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# Interacting crumpled manifolds: Exact results to all orders of perturbation theory

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# Interacting crumpled manifolds: Exact results to all orders of perturbation theory

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**Abstract.** – In this letter, we report progress on the field theory of polymerized tethered membranes. For the toy model of a manifold repelled by a single point, we are able to sum the perturbation expansion in the strength $g_0$ of the interaction *exactly* in the limit of internal dimension $D \to 2$. This exact solution is the starting point for an expansion in $2-D$, which aims at connecting to the well-studied case of polymers ($D = 1$). We here give results to order $(2-D)^2$, where again all orders in $g_0$ are resummed. This is a first step towards a more complete solution of the self-avoiding manifold problem, which might also prove valuable for polymers.

*Introduction.* – The statistical mechanics of fluctuating lines and surfaces is a subject of great interest, which poses fundamental problems and has remained challenging for more than 20 years. One particular universality class, which has been studied extensively in the past, are polymerized or "tethered" membranes [1–9]. These are two-dimensional networks, where the bond-length fluctuates, but never breaks up. In the high-temperature regime, nearest-neighbor interactions can be modeled by a harmonic potential. Neglecting self-avoidance, the membrane is extremely crumpled and highly folded, a property which is characterized by the universal radius-of-gyration exponent $\nu$, defined as

$$R_{\mathrm{g}} \sim L^{\nu}, \quad \nu = 0, \tag{1}$$

where $R_{\mathrm{g}}$ denotes the radius of gyration, and $L$ is the linear internal size. Physically, $0 \le \nu \le 1$, but in the absence of interactions, the radius of gyration grows only logarithmically with the internal size.

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For a more realistic description one has to take into account self-avoidance, whose continuum version can be modeled by the generalized Edwards Hamiltonian [10] with 2-particle contact interaction,

$$\mathcal{H}[r] = \frac{1}{2}\int_{x\in\mathcal{M}} (\nabla r(x))^2 + \frac{b_0}{2}\int_{x\in\mathcal{M}}\int_{y\in\mathcal{M}} \delta^d(r(x)-r(y)), \tag{2}$$

where $x \in \mathcal{M} \subset \mathbb{R}^D$ labels points in the manifold $\mathcal{M}$, while $r(x) \in \mathbb{R}^d$ points to their position in external space. The Edwards model successfully describes long polymers [11, 12]. Much effort has been spent to extend these results to membranes ($D = 2$). The problem is that the usual $\varepsilon$-expansion about the upper critical dimension is not feasible, since the latter is infinity. An important idea was therefore to generalize (2) to manifolds of arbitrary internal dimension $D$. One then studies the $D$-dimensional manifold problem, and finally continues analytically to $D = 2$. A major breakthrough was the proof of perturbative renormalizability [5, 6] to all orders in perturbation theory. This procedure was carried out to two loops [13, 14] resulting in a radius-of-gyration exponent of $\nu \approx 0.86$. This is a strong correction over the non-interacting theory with $\nu = 0$, but may still be in contradiction to Monte Carlo simulations, which often but not consistently find tethered membranes in a flat phase with $\nu = 1$ [15–18]. While simulations are very demanding and therefore not yet conclusive, it is nevertheless compelling to try to identify possible mechanisms, which might render flexible membranes flat at all scales. Such a mechanism has indeed been found for stiff membranes, where fluctuations strongly renormalize rigidity [1, 19].

Here we study a simplified model, and solve it *exactly* at $D = 2$. It corresponds to a Gaussian elastic manifold interacting by excluded volume with a single $\delta$-like impurity in external space [20],

$$\mathcal{H}[r] = \frac{1}{2}\int_{x\in\mathcal{M}} (\nabla r(x))^2 + g_0\int_{x\in\mathcal{M}} \delta^d(r(x)). \tag{3}$$

As a first step to prove renormalizability of the full problem, the authors of [3, 4] analysed (3) and indeed showed renormalizability to all orders in perturbation theory for all dimensions $0 < D < 2$. Expression (3) has essential features in common with SAM: Its critical embedding dimension tends to infinity as the internal dimension approaches $D = 2$. This can be read off from the dimension of the coupling $g_0$, which is $[g_0] =: \varepsilon = D - \frac{2-D}{2}d$. Thus, calculating universal quantities within the $\varepsilon$-expansion necessitates similar techniques as for SAM, and we expect to learn more from the solution of the toy model (3).

Summing the perturbative expansion *exactly* [21] in $D = 2$ led us to an outstanding result: The effective coupling (4) of the problem logarithmically diverges as the size of the membrane becomes large, which is the limiting case of a scale-invariant theory. This behavior reveals that the limit $D = 2$ is quite exceptional and that non-perturbative methods are needed. It might also explain why SAM are (mostly) seen flat in simulations; however, this remains to be clarified in an appropriate analysis of the full SAM problem.

The key-idea in [21] was that in the analytic continuation to $D = 2$, the correlator which enters all perturbative calculations becomes essentially a constant. The purpose of this letter is to check the consistency of these results in two ways: First, we rederive the results in $D = 2$, using an explicit UV cutoff instead of dimensional regularization in $D$. Second, we go away from $D = 2$, and hopefully smoothly connect to polymers in $D = 1$, which are well enough studied to check almost any quantity. In [21], we have done a first step in that direction, and obtained quite promising results in first order in $2 - D$. As we work in a “massive scheme”([1])

([1])Loop diagrams are calculated at finite $\varepsilon$.

the effective coupling is *a priori* shape dependent. However, we do not expect any shape dependence for the correction-to-scaling exponent $\omega$ as is the case both for $D = 1$ and $D = 2$, and therefore restrict the analysis to closed manifolds of toroidal or spherical shape([2]). This way we obtain the expansion up to order $(2-D)^4$ by summing all contributions in $g_0$ to the partition-function *exactly* at each order in $2-D$. This information can then be used to extrapolate away from $D = 2$. However, since we find that at $D = 2$, the fixed point is at infinity, one needs additional constraints, *i.e.* a scaling function, in order to be able to use this result. These scaling functions are severely restricted by the physics of the problem, but we have not been able to find enough constraints, despite the tremendous information contained in the resummed series.

*Perturbation theory.* – Physical observables are derived from the partition function $\mathcal{Z}(g_0)$. We use it to define the effective coupling of the problem,

$$g(z) := \frac{L^\varepsilon}{\mathcal{V}_\mathcal{M}}(\mathcal{Z}(0) - \mathcal{Z}(g_0)), \tag{4}$$

which only depends on the dimensionless combination $z := g_0 L^\varepsilon$. $\mathcal{V}_\mathcal{M}$ denotes the total internal volume of the manifold. Accordingly, the perturbation expansion reads

$$g(z) = \frac{g_0 L^\varepsilon}{\mathcal{V}_\mathcal{M}} \sum_{N=0}^{\infty} \frac{(-g_0)^N}{(N+1)!} \left\langle \prod_{i=1}^{N+1} \int_{x_i} \tilde{\delta}^d(r(x_i)) \right\rangle_0 , \tag{5}$$

where the normalization of the $\delta$-distribution has been chosen to be $\tilde{\delta}^d(r(x)) = (4\pi)^d \delta(r(x)) = \int_k \mathrm{e}^{ikr(x)}$ with $\int_k := \pi^{-d/2} \int \mathrm{d}^d k$. Performing the averages within the Gaussian theory with normalization $\frac{1}{\mathcal{V}_\mathcal{M}} \int_x \langle \tilde{\delta}^d(r(x)) \rangle_0 = 1$, one arrives at

$$g(z) = \frac{g_0 L^\varepsilon}{\mathcal{V}_\mathcal{M}} \sum_{N=0}^{\infty} \frac{(-g_0)^N}{(N+1)!} \left( \prod_{i=1}^{N+1} \int_{k_i} \int_{x_i} \right) \tilde{\delta}^d \left( \sum_i k_i \right) \mathrm{e}^{\frac{1}{2} \sum_{i,j=1}^{N+1} k_i k_j C(x_i - x_j)}, \tag{6}$$

where $C(x) := \frac{1}{2d} \langle (r(x) - r(0))^2 \rangle_0$ denotes the correlator, and the $\tilde{\delta}^d(\sum_i k_i)$ stems from the integration over the global translation. Performing the shift $k_{N+1} \to k_{N+1} - \sum_{i=1}^N k_i$ and integrating out the momenta $k_1, \dots, k_{N+1}$ one obtains

$$g(z) = z \sum_{N=0}^{\infty} \frac{(-z)^N}{(N+1)!} \left( \prod_{\ell=1}^{N} \int_{x_\ell} \right) (\det \mathfrak{D})^{-d/2}, \tag{7}$$

where we have factored out $L^\varepsilon$ from the loop integration (such that the integrals now run over a torus of size 1), and the matrix elements $\mathfrak{D}_{ij}$ are $\mathfrak{D}_{ij} = \frac{1}{2}[C(x_{N+1} - x_i) + C(x_{N+1} - x_j) - C(x_i - x_j)]$.

*Complete resummation of the perturbation series in $D = 2$.* – Let us compute the $N$-loop order of (7): The asymptotic behavior of the propagator $C(x)$ for large arguments is of the form $C(x) \simeq c_0 + \frac{1}{2\pi} \ln \frac{x}{a}$, where $c_0$ denotes some positive constant (note $C(x) \geq 0$), and the logarithmic growth (for large $x$) is universal. In $D = 2$ we need an additional short-distance

([2])For SAM, there are shape-dependent exponents, *e.g.* contact exponents or the divergence of the partition function [6].

cutoff $a$. The loop integrals, denoted by $I_N$, only depend on the dimensionless combination $L/a$. We can (somehow arbitrarily) decompose $\det \mathfrak{D} = (\prod_{i=1}^N \mathfrak{D}_{ii}) \det \tilde{\mathfrak{D}}$ into

$$\begin{aligned} \tilde{\mathfrak{D}}_{ij} &= \frac{1}{2}\left[1 + \frac{C(x_{N+1} - x_j) - C(x_i - x_j)}{C(x_{N+1} - x_i)}\right] \xrightarrow{a \to 0} \frac{1}{2}, \quad i \neq j, \\ \tilde{\mathfrak{D}}_{ii} &= 1. \end{aligned} \tag{8}$$

One has, in the limit of $a \to 0$,

$$\left(\prod_{\ell=1}^{N} \int_{x_\ell}\right) (\det \mathfrak{D})^{-d/2} =: I_N(L/a) = I_1^N(L/a)\big(\det \tilde{\mathfrak{D}}^{(0)}\big)^{-d/2}. \tag{9}$$

The matrix $\tilde{\mathfrak{D}}^{(0)}$ denotes the limit $a \to 0$ of (8). It can be written as $\tilde{\mathfrak{D}}^{(0)} = \frac{1}{2}(\mathbb{I} + N\mathbb{P})$, where $\mathbb{I}$ denotes the identity and $\mathbb{P}$ the projector onto $(1, 1, \ldots, 1)$, whose image has dimension 1, such that $\det \tilde{\mathfrak{D}}^{(0)} = \frac{1+N}{2^N}$. Furthermore, to one loop $I_1(L/a) \overset{a \to 0}{=} c_1(\ln \frac{L}{a})^{-d/2}$, where $c_1$ denotes some (finite) constant. One then arrives at

$$g(z) = z \sum_{N=0}^{\infty} \frac{\big(-z\big(\ln \frac{L}{a}\big)^{-d/2}\big)^N}{N!(1+N)^{d/2+1}}. \tag{10}$$

A factor $c_1 2^{d/2}$ has been absorbed into a rescaling of both $z$ and $g$. The above series can be analysed in the strong-coupling limit $z \to \infty$. For this purpose we define functions $f_k^d(z)$ together with their integral representation:

$$\begin{aligned} f_k^d(z) &:= z^k \sum_{N=0}^{\infty} \frac{(-z)^N}{N!(k+N)^{d/2}} = \frac{z^k}{\Gamma(\frac{d}{2})} \int_0^\infty \mathrm{d}r r^{d/2-1} \mathrm{e}^{-z\mathrm{e}^{-r} - kr} \\ &= \frac{(\ln z)^{d/2-1}}{\Gamma(\frac{d}{2})} \int_0^z \mathrm{d}y y^{k-1} \mathrm{e}^{-y} \left(1 - \frac{\ln y}{\ln z}\right)^{d/2-1} \xrightarrow{z \to \infty} \frac{\Gamma(k)}{\Gamma(\frac{d}{2})} (\ln z)^{d/2-1}. \end{aligned} \tag{11}$$

Thus, in the limit of large $z$, the effective coupling (10) approaches the asymptotic form

$$g(z) = \frac{\big(\ln \frac{L}{a}\big)^{d/2}}{\Gamma\big(\frac{d+2}{2}\big)} \left[\ln\left(z\left(\ln \frac{L}{a}\right)^{-d/2}\right)\right]^{d/2}. \tag{12}$$

*Observables.* – It immediately follows from this behavior that the correction-to-scaling exponent $\omega$, which is defined as the slope of the RG-$\beta$-function at the fixed point, equals zero. Here, it is useful to study the $\beta$-function as a function of the bare coupling $z$, which reads $\beta(z) = -\varepsilon z \partial g(z)/\partial z$. Then, the correction-to-scaling exponent is obtained from the limit $z \to \infty$ of $\omega(z) := -(\varepsilon z/\beta(z)) \cdot \partial\beta(z)/\partial z$.

The value of $\omega$ can be checked in a Monte Carlo experiment by considering plaquette density functions on a membrane with self-avoidance in only a single $\delta$-like defect. Be the partition function $\mathcal{Z}^\diamond = \int \mathcal{D}[r] \tilde{\delta}^d(r(y)) \exp[-\mathcal{H}[r]]$, then the plaquette density at the defect is obtained from $\langle n \rangle_\diamond = \frac{L^\varepsilon}{\partial g/\partial z} \frac{\partial}{\partial z}(\frac{\partial g}{\partial z})$, where $\frac{\partial g}{\partial z} = \mathcal{Z}^\diamond$. One furthermore needs the density-density correlation at this point, which is defined as $\langle n^2 \rangle_\diamond = \frac{L^{2\varepsilon}}{\partial g/\partial z} \frac{\partial^2}{\partial z^2}(\frac{\partial g}{\partial z})$. In the limit of strong coupling $\langle n \rangle_\diamond = \frac{1}{g_0}(1 + \frac{\omega}{\varepsilon})$ and $\langle n^2 \rangle_\diamond = \frac{1}{g_0^2}(2 + \frac{3\omega}{\varepsilon} + \frac{\omega^2}{\varepsilon^2})$, such that the ratio $\langle n \rangle_\diamond / \sqrt{\langle n^2 \rangle_\diamond} \xrightarrow{z \to \infty} \sqrt{(\varepsilon + \omega)/(2\varepsilon + \omega)} \xrightarrow{\omega = 0} \sqrt{1/2}$ becomes universal and should be measurable in simulations.

$(2-D)$-*expansion.* – Let us now analyse the theory below $D = 2$. Due to the renormalizability in $0 < D < 2$ and the existence of an $\varepsilon$-expansion we expect the renormalized coupling to reach a finite fixed point in the strong-coupling limit as soon as $D < 2$. This approach is characterized by a power law decay of the form

$$g(z) = g^* + S(\ln z) z^{-\omega/\varepsilon} + \mathrm{O}(z^{-\omega_1/\varepsilon}), \tag{13}$$

where $S$ is some scaling-function growing at most sub-exponentially and $\omega_1 > \omega > 0$, with $\omega$ defined in the previous section. In order to gain information about $g$ below $D = 2$ one has to expand the loop integrand $(\det \mathfrak{D})^{-d/2}$ in powers of $2 - D$. For convenience, we take $a \to 0$. The propagator takes in infinite $D$-space the form $C(x) = |x|^{2-D}/(S_D(2-D))$, where $S_D = 2\pi^{D/2}/\Gamma(\frac{D}{2})$ denotes the volume of the $D$-dimensional unit-sphere. The factor $(S_D(2-D))^{-1}$ replaces $\ln(\frac{L}{a})$ and is absorbed into a rescaling of the field and the coupling according to $r \to r(S_D(2-D))$ and $g_0 \to g_0(S_D(2-D))^{d/2}$, such that the factors of $(\ln \frac{L}{a})^{-d/2}$ in (10) and (12) are replaced by $(S_D(2-D))^{d/2}$. The propagator in the rescaled variable can then be written as

$$C(x) = 1 + (2-D)\mathbb{C}(x), \tag{14}$$

where, for convenience of notation, we allow $\mathbb{C}(x)$ to depend itself on $D$.

Of course, on a closed manifold of finite size, $C(x)$ needs to be modified, but the form (14) is independent of the shape of the manifold. Accordingly, one may expand the matrix $\mathfrak{D}$, which is $\mathfrak{D} = \tilde{\mathfrak{D}}^{(0)} + (2-D)\mathbb{D}$, where $\tilde{\mathfrak{D}}^{(0)}$ is defined as before and coincides with the limit $D \to 2$ when inserting the above $C(x)$ into $\mathfrak{D}$. Moreover, $\mathbb{D}$ is of the same form as $\mathfrak{D}$, but each $C(x)$ has been replaced with $\mathbb{C}(x)$: $\mathbb{D}_{ij} = \frac{1}{2}[\mathbb{C}(x_{N+1}-x_i) + \mathbb{C}(x_{N+1}-x_j) + \mathbb{C}(x_i - x_j)]$. Then,

$$\det \mathfrak{D} = \det \tilde{\mathfrak{D}}^{(0)} \exp\Big[\mathrm{Tr}\Big[\ln\Big(1 + (2-D)\big[\tilde{\mathfrak{D}}^{(0)}\big]^{-1}\mathbb{D}\Big)\Big]\Big], \tag{15}$$

where $[\tilde{\mathfrak{D}}^{(0)}]^{-1} = 2(\mathbb{I} - \frac{N}{N+1}\mathbb{P})$ denotes the inverse matrix of $\tilde{\mathfrak{D}}^{(0)}$. Expanding the integrand (15) in powers of $(2-D)$ and the coupling $g_0$, all orders in $g_0$ can again be summed, with the difference that the integrands are no longer constant. Expanding up to the $n$-th order in $2-D$ involves $n$ powers of $\mathbb{C}(x)$. Introducing the notation $\overline{f(x_1, \ldots, x_k)} := \int_{x_1} \cdots \int_{x_k} f(x_1, \ldots, x_k)$ with the integration defined as $\int_x := \int \mathrm{d}^D x$ (on the torus), the overbar can be thought of as an averaging procedure. To first and second order in $2-D$, the only integrals to be evaluated are $\overline{\mathbb{C}(x)}$ and $\overline{\mathbb{C}^2(x)}$. The renormalized coupling then reads (note that we have absorbed a factor of $2^{d/2}$ in both $g$ and $z$):

$$\begin{aligned} g(z) \; = \; & f_1^{d+2}(z) - (2-D)\frac{d}{2}\big[\overline{\mathbb{C}} f_1^{d+2}(z) - \overline{\mathbb{C}} f_1^{d}(z)\big] + \\ & +(2-D)^2 \frac{d}{4}\Big[2\overline{\mathbb{C}^2_c} f_1^{d+4}(z) + \Big(\overline{\mathbb{C}}^2 - 4\overline{\mathbb{C}^2_c}\Big) f_1^{d+2}(z) + \Big(3\overline{\mathbb{C}^2_c} - \overline{\mathbb{C}}^2\Big) f_1^{d}(z) - \overline{\mathbb{C}^2_c} f_1^{d-2}(z)\Big] + \\ & +(2-D)^2 \frac{d^2}{8}\Big[2\overline{\mathbb{C}^2_c} f_1^{d+4}(z) - \Big(2\overline{\mathbb{C}^2_c} + \overline{\mathbb{C}}^2\Big) f_1^{d+2}(z) + 2\overline{\mathbb{C}}^2 f_1^{d}(z) - \overline{\mathbb{C}}^2 f_1^{d-2}(z)\Big] + \\ & +\mathrm{O}(2-D)^3. \end{aligned} \tag{16}$$

In order to reveal the structure of the expansion, we have calculated all terms up to fourth order, which will be reported elsewhere [22]. From the integral representation (11) of $f_1^{d+j}(z)$ and the above expansion, it follows immediately that the exact renormalized coupling can be written as

$$g(z) = z \int_0^\infty \mathrm{d}r \, \tilde{g}(r) \mathrm{e}^{-z\mathrm{e}^{-r} - r}, \tag{17}$$

where $\tilde{g}(r)$ is of the form

$$\tilde{g}(r) = r^{d/2}\left[\frac{1}{\Gamma(\frac{d+2}{2})} + (2-D)\sum_{n=0}^{\infty}\sum_{j=-n_{\max}}^{n} p_{n_j} r^j (2-D)^n\right]. \tag{18}$$

Let us try to gain more information about the power law behavior in (13), that is about the expansion in $2-D$ of the correction-to-scaling exponent $\omega$. Power law behavior forces the series (18) to turn into some exponentially decaying function $\tilde{g}(r)$ as can be seen from the asymptotic form of $g(z)$:

$$g(z) \simeq \mathcal{A} + \mathcal{B}z^{-\omega/\varepsilon} = z\int_0^{\infty} \mathrm{d}r \mathrm{e}^{-z\mathrm{e}^{-r}-r}\left(\mathcal{A} + \frac{\mathcal{B}\mathrm{e}^{-r\omega/\varepsilon}}{\Gamma(1+\frac{\omega}{\varepsilon})}\right) + \mathrm{O}(\mathrm{e}^{-z}). \tag{19}$$

Now, we test a possible form of the exact $\tilde{g}(r)$, which satisfies the following properties: i) In the limit of $D = 2$ the exact form $r^{d/2}/\Gamma(\frac{d+2}{2})$ emerges, ii) for $D < 2$ the corresponding $g(z)$ has a finite fixed-point value together with a strong-coupling expansion and iii) it is consistent with the expansion (16). The (non-unique) ansatz is

$$\tilde{g}(r) = \mathcal{C}\left(\frac{1-\mathcal{S}(D,r)\mathrm{e}^{-\frac{\omega}{\varepsilon}r}}{\omega/\varepsilon}\right)^{d/2}, \tag{20}$$

where $\mathcal{S}(D,r)$ is analytic in $D = 2$ of the form $\mathcal{S}(D,r) = 1+\frac{\omega}{\varepsilon}r\sum_{n=1}^{\infty}\mathcal{S}_n(r)(2-D)^n$, and each $\mathcal{S}_n(r)$ has a Laurent expansion $\mathcal{S}_n(r) = \sum_{j=-n_{\min}}^{n_{\max}} s_{n,j}r^j$. Note that, in the limit of $D \to 2$, the expression (20) gives $r^{d/2}$, while for $D < 2$ it yields upon integration the form (19), ensuring both properties i) and ii). Inserting $\omega/\varepsilon = \omega_2(2-D)^2 + \mathrm{O}(2-D)^3$ (the linear term in $(2-D)$ has to vanish) into the ansatz (20) and expanding to second order in $2-D$ provides

$$\tilde{g}(r) = \mathcal{C}r^{d/2}\left[1 - \frac{d}{2}\left(\mathcal{S}_1(r)(2-D) + \left(\frac{\omega_2}{2}r - \frac{d-2}{4}\mathcal{S}_1(r)^2 + \mathcal{S}_2(r)\right)(2-D)^2 + \cdots\right)\right]. \tag{21}$$

The first coefficients of the $(2-D)$-expansion of $\tilde{g}(r)$ obtained from (16) read

$$\begin{aligned}\tilde{g}(r) = \frac{r^{d/2}}{\Gamma(\frac{d+2}{2})}\Bigg\{&1+(2-D)\frac{d}{2}\overline{\mathbb{C}}\left(1-\frac{d}{2r}\right)-(2-D)^2\Bigg[\frac{d}{2}\overline{\mathbb{C}_c^2}r+\frac{d}{4}\left(\overline{\mathbb{C}}^2-4\overline{\mathbb{C}_c^2}\right)-\frac{d^2}{8}\left(2\overline{\mathbb{C}_c^2}+\overline{\mathbb{C}}^2\right)+\\ &+\left(\frac{d^2}{8}\left(-\overline{\mathbb{C}}^2+3\overline{\mathbb{C}_c^2}\right)+\frac{d^3}{8}\overline{\mathbb{C}}^2\right)r^{-1}-\frac{d^2}{8}\left(\frac{d}{2}-1\right)\left(\overline{\mathbb{C}_c^2}+\frac{d}{2}\overline{\mathbb{C}}^2\right)r^{-2}\Bigg]\Bigg\}.\end{aligned} \tag{22}$$

Comparing (21) and (22), one identifies $\mathcal{C} = 1/\Gamma(\frac{d+2}{2})$, $\mathcal{S}_1 = -\overline{\mathbb{C}}(1-\frac{d}{2}\frac{1}{r})$ and $\omega_2 = 2\overline{\mathbb{C}_c^2}$, where $\mathbb{C}_c(x) := \mathbb{C}(x)-\overline{\mathbb{C}}$. Note that the terms proportional to $\overline{\mathbb{C}}^2$ in $\mathcal{S}_2(r)$ mostly cancel with $\mathcal{S}_1(r)^2$, a sign that the ansatz catches some structure.

The diagrams to be calculated at this order are $\overline{\mathbb{C}}$ and $\overline{\mathbb{C}_c^2}$. On a manifold of toroidal shape, which is equivalent to periodic boundary conditions, two discrete sums have to be evaluated:

$$\overline{\mathbb{C}} = \frac{S_D}{4\pi^2}\left[\sum_{k\in\mathbb{Z}^D,k\neq 0}\frac{1}{k^2} - \frac{2\pi}{(2-D)}\right] = -0.44956 + 0.3583(2-D) + \mathrm{O}(2-D)^2, \tag{23}$$

$$\overline{\mathbb{C}_c^2} = \frac{S_D^2}{16\pi^4}\sum_{k\in\mathbb{Z}^D,k\neq 0}\frac{1}{k^4} = 0.152661 + \mathrm{O}(2-D). \tag{24}$$

With the results given above, this leads to

$$\omega = 2\varepsilon\overline{\mathbb{C}_c^2}(2-D)^2 + \mathrm{O}(2-D)^3 = 0.305322\varepsilon(2-D)^2 + \mathrm{O}(2-D)^3, \tag{25}$$

which can be compared to the exact result for $D=1$ (polymers): $\omega = \varepsilon$. As a caveat, note that the above scheme is not unambiguous in the sense that the second-order term proportional to $r$ in (22) could in principle either be attributed to $\omega_2$ or $\mathcal{S}_2$. However, any ansatz in (20) will provide an $\omega$, whose expansion starts at least quadratically in $2-D$. Though (20) is the best ansatz that could yet be found ensuring properties i)-iii), the precise form of constraints on the scaling function $\mathcal{S}$ remains to be discussed in order to settle this question.

In summary: we have presented a complementary approach to treat the problem of tethered membranes in interaction. We hope that this approach will prove fruitful for self-avoiding tethered membranes with eventual applications for polymers, as well as other 2D systems.

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