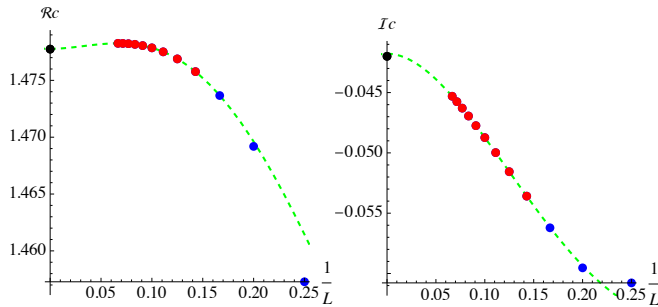


FIG. 93. Extrapolation of c , LHP.

$$\begin{aligned} c_{L \rightarrow \infty}^{\text{lower}} &= 1.47777 - 0.0417788i \\ c_{\text{ana}}^{\text{lower}} &= 1.47775 - 0.0419944i. \end{aligned} \quad (\text{E19})$$

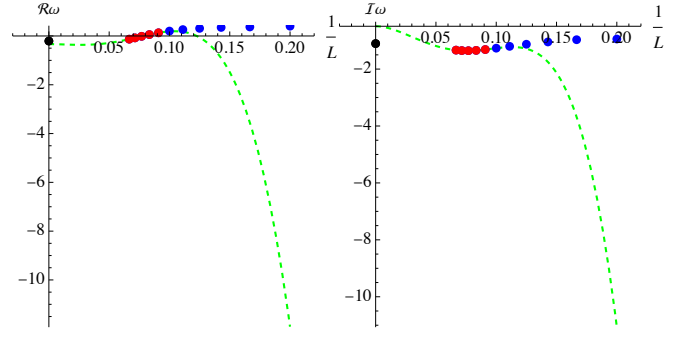


FIG. 94. Extrapolation of ω , LHP.

$$\omega_{L \rightarrow \infty}^{\text{lower}} = -0.333735 - 0.495648i, \quad (\text{E20})$$

$$\begin{aligned} \Delta_{L \rightarrow \infty}^{\text{lower}} &= 2 + \omega_{L \rightarrow \infty}^{\text{lower}} \\ &= 1.66627 - 0.495648i \end{aligned} \quad (\text{E21})$$

$$2h_{3,1} = 1.79269 - 1.10454i. \quad (\text{E22})$$

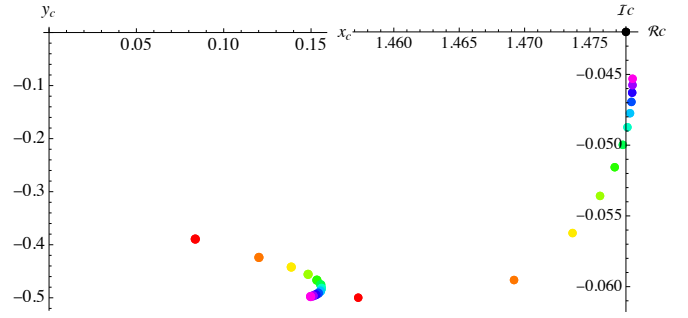


FIG. 95. Movement of z_c (left) and c (right) as a function of L .

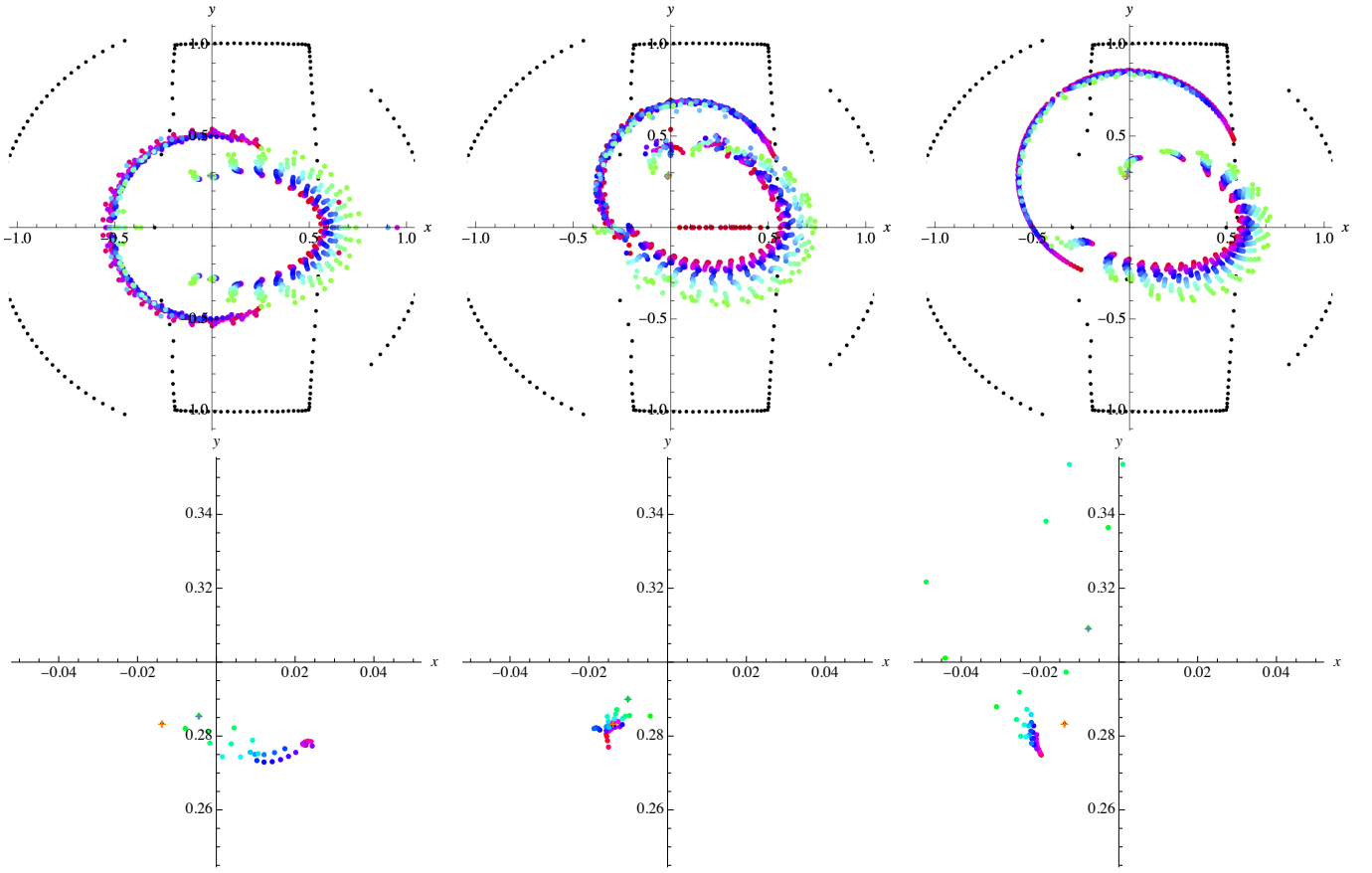


FIG. 96. Top line: solutions of $\partial_z c(z) = 0$. In black from the plan-data, in color from the line data. The colors indicate the order of the fit, with green the smallest order 4, and red the largest order 12; larger orders are not necessarily better, as they introduce artifacts. The expansion points are $z = 0$ (left), $z = 0.1 + 0.25i$ (middle) and $z = 0.3i$ (right). Bottom line: the region around the fixed point $z_c = -0.0138 + 0.2833i$. The solution (F1) is indicated by an orange cross.

Appendix F: Can one find $z_c(Q = 5)$ from data on the real line only?

An important question, relevant e.g. for the analysis of the 3-dimensional 3-state Potts model [83], is whether one can find the location of the critical point using only the information on the real line. We show below how this can be done, and what the artifacts are.

For this exercise we use our plane data with $L = 10$. By a fit to all data points, we find the reference point for z_c and $f''(z_c)$:

$$z_c = -0.0137785 + 0.283345i, \quad (\text{F1})$$

$$f''(z_c) = 0.552501 - 0.664543i. \quad (\text{F2})$$

Let us now try to reconstruct this from data on the real line. In order to eliminate artifacts from the singularity at $x \approx -0.48$, we retain only $x \geq 0$. We also have the choice to vary the degree of the polynomial fit, and its expansion point x_0 . Our first example is for $x_0 = 0$, see Fig. 96, left column. Retaining the solution which has the lowest $f''(z_c)$ gives

$$z_c \approx -0.0044 + 0.2854i, \quad (\text{F3})$$

$$f''(z_c) \approx 0.803 - 0.456i. \quad (\text{F4})$$

This uses 50 data points in the interval $[0, 1]$; it should be compared to Eqs. (F1)-(F2). Precision might increase when using more data points in the same interval. The procedure breaks down e.g. when using data for $x > -0.2$.

In our next test, shown in the middle column of Fig. 96, we use a generic expansion point somewhere in the upper half plane, namely $x_0 = 0.1 + 0.25i$. (The positive real part was chosen s.t. $\Re x_0 > 0$, the first point kept on the real line.) The extrapolation changes to

$$z_c \approx -0.0100 + 0.2900i, \quad (\text{F5})$$

$$f''(z_c) \approx 0.574 - 0.422i. \quad (\text{F6})$$

This is rather close to the solution (F1)-(F2), as can be seen on Fig. 96 (bottom line).

Now let us assume that we know approximately where the critical point is located, and use $z_0 = 0.3i$. The extrapolation changes to

$$z_c \approx -0.0077 + 0.3091i, \quad (\text{F7})$$

$$f''(z_c) \approx 0.453 + 0.054i. \quad (\text{F8})$$

This is good, but no improvement over Eqs. (F5)-(F6); probably the former is working better since the expansion point is further away from the boundary of retained points, i.e. $x = 0$.

To conclude: it is possible to reconstruct the critical point solely from points for real couplings. If need be, the procedure could be made more systematic, and more data points could be calculated to increase the precision. One should be careful to not use a fitting polynomial of too high degree. E.g. the dots in the middle top plot of Fig. 96 are artifacts of this.

Appendix G: OPE-coefficients following Dotsenko-Fateev

The evaluation of OPE coefficients was pioneered by Dotsenko and Fateev [84] (DF), and later continued in [85]. While the formalism we develop in section III is able to evaluate more coefficients than is possible in the original framework, we keep some results here. There are two main reasons: First of all, it is instructive to see how this works out in [84], with some simplifications over the original work. Secondly, DF give explicit analytical results in terms of Γ -functions, while the method in section III uses the Barnes double-Gamma function, which is hard to evaluate numerically, and which in the cases shown can be expressed in terms of Gamma-functions. All formulas below are explicit enough to infer where there are non-analyticities in the complex b -plane. This also allows us to evaluate them in a series in $Q - 4$.

1. General formulas

We follow Dotsenko and Fateev (DF) [84]. Their Eqs. (6) and (7) read in our conventions

$$h_{n,n'} = \frac{1}{4} [(\alpha_- n' + \alpha_+ n)^2 - (\alpha_+ + \alpha_-)^2], \quad (\text{G1})$$

$$c = 1 - 24\alpha_0^2, \quad (\text{G2})$$

$$\alpha_{\pm} = \alpha_0 \pm \sqrt{1 + \alpha_0^2}, \quad (\text{G3})$$

$$1 = \alpha_+ \alpha_-. \quad (\text{G4})$$

Comparing to our formulas implies that

$$4\alpha_0^2 b(b-1) = 1. \quad (\text{G5})$$

Let us suppose we consider the series of minimal models with $b > 1$, then the appropriate branch is

$$\alpha_0 > 0 \iff b > 1. \quad (\text{G6})$$

This implies

$$\alpha_0 = \frac{1}{2\sqrt{b(b-1)}}, \quad (\text{G7})$$

$$\alpha_+ = \sqrt{\frac{b}{b-1}}, \quad (\text{G8})$$

$$\alpha_- = \sqrt{\frac{b-1}{b}}. \quad (\text{G9})$$

$$(\text{G10})$$

DF [84] further use

$$\rho := \alpha_+^2 \equiv \frac{b}{b-1}, \quad (\text{G11})$$

$$\rho' := \alpha_-^2 \equiv \frac{b-1}{b}. \quad (\text{G12})$$

($\rho = 1/g$ and $\rho' = g$ in the notations of Sec. IID). Note that $b \rightarrow b+1$ (from critical to tricritical point) is equivalent to $b \rightarrow -b$ and exchange of ρ and ρ' .

DF's expressions for the structure constants are given in their Eqs. (15)–(18). Below in Eq. (G14) we give a slightly more compact and simplified form. To achieve this, define

$$\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}. \quad (\text{G13})$$

The second simplification is to shift in their Eq. (15) the indices i and j by 1; this simplifies their expressions to

$$\begin{aligned} \widehat{C}_{(n,n'),(s,s')}^{(p,p')} &= \rho^{4(l-1)(l'-1)} \\ &\times \prod_{i=1}^{l-1} \prod_{j=1}^{l'-1} \frac{1}{[i - \rho j]^2 [s - i - \rho(s' - j)]^2} \\ &\times \frac{1}{[n - i - \rho(n' - j)]^2 [p + i - \rho(j + p')]^2} \\ &\times \prod_{i=1}^{l-1} \gamma(i\rho') \gamma(s' - (s - i)\rho') \\ &\times \gamma(n' - (n - i)\rho') \gamma((i + p)\rho' - p') \\ &\times \prod_{j=1}^{l'-1} \gamma(j\rho) \gamma(s - (s' - j)\rho) \\ &\times \gamma(n - (n' - j)\rho) \gamma((j + p')\rho - p) \end{aligned} \quad (\text{G14})$$

$$l = \frac{1}{2}(s + n - p + 1), \quad l' = \frac{1}{2}(s' + n' - p' + 1) \quad (\text{G15})$$

Note that these formulas can only be evaluated for integer l and l' . For other values, one needs to continue the products of Γ functions analytically, in the same spirit as $n! = \prod_{i=1}^n i$ is continued analytically to $\Gamma(n+1)$. This is achieved by the Barnes double gamma function, as discussed in section III. The structure constants $\omega_{(n,n'),(s,s')}^{(p,p')}$ (denoted D by DF) contain an additional normalization factor a ,

$$[\omega_{(n,n'),(s,s')}^{(p,p')}]^2 = [\widehat{C}_{(n,n'),(s,s')}^{(p,p')}]^2 \frac{a_{n,n'} a_{s,s'}}{a_{p,p'}}. \quad (\text{G16})$$

The latter is defined as⁹

$$\begin{aligned} a_{n,n'} &= \prod_{i=1}^{n-1} \prod_{j=1}^{n'-1} \frac{[1 + i - (1 + j)\rho]^2}{(i - j\rho)^2} \\ &\times \prod_{i=1}^{n-1} \gamma(i\rho') \gamma(2 - (i + 1)\rho') \\ &\times \prod_{j=1}^{n'-1} \gamma(j\rho) \gamma(2 - (j + 1)\rho). \end{aligned} \quad (\text{G17})$$

2. Examples of OPE coefficients following DF, physical branch

We now give some examples for the formulas in section G. Due to Eq. (G15), $2l = s + n - p + 1$ as well as $2l' =$

⁹ Attention, there is a misprint in DF in the first product in the first line after Eq (18).

$s' + n' - p' + 1$ need to be even. For integer indices this implies that within each group $\{s, n, p\}$ and $\{s', n', p'\}$, either one index is odd and two even, or all three indices are odd. For non-integer indices this is more complicated. Below we give results for the relevant physical branch; the other branch is treated in section G 3.

$$\begin{aligned}\omega_{\varepsilon\varepsilon\varepsilon'} &= \omega_{(2,1)(2,1)(3,1)} \\ &= \sqrt{\frac{\Gamma(\frac{1+2b}{b-1})\Gamma(\frac{-2}{b-1})\Gamma(\frac{b}{b-1})\Gamma(\frac{b+1}{1-b})}{\Gamma(\frac{1}{1-b})\Gamma(\frac{2b}{b-1})\Gamma(\frac{b+1}{b-1})\Gamma(\frac{b+2}{1-b})}}. \quad (\text{G18})\end{aligned}$$

$$\begin{aligned}\omega_{\varepsilon'\varepsilon'\varepsilon'} &= \omega_{(3,1)(3,1)(3,1)} \\ &= \frac{\Gamma(\frac{1+3b}{b-1})}{\Gamma(\frac{2(b+1)}{1-b})} \left[\frac{\Gamma(\frac{b}{b-1})\Gamma(\frac{b+1}{1-b})}{\Gamma(\frac{1}{1-b})\Gamma(\frac{2b}{b-1})} \right]^{3/2} \\ &\quad \times \sqrt{\frac{\Gamma(\frac{2}{1-b})\Gamma(\frac{b+2}{1-b})}{\Gamma(\frac{2b+1}{b-1})\Gamma(\frac{b+1}{b-1})}}. \quad (\text{G19})\end{aligned}$$

$$\begin{aligned}\omega_{\varepsilon',\varepsilon'',\varepsilon''} &= \omega_{(3,1)(4,1)(4,1)} \\ &= \sqrt{\frac{\Gamma(\frac{2}{1-b})\Gamma(\frac{b+2}{1-b})\Gamma(\frac{b+1}{1-b})}{\Gamma(\frac{2b}{b-1})\Gamma(\frac{b+1}{b-1})\Gamma(\frac{2b+1}{b-1})}} \\ &\quad \times \frac{\Gamma(\frac{2b+1}{1-b})\Gamma(\frac{4b+1}{b-1})}{\Gamma(\frac{3b+2}{1-b})\Gamma(\frac{3b}{b-1})} \left[\frac{\Gamma(\frac{b}{b-1})}{\Gamma(\frac{1}{1-b})} \right]^{3/2}. \quad (\text{G20})\end{aligned}$$

$$\begin{aligned}\omega_{\varepsilon,\varepsilon',\varepsilon''} &= \omega_{(2,1)(3,1)(4,1)} \\ &= \sqrt{\frac{\Gamma(\frac{2}{1-b})\Gamma(\frac{2b+1}{1-b})\Gamma(\frac{b}{b-1})\Gamma(\frac{3b+1}{b-1})}{\Gamma(\frac{1}{1-b})\Gamma(\frac{3b}{b-1})\Gamma(\frac{2(b+1)}{1-b})\Gamma(\frac{b+1}{b-1})}}. \quad (\text{G21})\end{aligned}$$

$$\begin{aligned}\omega_{(0,1)(0,1)(3,1)} &= \frac{\Gamma(\frac{1}{b-1})\Gamma(\frac{2b-1}{b-1})}{\Gamma(\frac{b-2}{b-1})\Gamma(\frac{b}{1-b})} \left[\frac{\Gamma(\frac{b}{b-1})}{\Gamma(\frac{1}{1-b})} \right]^{3/2} \\ &\quad \times \sqrt{\frac{\Gamma(\frac{2}{1-b})\Gamma(\frac{b+2}{1-b})\Gamma(\frac{b+1}{1-b})}{\Gamma(\frac{2b}{b-1})\Gamma(\frac{b+1}{b-1})\Gamma(\frac{2b+1}{b-1})}}. \quad (\text{G22})\end{aligned}$$

At $Q = 5$, this yields

$$\omega_{(2,1)(2,1)(3,1)} = 0.879108 + 0.140371i \quad (\text{G23})$$

$$\omega_{(3,1)(3,1)(3,1)} = 2.26873 + 1.1967i \quad (\text{G24})$$

$$\omega_{(3,1)(4,1)(4,1)} = 3.92606 + 3.32608i \quad (\text{G25})$$

$$\omega_{(2,1)(3,1)(4,1)} = 0.831751 + 0.202731i \quad (\text{G26})$$

$$\omega_{(0,1)(0,1)(3,1)} = -0.245254 + 0.131105i \quad (\text{G27})$$

3. Examples of OPE coefficients following DF, non-physical branch

Below are OPE coefficients for operators not realized in the Potts model. They are relevant for other models; some of them appear in [14]. Note that this list contains some OPE coefficients with non-integer entries, see Eqs. (G32)–(G33).

To compare to the perturbative expansion of [14], we also give its $(4 - Q)$ expansion. This is achieved by first establishing the $1/b$ expansion, and then expressing it in powers of

$(4 - Q)$.

$$\begin{aligned}\omega_{(1,2)(1,2)(1,3)} &= \\ &= \frac{\Gamma(\frac{2}{b})\Gamma(\frac{b-1}{b})}{\Gamma(\frac{1}{b})\Gamma(\frac{b-2}{b})} \sqrt{\frac{b^3\Gamma(2-\frac{3}{b})\Gamma(\frac{1}{b})}{(b-2)^2\Gamma(\frac{3}{b}-1)\Gamma(-\frac{1}{b})}} \\ &= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2b} - \frac{\sqrt{3}}{b^2} + \frac{2\sqrt{3}[\zeta(3)-1]}{b^3} \\ &\quad - \frac{2\sqrt{3}[\zeta(3)+2]}{b^4} + \mathcal{O}(b^{-5}) \\ &= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}\sqrt{4-Q}}{4\pi} - \frac{\sqrt{3}(4-Q)}{4\pi^2} \\ &\quad - \frac{(4-Q)^{3/2}[24+\pi^2-24\zeta(3)]}{32\sqrt{3}\pi^3} \\ &\quad - \frac{(4-Q)^2[6\zeta(3)+12+\pi^2]}{16\sqrt{3}\pi^4} + \mathcal{O}(4-Q)^{5/2}. \quad (\text{G28})\end{aligned}$$

The first two terms of this OPE-coefficient agree with [14] Eq. (4.19), who obtained their results via a perturbative expansion in $Q - 4$.

$$\begin{aligned}\omega_{(1,3)(1,3)(1,3)} &= \\ &= \frac{2^{2-\frac{4}{b}}\Gamma(\frac{3}{2}-\frac{2}{b})\Gamma(\frac{1}{b}-\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{b})\Gamma(\frac{2}{b}-\frac{1}{2})} \sqrt{\frac{(b-2)^2\Gamma(\frac{3}{b}-1)\Gamma(-\frac{1}{b})}{b^3\Gamma(2-\frac{3}{b})\Gamma(\frac{1}{b})}} \\ &= \frac{4}{\sqrt{3}} - \frac{4\sqrt{3}}{b} - \frac{20}{\sqrt{3}b^2} + \frac{4[24\zeta(3)-7]}{\sqrt{3}b^3} - \frac{4[5+72\zeta(3)]}{\sqrt{3}b^4} \\ &\quad + \mathcal{O}(b^{-5}) \\ &= \frac{4}{\sqrt{3}} - \frac{2\sqrt{3}\sqrt{4-Q}}{\pi} - \frac{5(4-Q)}{\sqrt{3}\pi^2} \\ &\quad - \frac{(4-Q)^{3/2}(\pi^2-48\zeta(3)+14)}{4\sqrt{3}\pi^3} \\ &\quad - \frac{(4-Q)^2[15+5\pi^2+216\zeta(3)]}{12\sqrt{3}\pi^4} + \mathcal{O}(4-Q)^{5/2}. \quad (\text{G29})\end{aligned}$$

The first two terms of the OPE-coefficients above agree with Eq. (4.11) in [14].

$$\begin{aligned}\omega_{(1,2)(1,3)(1,4)} &= \\ &= \sqrt{\frac{2^{\frac{4}{b}-1}\Gamma(3-\frac{4}{b})\Gamma(\frac{1}{2}+\frac{1}{b})\Gamma(\frac{3}{b}-2)}{\Gamma(3-\frac{3}{b})\Gamma(\frac{1}{2}-\frac{1}{b})\Gamma(\frac{4}{b}-2)}} \\ &= \sqrt{\frac{2}{3}} - \frac{\sqrt{\frac{3}{2}}}{b} - \frac{13}{2\sqrt{6}b^2} + \frac{(80\zeta(3)-63)}{4\sqrt{6}b^3} \\ &\quad - \sqrt{\frac{3}{2}} \frac{80\zeta(3)+111}{8b^4} + \mathcal{O}(b^{-5})\end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{2}{3}} - \sqrt{\frac{3}{2}} \frac{\sqrt{4-Q}}{2\pi} - \frac{13(4-Q)}{8\sqrt{6}\pi^2} \\
&\quad - \frac{(4-Q)^{3/2}[63+2\pi^2-80\zeta(3)]}{32\sqrt{6}\pi^3} \\
&\quad - \frac{(4-Q)^2[999+52\pi^2+720\zeta(3)]}{384\sqrt{6}\pi^4} \\
&\quad + \mathcal{O}(4-Q)^{5/2}. \tag{G30}
\end{aligned}$$

$$\begin{aligned}
&\omega_{(1,3)(1,4)(1,4)} \\
&= \frac{2^{\frac{4}{b}-2}\Gamma(4-\frac{5}{b})\Gamma(\frac{1}{b}+\frac{1}{2})\Gamma(\frac{3}{b}-2)}{\Gamma(3-\frac{3}{b})\Gamma(\frac{3}{2}-\frac{1}{b})\Gamma(\frac{5}{b}-3)} \sqrt{\frac{\Gamma(-\frac{1}{b})\Gamma(\frac{3}{b}-1)}{b\Gamma(2-\frac{3}{b})\Gamma(\frac{1}{b})}} \\
&= \frac{5\sqrt{3}}{2} - \frac{65}{2\sqrt{3}b} - \frac{155}{3\sqrt{3}b^2} + \frac{65[72\zeta(3)-11]}{12\sqrt{3}b^3} \\
&\quad + \frac{5[127-4056\zeta(3)]}{12\sqrt{3}b^4} + \mathcal{O}(b^{-5}) \\
&= \frac{5\sqrt{3}}{2} - \frac{65\sqrt{4-Q}}{4\sqrt{3}\pi} - \frac{155(4-Q)}{12\sqrt{3}\pi^2} \\
&\quad - \frac{65(4-Q)^{3/2}[11+\pi^2-72\zeta(3)]}{96\sqrt{3}\pi^3} \\
&\quad - \frac{5(4-Q)^2[12168\zeta(3)-381+124\pi^2]}{576\sqrt{3}\pi^4} \\
&\quad + \mathcal{O}(4-Q)^{5/2}. \tag{G31}
\end{aligned}$$

$$\begin{aligned}
\omega_{(0,\frac{1}{2})(0,\frac{1}{2})(3,1)} &= \sqrt{\frac{\Gamma(\frac{b+2}{1-b})\Gamma(\frac{b+1}{1-b})\Gamma(\frac{b}{b-1})\Gamma(\frac{2b}{b-1})}{\Gamma(\frac{1}{1-b})\Gamma(\frac{2}{1-b})\Gamma(\frac{b+1}{b-1})\Gamma(\frac{2b+1}{b-1})}} \\
&\quad \times 4^{\frac{b+1}{1-b}} \frac{\Gamma(\frac{b+1}{2(b-1)})}{\Gamma(\frac{b+1}{2(1-b)})} \tag{G32}
\end{aligned}$$

$$\begin{aligned}
\omega_{(0,\frac{3}{2})(0,\frac{3}{2})(3,1)} &= \sqrt{\frac{\Gamma(\frac{b+2}{1-b})\Gamma(\frac{b-3}{2(b-1)})\Gamma(\frac{b}{b-1})\Gamma(\frac{2b}{b-1})}{\Gamma(\frac{b+1}{1-b})\Gamma(\frac{b+1}{b-1})\Gamma(\frac{2b+1}{b-1})}} \\
&\quad \times \frac{2^{-\frac{5}{b-1}-\frac{1}{2}}(1-3b)^2\Gamma(\frac{3-b}{2(b-1)})}{\sqrt[4]{\pi}(b-1)^2\Gamma(\frac{3b-5}{2(b-1)})} \tag{G33}
\end{aligned}$$

Numerically we find

$$\omega_{(1,2)(1,2)(1,3)}(Q=5) = 0.90131 + 0.134446i, \tag{G34}$$

$$\omega_{(1,2)(1,2)(1,3)}(Q=5) = 2.48214 + 1.20601i, \tag{G35}$$

$$\omega_{(1,2)(1,3)(1,4)}(Q=5) = 0.864836 + 0.196605i, \tag{G36}$$

$$\omega_{(1,3)(1,4)(1,4)}(Q=5) = 4.53186 + 3.48543i \tag{G37}$$

$$\omega_{(0,\frac{1}{2})(0,\frac{1}{2})(3,1)} = 0.0438596 - 0.0665729i, \tag{G38}$$

$$\omega_{(0,\frac{3}{2})(0,\frac{3}{2})(3,1)} = 0.578473 - 0.134703i. \tag{G39}$$

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