# Functional renormalization group at large $N$ for random manifolds

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We introduce a method, based on an exact calculation of the effective action at large $N$, to bridge the gap between mean field theory and renormalization in complex systems. We apply it to a $d$-dimensional manifold in a random potential for large embedding space dimension $N$. This yields a functional renormalization group equation valid for any $d$, which contains both the $O(\epsilon = 4-d)$ results of Balents-Fisher and some of the non-trivial results of the Mezard-Parisi solution thus shedding light on both. Corrections are computed at order $O(1/N)$. Applications to the problems of KPZ, random field and mode coupling in glasses are mentioned.

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The random manifold problem, i.e. the behavior of an elastic interface in a random potential is important for many experimental systems and still offers a considerable theoretical challenge [1]. It is the simplest example of a class of disordered systems, including random field magnets, where the so called dimensional reduction [2] renders conventional perturbation theory trivial and useless. It also provides powerful analogies, via mode coupling theory, to complex systems such as structural glasses [1]. Two analytical approaches have been devised so far, in limits where the problem *appears* to simplify while remaining non-trivial: the functional renormalization group (FRG) [3, 4, 10] and the (mean field) replica gaussian variational method (GVM) [5], together with their dynamical versions [6, 7, 8]. The FRG is hoped to be controlled for small $\epsilon = 4-d$, where $d$ is the internal dimension of the interface (parameterized by a $N$ component vector $\vec{u}(x)$ in the embedding space). It follows the second cumulant of the random potential $R(u)$ under coarse graining, which becomes non-analytic at $T = 0$ beyond the Larkin scale. The GVM approximates the replica measure by a replica symmetry broken (RSB) gaussian, equivalently, the Gibbs measure for $u$ as a random superposition of gaussians [5], and is claimed to be exact for $N = \infty$. Computing next order corrections is fraught with difficulties [13], and it is still unclear in which sense both methods describe small but *finite* $\epsilon$ or $1/N$. The GVM for instance predicts a transition for $d = 0$ which must disappear for any finite $N$. We found recently [12, 16] that higher loop FRG equations for $R(u)$ at $u \neq 0$ contain non-trivial, potentially ambiguous "anomalous terms" involving the non-analytic structure of $R(u)$ at $u = 0$. Although a solution was found to two loops [12, 16], the many loop structure remains mysterious. While both methods circumvent dimensional reduction by providing a non-perturbative mechanism, the GVM via replica symmetry breaking and the FRG via the generation of a cusp-like non-analyticity in $R''(u)$, they are disconcertingly different in spirit. Physically, however, both capture the metastable states beyond the Larkin scale $R_c$ and should thus be related [9]. A useful, quantitative and more general method, where this connection appears naturally, is still lacking.

In this Letter we introduce such a method, encompassing both the FRG and the GVM. Via an exact calculation of the effective action $\Gamma[u]$ at large $N$, we obtain the FRG $\beta$-function in any $d$ at large $N$. Its detailed analysis at dominant order, $N = +\infty$, reveals that the FRG *exactly* reproduces (without invoking spontaneous RSB) the non-trivial result of the GVM for small overlap. The connections can be clarified using that $\Gamma[u]$ also gives the probability distribution of a given mode $u_q$. $O(1/N)$ corrections are computed, with the aim of understanding finite but large $N$. Further results, extensions and discussions will appear in [11].

We start from the partition sum of an interface $\mathcal{Z}_V = \int \mathcal{D}[u]\, \mathrm{e}^{-\mathcal{H}_V[u]/T}$ in a given sample, with energy:

$$\mathcal{H}_V[u] = \int_q \frac{1}{2}(q^2+m^2) u_{-q}\cdot u_q + \int_x V_x(u(x)) \qquad (1)$$

where $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$, $\int_x \equiv \int d^d x$. The small confining mass $m$ provides a scale. To obtain a non-trivial large $N$ limit one defines the scaled field $v = u/\sqrt{N}$ and chooses the distribution of the random potential $O(N)$ rotationally invariant, e.g. its second cumulant as:

$$\overline{V_x(u)V_{x'}(u')} = R(u-u')\delta_{xx'} = NB((v-v')^2)\delta_{xx'} \qquad (2)$$

in terms of a function $B(z)$. Higher connected cumulants are scaled as $\overline{V_{x_1}(u_1)\dots V_{x_p}(u_p)}^{\text{conn}} = N\delta_{x_1,\dots,x_p} S^{(p)}(v_1,\dots,v_p)$. Physical observables can be obtained for any $N$ from the replicated action at $n = 0$ with a source $\mathcal{Z}[j] = \int \mathcal{D}[u]\mathcal{D}[\chi]\mathcal{D}[\lambda]\mathrm{e}^{-N\mathcal{S}[u,\chi,\lambda,j]}$

$$\begin{aligned}\mathcal{S}[u,\chi,\lambda,j] = {} & \frac{1}{2}\int_q (q^2+m^2) v^a_{-q}\cdot v^a_q \qquad (3)\\ & + \int_x [U(\chi_x) - \frac{1}{2} i\lambda^{ab}_x(\chi^{ab}_x - v^a_x\cdot v^b_x) - j^a_x\cdot v^a_x]\end{aligned}$$

where the replica matrix field $\chi_x \equiv \chi^{ab}_x$ has been introduced through a Lagrange multiplier. The bare interaction matrix potential $U(\chi) = \frac{-1}{2T^2}\sum_{ab} B(\tilde{\chi}_{ab}) - \frac{1}{3!T^3}\sum_{abc} S(\tilde{\chi}_{ab},\tilde{\chi}_{bc},\tilde{\chi}_{ca}) + \dots$ depends only on $\tilde{\chi}_{ab} = \chi_{aa} + \chi_{bb} - \chi_{ab} - \chi_{ba}$ and has a cumulant expansion in terms of sums with higher numbers of replicas.

The effective action functional is defined as Legendre transform [14] $\Gamma[u] + \mathcal{W}[J] = \int J\cdot u$, with $\mathcal{W}[J] = \ln \mathcal{Z}[j]$,

$J = \sqrt{N} j$. Its full calculation is given in [11]. Since $\Gamma[u]$ defines the renormalized 1PI vertices, its zero momentum limit defines the *renormalized disorder*. Thus we only need the result (per unit volume) for a *uniform* configuration of the replica field $u^a_x = u^a = \sqrt{N} v^a$:

$$\tilde{\Gamma}(v) = \frac{1}{L^d N}\Gamma(u) = \frac{1}{2T} m^2 v_a^2 + \tilde{U}(vv) \qquad (4)$$

where $vv$ stands for the matrix $v_a \cdot v_b$. We have computed the two first coefficients of the renormalized disorder in the $1/N$ expansion $\tilde{U} = \tilde{U}^0 + \frac{1}{N}\tilde{U}^1 + \ldots$. Defining the notation $\partial_{ab} U(\phi) \equiv \partial_{\phi_{ab}} U(\phi)$ for any matrix $\phi$ with components $\phi_{ab}$, we find at dominant order

$$\partial_{ab}\tilde{U}^0(vv) = \partial_{ab} U(\chi(v)) \qquad (5)$$

$$\chi(v) = vv + T\int_q [(q^2+m^2)\delta + 2T\partial U(\chi(v))]^{-1} \qquad (6)$$

i.e. a self consistent equation for $\partial \tilde{U}^0(vv)$, which, as we now show, contains both the GVM and the FRG.

For simplicity, we now set all bare cumulants except $B$ to zero. The above equations contain a huge amount of information, since they encode the full distribution (i.e. all cumulants) of the renormalized disorder, and are thus quite non-trivial to analyze. One limit where they "simplify" is when $v$ is set to zero, since they then reproduce the Mezard Parisi (MP) equations [5] with $\chi(v{=}0)_{ab} = \int_k G_{ab}(k)$. These exhibit spontaneous RSB (with multiple solutions [15]) and are solved by a hierarchical Parisi ansatz for $\chi(v{=}0)_{ab} = \chi(v{=}0)(\mathsf{u})$ where $0 \le \mathsf{u} \le 1$ is the overlap between replicas $a$ and $b$. In the opposite limit of "strong" explicit symmetry breaking field (all $v_{ab} \equiv v_a - v_b \neq 0$) we expect that the renormalized disorder $\tilde{U}(vv)$ is given by a single saddle point and can be expanded in replica sums in terms of unambiguous renormalized cumulants, i.e. up to a constant:

$$\tilde{U}(vv) = \frac{-1}{2T^2}\sum_{ab}\tilde{B}(v_{ab}^2) - \frac{1}{3!T^3}\sum_{abc}\tilde{S}(v_{ab}^2, v_{bc}^2, v_{ca}^2) + \ldots$$

This is the limit solved here, which we show is the one natural in the FRG, and amounts (in the RSB picture) to forcing the manifold in distant states. Work is in progress to analyze the rich crossover to RSB contained in (6), when some of the $v_{ab}$ are set to zero.

We can now expand, as detailed in [11], any quantity in (6) (e.g. a replica matrix $M_{ab} = M^0_{ab} + \sum_f M^1_{abf} + \ldots$ and its powers) in sums over an increasing number of free replica indices. This yields *closed* equations for the second cumulant (with $I_n := \int_k 1/(k^2+m^2)^n$ )

$$\tilde{B}'(v_{ab}^2) = B'(v_{ab}^2 + 2TI_1 + 4I_2(\tilde{B}'(v_{ab}^2) - \tilde{B}'(0))) \qquad (7)$$

with no other contributions from higher cumulants at any $T$. This is illustrated graphically in figure 1. The three replica term $\tilde{S}$ satisfies a closed equation involving only $\tilde{B}$, and all cumulants can be determined iteratively.

The self consistent equation (7) for the renormalized disorder can either be inverted directly (done below) or,

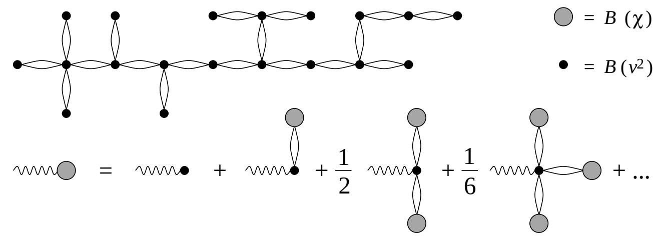


FIG. 1: Top: typical $T = 0$ contribution to $\tilde{B}(v_{ab})$. Bottom: self-consistent equation at leading order for $\tilde{B}'(v_{ab}^2) = B'(\chi_{ab})$. The wiggly line denotes a derivative, and is combinatorially equivalent to choosing one $B$. At finite $T$ one can attach an additional arbitrary number of tadpoles to any $B$.

equivalently, turned into a FRG equation. We start with the solution $\tilde{B}(x) = B(x)$ for $m = \infty$ (in presence of a fixed ultraviolet (UV) cutoff $\Lambda$) and then decrease $m$. Taking the derivative $m\partial_m$ and rearranging gives:

$$m\partial_m \tilde{B}'(x) = \tilde{B}''(x)\big[4(m\partial_m I_2)(\tilde{B}'(x) - \tilde{B}'(0)) + 2(m\partial_m T I_1)(1 + 4I_2\tilde{B}''(0))^{-1}\big] \qquad (8)$$

valid for any $d$ [17]. Since $-\frac{1}{2} m\partial_m I_1 = m^2 I_2$, (8) has a well defined $\Lambda \to +\infty$ limit for $d < 4$. Then $I_2 = A_d \frac{m^{-\epsilon}}{\epsilon}$ with $A_d = 2(4\pi)^{-d/2}\Gamma[3 - d/2]$ and we can define the dimensionless function $b(x) = 4A_d m^{4\zeta-\epsilon}\tilde{B}(xm^{-2\zeta})$, $\zeta$ for now arbitrary, which satisfies:

$$-m\partial_m b(x) = (\epsilon - 4\zeta)b(x) + 2\zeta x b'(x) + \tfrac{1}{2}b'(x)^2 - b'(x)b'(0) + T_m b'(x)(1 + b''(0)/\epsilon)^{-1} \qquad (9)$$

where $T_m = T\frac{4A_d}{\epsilon}m^\theta$, $\theta = d - 2 + 2\zeta$. The FRG equation that we have derived is valid, to dominant order in $1/N$, *in any dimension* $d < 4$ and at any temperature $T$. Restricted to $T = 0$ it correctly matches the one obtained by Balents and Fisher [4] at any $N$ but to lowest order in $\epsilon = 4 - d$. Furthermore, due to the self consistent equation (7), it is fully integrable (not noted in [4]). Indeed, (7) can be inverted into

$$x = m^{2\zeta}\Phi\Big[\frac{y}{4A_d m^{2\zeta-\epsilon}}\Big] + \frac{1}{\epsilon}(y - y_0) - \tilde{T}_m \qquad (10)$$

where $y = -b'(x)$, $y_0 = -b'(0) = -4A_d m^{2\zeta-\epsilon}B'(0)$, $\Phi$ is the inverse function of $-B'(x)$, i.e. $(-B')(\Phi(y)) = y$ and $\tilde{T}_m = 2TI_1 m^{2\zeta}$ ($= T_m/(2-d)$ for $d < 2$). That this is also the general solution of the FRG equation can be seen by noting that (9) can be transformed into a *linear* equation for the inverse function $x(y)$

$$m\partial_m x = (\epsilon - 2\zeta)yx' + 2\zeta x - (y - y_0) + \frac{T_m \epsilon x_0'}{1 - \epsilon x_0'} \qquad (11)$$

with $x_0' = x'(y_0)$. The general $T = 0$ solution of the homogeneous part reproduces the $\Phi$ term while a particular solution is $x = (y - y_0)/\epsilon$ using that $m\partial_m y_0 = (2\zeta - \epsilon)y_0$.

To analyze the solutions of the large $N$ FRG equation (9), two approaches are legitimate, corresponding to different points of view. The first, natural in mean field, is exact integration. One discovers that (7),(9) admit an analytic function as a solution, given by (10), only for $m > m_c$ where $m_c$ is the Larkin mass (more generally a Larkin scale [17]). Indeed its second derivative, $\tilde{B}''(0)^{-1} = B''(2TI_1)^{-1} - 4I_2$, diverges (always for $T = 0$, $d < 4$, in some cases for $T > 0$) when $m$ is lowered down to $m_c$ (which defines $m_c$). Since this expression is proportional to the replicon eigenvalue of the replica symmetric (RS) solution in the GVM, which exists for $m > m_c$, the generation of a cusp in the FRG exactly coincides at large $N$ with the instability of the RS solution.

The second approach, natural in the RG [4], is to view the r.h.s. of (9) as the large-$N$ limit of the true $\beta$-function and to search for a zero. The general solution for $\theta > 0$, obtained from (11), is parametrized by $\zeta$:

$$x = \frac{y}{\epsilon} - \frac{y_0}{2\zeta} + \frac{\epsilon - 2\zeta}{2\zeta\epsilon} y_0^{\frac{\epsilon}{\epsilon-2\zeta}} y^{-\frac{2\zeta}{\epsilon-2\zeta}} \tag{12}$$

for $\zeta > 0$ and $\epsilon x = y - y_0 - y_0 \ln(y/y_0)$ for $\zeta = 0$, with $y = -b'(x)$. Here $y_0 = -b'(0)$ is a fixed number. The value of the roughness exponent $\zeta$ [19] is selected by the decay of $R(u)$ in (2) at large $u$, argued to be identical for $B$ and $\tilde{B}$, i.e. if $B'(z) \sim z^{-\gamma}$ one finds $\zeta = \zeta(\gamma) \equiv (4-d)/2(1+\gamma)$ or $\zeta = 0$ for shorter range correlations, to this order in $1/N$. All the fixed points (12) have a cusp $x'(y_0) = 0$ (for $\zeta \neq \epsilon/2$) and are expected to be the physically correct solutions at small $m$.

To show how to reconcile these two results, we study specific models, the long range (LR) correlations [5] $B(z) = \frac{\tilde{g}}{4(\gamma-1)A_d}(a^2+z)^{1-\gamma}$ and the short range (SR) gaussian correlator $B(z) = \frac{\tilde{g}}{4A_d} e^{-z}$. Choosing $\zeta = \zeta(\gamma)$ (10) yields:

$$x = (y/\tilde{g})^{-1/\gamma} + \epsilon^{-1}(y - y_0) - m^{2\zeta}a^2 - \tilde{T}_m \tag{13}$$

(and $\ln(\tilde{g}m^{-\epsilon}/y)$ in the r.h.s. for SR gaussian with $\zeta = 0$, $a = 0$). As one sees from Fig. 2, the r.h.s. of (13) has a minimum and decreasing $m$ the curve $x(y)$ cuts the axis $x = 0$ closer to the minimum. It reaches it at $m = m_c$ where the solution acquires a cusp $b'(0) - b'(x) \approx \sqrt{-2(\epsilon - 2\zeta)b'(0)x}$ and $m_c^{2\zeta}a^2 + \tilde{T}_{m_c} = (\tilde{g}\gamma/\epsilon)^{1/(1+\gamma)} \equiv \tilde{T}_c$, $b'(0) = -\tilde{g}^{1/(1+\gamma)}(\epsilon/\gamma)^{\gamma/(1+\gamma)}$ ($= -\epsilon$ for SR with $m_c^{\epsilon} = \tilde{g}/\epsilon$). Although it is a priori unclear how to follow this solution for $m < m_c$, the following remarkable property indicates how one may proceed. If we compute the $\beta$-function, i.e. the r.h.s. of (9), using (13) at $m = m_c$ and $\zeta = \zeta(\gamma)$ we find that it *exactly vanishes* for all $x > 0$. Thus, for the potentials studied here, $b(x)$ evolves according to (13) until $m_c$ where it reaches its fixed point, and does not evolve for $m < m_c$. This provides unambiguously a solution beyond the Larkin scale, reconciles the two approaches and justifies the value obtained for $\zeta$. The quantity $\tilde{B}''(v^2)^{-1} = B''(\chi(v))^{-1} - 4I_2$ plays the role of a replicon eigenvalue and remains frozen and positive for $v > 0$.

Here, for $N = \infty$, temperature plays only a minor role in the case where disorder is relevant, i.e. for $\theta(\gamma) > 0$

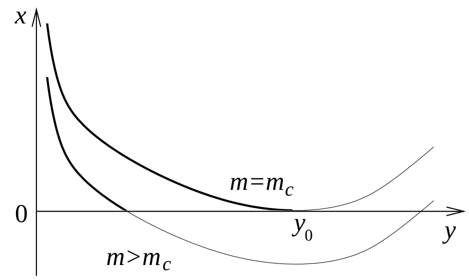


FIG. 2: The function $x(y)$ given by (13).

(i.e. $2 < d < 4$; $d < 2$ for $\gamma < \gamma_c = 2/(2-d)$), which is described by the $T = 0$ fixed point (12). Contrarily to the one loop FRG, where $b(x)$ remains analytic in a boundary layer $x \sim T_m$ at $T > 0$, here the denominator in (9) (resumming all orders in $\epsilon$) blows up and a cusp arises. Only in the marginal case ($\gamma = \gamma_c$; $d = 2$ for SR disorder) we find a line of fixed points of (9) with $\zeta = (2-d)/2$. For $T > T_c$ the disorder is analytic, given by (13) for all $m$ ($\tilde{T}_m/\tilde{T}_c \equiv T/T_c$ does not flow, and $T_c = 2\pi$ in $d = 2$ for SR disorder). Below $T_c$ one recovers a cusp and the $T = 0$ fixed point. Finally, for $\gamma > \gamma_c$ ($d < 2$) no cusp is generated as $m \to 0$ and disorder is irrelevant, as in the corresponding RS solution of MP.

Since the FRG aims at obtaining the universal behaviour at small $m, k$ of the correlation function of the manifold $\langle v_k^a v_{-k}^b \rangle = m^{d-2\zeta} g(k/m)$, we can now compute its zero momentum limit $g(0)$, which is universal in the LR case, and compare with the GVM result $g_{\rm RSB}(0)$. Extending [5] in presence of a mass, one shows that $m\partial_m \sigma_m(\mathsf{u}) = 0$ and thus the MP self energy is $\sigma_m(\mathsf{u}) = C\mathsf{u}^{2/\theta - 1}$ if $\mathsf{u} > \mathsf{u}_m$, $\sigma_m(\mathsf{u}) = \sigma_m(0)$ if $\mathsf{u} < \mathsf{u}_m$. Within the FRG, $g_{\rm FRG}(0) = -b'(0^+)/(2A_d)$ and substituting, we discover that $g_{\rm FRG}(0) = T\sigma_m(0)/m^4 < g_{\rm RSB}(0)$, i.e. the FRG gives only but exactly the contribution from the overlaps $\mathsf{u} < \mathsf{u}_m$ (the distant states). This non-trivial information which within the MP approach *requires* a *full* RSB calculation, is obtained here *without any* RSB. Moreover using $m\partial_m \sigma_m(\mathsf{u}) = 0$ one shows that contributions of larger overlaps in the MP result can be obtained by integrating the $m$ dependent FRG result as $g_{\rm RSB}(0) = T[\sigma_m(0)/m^4 + \int_m^{m_c} {\rm d}\sigma_{m'}(0)/m'^4 + m_c^{-2} - m^{-2}]$.

These results can be understood as follows. The effective action also describes the probability distribution $P_V(w)$ in a given environment $V$, of the center of mass of the interface $w = \frac{1}{N^{1/2}L^d}\int_x u_x$, i.e. one has $\tilde{\Gamma}[\{w_a\}] = -\lim_{L\to\infty} \frac{1}{NL^d} \ln \overline{P_V(w_1)\dots P_V(w_n)}$. Extension of the FRG beyond the Larkin scale requires giving a meaning to the $u = 0^+$ limit. We find here that what the FRG actually computes (from $b'(0^+)$) is a second moment of $w$ in presence of a small extra field $\sqrt{N}j_a$ such that all $v_{ab} \neq 0$, i.e. an average such that when there are several states the different replicas are chosen in maximally separated states ($\mathsf{u} = 0$).

In a previous study aiming to connect the RSB solution to the FRG [9] $\overline{\ln P_V(w) \ln P_V(-w)}$ was computed and

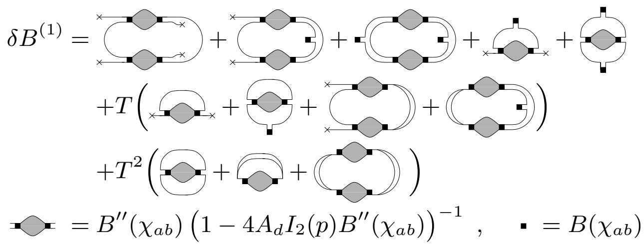


FIG. 3: Contribution to the second cumulant at order $1/N$.

used to define a renormalized second cumulant of the disorder. This quantity is however *different* from the one in the FRG, obtained here, and does not reproduce the second moment of $w$, neither $g(0)_{\rm RSB}$, nor $g(0)_{\rm FRG}$. In addition, since [9] used the unperturbed MP saddle point, the two calculations focus on different regimes ($v_{ab}^2 \sim 1$ here, $v_{ab}^2 \sim 1/N$ there, $v_a \sim 1$ in both) [18]. Work is in progress to connect these regimes, and obtain a more complete version of the FRG, using our equations (6) and summing over RSB saddle points [15].

The FRG approach should allow a quantitative study of finite $N$ beyond possible artifacts of $N = \infty$, and extension of [4] to any $d$. The calculation of the correction at order $1/N$ to $\tilde{B}$ is involved, and was performed using two complementary methods, a graphical one, see Fig. 3, and the algebraic expansion in number of replica sums. The resulting expression for the $\beta$-function at order $1/N$ and $T = 0$ is UV-convergent and reads for $\Lambda \to \infty$

$$\begin{aligned}\beta(b) = \epsilon b + \frac{1}{2} b'^2 - b' b'_0 + \frac{1}{N A_d} \int_p \Big[ &-4 I_0^p I_3^p (b'_0 - b')^2 b''^2 H_p^{-2} - 2\epsilon x I_0^{p2} (b'_0 - b') b'' (1 - I_2 b'') H_p^{-2} \\ &+ \epsilon I_4^p (b'_0 - b')^2 b'' (2 + (2I_2 - I_2^p) b'') H_p^{-2} - 8\epsilon x I_0^p I_3^p (b'_0 - b') b''^2 (1 + I_2 b'') H_p^{-3} + 2\epsilon I_3^{p2} (b'_0 - b')^2 b''^2 (3 + (3I_2 - I_2^p) b'') H_p^{-3} \\ &- 2 I_0^{p2} (b'_0 - b')^2 b'' H_p^{-1} + 2\epsilon x^2 I_0^{p2} b''^2 (1 + (I_2 + I_2^p) b'') H_p^{-3} \Big] , \qquad H_p = 1 + (I_2 - I_2^p) b'', \qquad I_0^p = (1 + p^2)^{-1} \end{aligned} \tag{14}$$

with $I_2 = \epsilon^{-1}$, $I_2^p = J_{11}^p$, $I_3^p = J_{12}^p$, $I_4^p = J_{22}^p$ and $J_{nm}^p = \frac{1}{A_d} \int_k (I_0^k)^n (I_0^{k+p})^m$, $b' \equiv b'(x)$, $b'_0 \equiv b'(0)$, etc... The corresponding expressions at $T > 0$ have been obtained. Analysis of these formidable expressions is in progress. In particular, we have not included in (14) anomalous terms arising from the non-analytic structure. We have checked that (14) non-trivially reproduces the two loop FRG equation for the $N$-component model of [10, 11].

In summary, via an exact calculation of the effective action at large $N$ we have derived equations valid in any $d$ containing both the GVM and the FRG. The FRG and its continuation to $m < m_c$ are consistent with the main results of the full and the marginal one step RSB-solutions of MP. Since it reproduces the non-trivial small overlap results it provides another way to attack finite $N$. $1/N$ corrections have been obtained. Our study hints at further connections between: $1/N$ and thermal boundary layers, RSB, and how to fix the ambiguities in the anomalous terms in the $\beta$-function. Their understanding should allow quantitative progress in the SR case (e.g. for $d = 1$, equivalent to the $N$-dimensional KPZ equation). Applications of the method to other complex systems is in progress. We have also computed the effective action for the random field $O(N)$-model at large $N$ [11]. Finally, applications to the dynamics offer the hope to go systematically beyond mode-coupling approximations.

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[17] One can also derive a FRG equation by varying the UV cutoff from 0 to $\Lambda$ and a Larkin scale $\Lambda_c = R_c^{-1}$.
[18] with a $T > 0$ boundary layer there. We thank L. Balents for help in clarifying these issues.
[19] and of $y_0$ in the LR case.