Eur. Phys. J. B **7**, 187–190 (1999)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Rapid Note

A geometric generalization of field theory to manifolds of arbitrary dimension

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Received: 15 October 1998 / Accepted: 4 November 1998

Abstract. We introduce a generalization of the $O(N)$ field theory to N-colored membranes of arbitrary inner dimension D. The $O(N)$ model is obtained for $D \to 1$, while $N \to 0$ leads to self-avoiding tethered membranes (as the $O(N)$ model reduces to self-avoiding polymers). The model is studied perturbatively by a 1-loop renormalization group analysis, and exactly as $N \to \infty$. Freedom to choose the expansion point D, leads to precise estimates of critical exponents of the $O(N)$ model. Insights gained from this generalization include a conjecture on the nature of droplets dominating the $3d$-Ising model at criticality; and the fixed point governing the random bond Ising model.

PACS. 05.70.Jk Critical point phenomena – 11.10.Gh Renormalization – 64.60.Ak Renormalization-group, fractal, and percolation studies of phase transitions – 75.10.Hk Classical spin models

Field theories have strong connections to geometrical problems involving fluctuating lines. For example, summing over all world-lines representing the motion of particles in space-time, is the Feynman path integral approach to calculating transition probabilities, which can also be obtained from a quantum field theory. Another example is the high-temperature expansion of the Ising model, where the energy-energy correlation function is a sum over all self-avoiding closed loops which pass through two given points. Generalizing from the Ising model to N component spins, the partition function of a corresponding $O(N)$ "loop model" is obtained by summing over all configurations of a gas of closed loops, where each loop comes in N-colors, or has a fugacity of N. In the limit $N \to 0$, only a single loop contributes, giving the partition function of a closed self-avoiding polymer [1].

There are several approaches to generalizing fluctuating lines to entities of other internal dimensions D; it is important to note that such extensions *are not unique*. The most prominent generalizations are string theories and lattice gauge theories, both describing $D = 2$ world sheets [2]. The low temperature expansion of the Ising model in d-dimension also results in a sum over surfaces that are $(d-1)$-dimensional. Each of these extensions has its own strengths, and offers new insights on field theory. Here we introduce a generalization based on a class of D-dimensional manifolds called "tethered" (or polymerized) membranes, which have fixed internal connectivity,

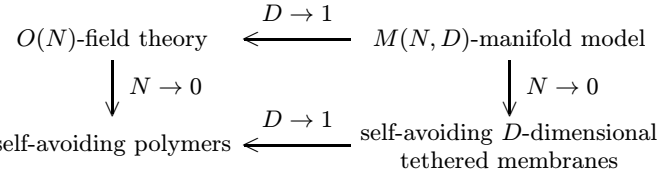

Fig. 1. Schematic description of the model, and its limits.

and are the simplest generalization of linear polymers [3]. The resulting *manifold theory* depends on two parameters N and D, with limiting behaviors related to well-known models as depicted in Figure 1. The model is defined by its perturbation series, and as in string theory not obviously derivable from a local Hamiltonian.

We start with a brief review of the *high temperature expansion* for an N-component (spin) field $\mathbf{s}$. Corrections to the infinite temperature partition function $\mathcal{Z}$, set to unity for convenience, can be represented diagrammatically by closed graphs. The terms contributing to the free energy are then grouped as

$$\frac{\ln \mathcal{Z}}{V} = \quad + \quad + \quad + \cdots \qquad (1)$$

The first term is the sum of all closed random walks. There is a trivial factor of volume V, since the same walk can start at any point. Having divided by this factor, the black indicates that the walk starts from a specific point.

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A walk of length ℓ is weighted by $e^{-t\ell}$, where e^{-t} is the high temperature expansion parameter. Since the probability that a walk of ℓ steps returns to its origin in d-dimensions is proportional to $\ell^{-d/2}$,

$$\bigcirc = N \int_0^\infty \frac{\mathrm{d}\ell}{\ell} \ell \ell^{-d/2} e^{-t\ell} \sim N \int \mathrm{d}^d k \frac{1}{t + k^2}. \quad (2)$$

The final equality indicates the connection between random walks and propagators of free field theory. Each loop carries a factor of N, for the number of field components.

The graphs in the high temperature series are distinct from random walks, in that their intersections are either forbidden or penalized. Subsequent terms in equation (1) correct for this by subtracting random walks with intersection points indicated by dashed lines. The second term in this expression indicates two intersecting loops (factor of N^2), and the third one a single loop with a self-intersection (factor of N). In the standard field theory, with "interaction" $b\,(\mathbf{s}\cdot\mathbf{s})^2$, each dashed line carries a factor of $-b$. The high temperature expansion thus provides a geometrical representation of field theory as a grand canonical gas of non-intersecting N-colored loops.

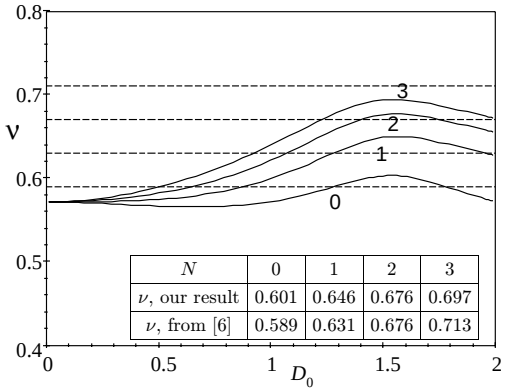
It is natural to attempt a generalization of the above description in which the basic entities are higher-dimensional manifolds of internal dimension D. *Such extensions are not unique*, and we shall outline the steps that lead to our choice: (a) regard the manifold as a collection of points, whose locations are the basic degrees of freedom. In a fluid manifold the relative positions of these points can change. Summing over all particle locations consistent with a given shape is a difficult problem. Encouraged by its success in polymer theory, we study instead *tethered* manifolds, in which particles are permanently linked as in a net. The surface is now described by a d-dimensional vector $\mathbf{r}(\mathbf{x})$, with a flat metric. (b) The fixed local connectivity is still consistent with many possible global shapes. As the simplest extension of loops we select hyper-spherical manifolds. (We have confirmed that surfaces of other genus are at least perturbatively irrelevant [4].) (c) A manifold of volume Ω is now weighted by a factor $e^{-t\Omega}$, where t is a chemical potential per particle. The probability that a tethered manifold is attached to a given point in space is [5]

$$\left\langle \tilde{\delta}^d(r(x_0)) \right\rangle_0 \sim \Omega^{-\frac{2-D}{D}d}. \quad (3)$$

As any one of ω points can be attached to the origin, equation (2) is generalized to

$$\bigcirc = N \frac{c(D)}{D} \int_0^\infty \frac{\mathrm{d}\Omega}{\Omega}\, \Omega\, \Omega^{-\frac{2-D}{D}d} e^{-t\Omega}. \quad (4)$$

Note that the final step is an integration over all manifold sizes. Should we weigh each size according to its linear dimension X, or volume $\Omega \propto X^D$, with $\mathrm{d}\Omega/\Omega = D\mathrm{d}X/X$? The arbitrariness of this choice is the origin of the parameter $c(D)$ introduced in equation (4). Any choice of $c(D)$ which reproduces the polymer partition function with $c(1) = 1$, is acceptable. In the remainder, we will mainly focus on $c(D) = D$, equivalent to the integral $\mathrm{d}\Omega$ over all scales. The same result is obtained by demanding that equation (4) only depend on one parameter, namely $2 - d(2-D)/2D$.



N	0	1	2	3
ν, our result	0.601	0.646	0.676	0.697
ν, from [6]	0.589	0.631	0.676	0.713

Fig. 2. Extrapolations for the exponent ν of the $O(N)$ model in $d = 3$, using the expansion of νd with $c(D) = D$. The dashed lines represent the best known values from reference [6].

Higher order terms in equation (1) have similar interpretations as intersecting manifolds. By construction, the perturbative series reduces to that of the $O(N)$ model for $D \to 1$, while the leading diagrams in the limit of $N \to 0$ describe a single tethered manifold (see Fig. 1). What can be learned from this generalized theory? We can actually take advantage of the non-uniqueness of the generalization to obtain good estimates of the critical exponents of the standard field theory! Simple power counting indicates that intersections are relevant only for dimensions $d < d_c = 4D/(2-D)$. This is reflected in divergences in the perturbative series which are removed by renormalization (for details see Ref. [4]). For example, the divergence of the correlation length at a critical point is described by the exponent $\nu(D, N, d)$, which has an expansion in $\varepsilon = 2D - d(2-D)/2 \sim (d_c(D) - d)$, as

$$\nu(D,N,d) = \frac{2-D}{2} \times \left[1 + \frac{\varepsilon}{2D} \frac{1 + c(D)\frac{N}{2}}{\frac{1}{2-D}\Gamma\left(\frac{D}{2-D}\right)^2 \Gamma\left(\frac{2D}{2-D}\right)^{-1} + 1 + c(D)\frac{N}{4}}\right]. \quad (5)$$

This expression reduces to the well-known ε-expansion [6] around $d = 4$ for lines ($D = 1$), while the $N \to 0$ limit reproduces the result for self-avoiding manifolds [7], recently generalized to 2-loop order [8].

To extract the physically relevant $O(N)$ exponent for $D = 1$, one has the freedom to expand equation (5) about *any* point (D_0, d_0) on the critical curve $\varepsilon(D_0, d_0) = 0$ [9]. As depicted in Figure 2, the resulting extrapolation for ν varies with the extrapolation point. (For technical reasons, this figure is obtained by reorganizing the expansion as $\nu d = 2D + a(D)\varepsilon$, before extracting ν.) Guided by previous results for polymers and membranes [8], the criterion for selecting a particular value from such curves is that of minimal sensitivity to the expansion point, and we thus evaluate ν at the extrema. The broadness of the extremum

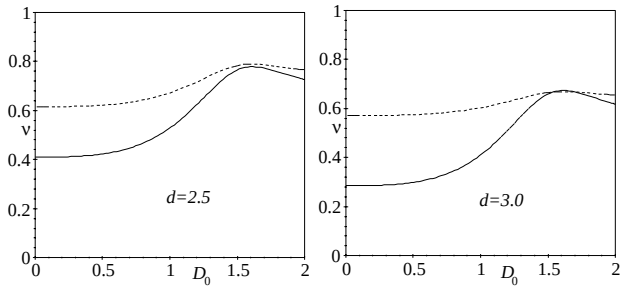


Fig. 3. Test of equation (6) for Ising models in $d=2.5$ and 3. The upper curves are from the high temperature representation ($D=1$), while the lower curves are from the low temperature description ($D=d-1$). The exponent ν is estimated from the maximum of each curve, which are obtained by extrapolating νd with $c(D)=D$, linearizing in N, and dividing the result by dD.

then provides a measure for the goodness of the result, and the expansion scheme. Although we examined several such curves, only a selection is reproduced in Figure 2. Our results are clearly better than the standard 1-loop expansion of $\nu = 1/2 + (N+2)/[4(N+8)]$.

In analogy to tethered membranes [10], we expect the above expansion scheme to be better controlled than the traditional ε-expansion. (The ε-expansion should become quasi-convergent for $D \to 2$.) However, since the exponents of the $O(N)$ model are already known to high accuracy, the generalization to $M(N,D)$ is valuable if it offers insights beyond the standard field theory. Furthermore, the scheme will have limited appeal, if cannot be extended to other types of field theories. In the rest of the article we shall demonstrate that: (a) The model provides insights about the boundaries of droplets at criticality in Ising models. (b) A generalized manifold model is constructed with cubic anisotropy, which exhibits a reverse Coleman-Weinberg mechanism not present in standard field theory. Furthermore, it provides us with a 1-loop fixed point for the random bond Ising model.

For the Ising model ($N=1$), a different geometrical description is obtained from a low temperature expansion: excitations to the uniform ground state are droplets of spins of opposite sign. The energy cost of each droplet is proportional to its boundary, *i.e.* again weighted by a Boltzmann factor of $e^{-t\Omega}$. Thus, a low temperature series for the d-dimensional Ising partition function is obtained by summing over closed surfaces of dimension $D=d-1$. For $d=2$, the high and low temperature series are similar, due to self-duality. For $d=3$, the low temperature description is a sum over surfaces. What types of surfaces dominate the above sum? Since there is no constraint on the internal metric, it may be appropriate to examine *fluid membranes*. However, there is no practical scheme for treating interacting fluid membranes, and the excluded volume interactions are certainly essential in this case. Configurations of a single surface for $N=0$, self-avoiding or not, are dominated by tubular shapes (spikes) which have very large entropy [11]. Such "branched polymer" configurations are very different from tethered surfaces. However, for $N \neq 0$, it may be entropically advantageous to break up a singular spike into a string of many bubbles. If so, describing the collection of bubbles by fluctuating hyper-spherical (tethered) manifolds may not be too off the mark [12]. To test this conjecture, we compare the predictions of the dual high and low-temperature descriptions.

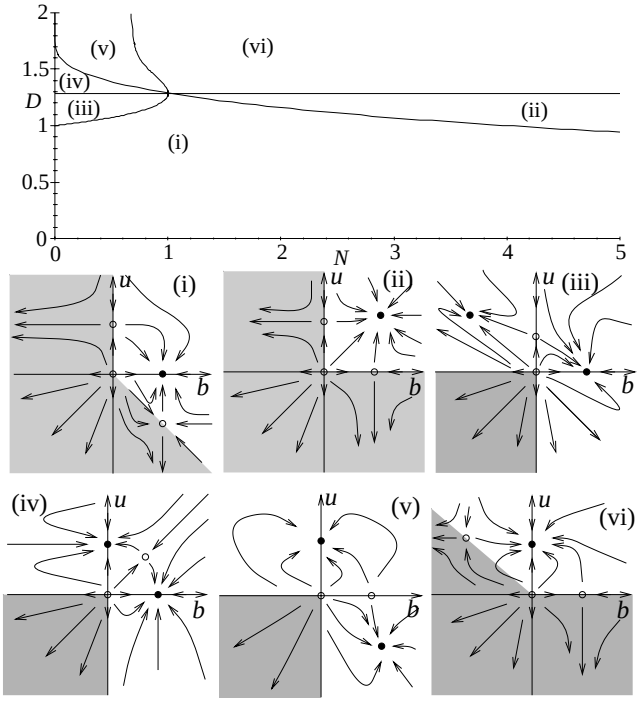


Fig. 4. Regions with different RG-flow patterns in the (N,D)-plane (top), and the corresponding RG-flows (bottom); shaded regions are unstable.

Singularities of the partition function are characterized by the critical exponent $\alpha(D,d,N)$, or (using hyperscaling) through $\ln \mathcal{Z}_{singular} \sim |t-t_c|^{\nu d/D}$. The equality of the singularities on approaching the critical point from low or high temperature sides, leads to a putative identity

$$\nu(1,d,1) = \frac{\nu(d-1,d,1)}{d-1}\,. \tag{6}$$

Numerical tests of the conjecture in equation (6) are presented in Figure 3. The extrapolated exponents (the maxima of the curves) from the dual expansions are in excellent agreement. Nevertheless, higher-loop calculations would be useful to check this surprising hypothesis.

The simplest extension of the $O(N)$ model breaks the rotational symmetry by inclusion of cubic anisotropy [13]. In the field theory language, cubic anisotropy is represented by a term $u\sum_i s_i^4$, in addition to the usual interaction of $b\sum_{ij} s_i^2 s_j^2$. In the geometric prescription of high temperature expansions, the anisotropic coupling u acts only between membranes of the same color, while the interaction b acts irrespective of color. Stability of the system of colored membranes places constraints on possible values of b and u. To avoid collapse of the system, energetic considerations imply that if $u<0$, the condition $u+b>0$ must hold, while if $u>0$, we must have

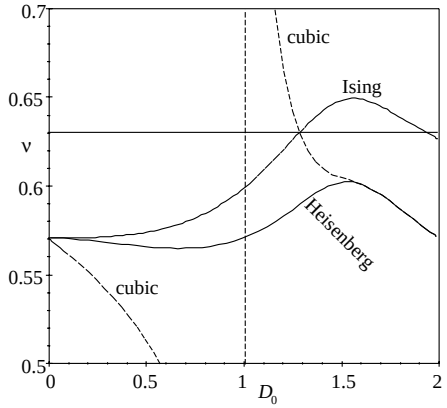


Fig. 5. Extrapolations of ν from the expansion of νd with $c(D) = D$, for the $O(N)$ model in $d = 3$. Exponents at the Heisenberg fixed point for $N = 0$ are compared to those of the Ising and cubic fixed points. The crossing of the latter curves yields an estimate of $\nu = 0.6315$ for the $3d$ Ising model.

$u + Nb > 0$ [4,13]. These stability arguments may be modified upon the inclusion of fluctuations: in the well-known Coleman-Weinberg mechanism [13], the RG flows take an apparently stable combination of b and u into an unstable regime, indicating that fluctuations destabilize the system. In the flow diagrams described below, we also find the reverse behavior (see (iii) to (v)) in which an apparently unstable combination of b and u flows to a stable fixed point. We interpret this as indicating that fluctuations stabilize the model, a reverse Coleman-Weinberg effect, which to our knowledge is new. We have shaded in grey, the unphysical regions in the flow diagrams of Figure 4. As in their $O(N)$ counterpart, the RG equations admit 4 fixed points: the *Gaussian* fixed point with $b^*_G = u^*_G = 0$; the *Heisenberg* fixed point located at $b^*_H \neq 0$, $u^*_H = 0$; the *Ising* fixed point with $b^*_I = 0$, $u^*_I \neq 0$; and the *cubic* fixed point at $b^*_c \neq 0$, $u^*_c \neq 0$. Furthermore, as depicted in Figure 4, there are six different possible flow patterns. In the $O(N)$ model, the flows in (i) and (ii) occur for $N < 4$ and $N > 4$, respectively. The other patterns do not appear in the standard field theory, as is apparent from their domain of applicability in the (N, D) plane in Figure 4. Note that there are *two stable fixed points* in three out of these four cases.

The $N \to 0$ limit of the above models is interesting, not only because of its relevance to self-avoiding polymers and membranes, but also for its relation to the Ising model with bond disorder. The latter connection can be shown by starting with the field theory description of the random bond Ising model, replicating it N times, and averaging over disorder [14]. The replicated system is controlled by a Hamiltonian with positive cubic anisotropy u, but negative $b = -\sigma$ (σ is related to the variance of bond disorder). From the "Harris criterion" [15], new critical behavior is expected for the random bond Ising system. But in the usual field theory treatments [14], there is no fixed point at the 1-loop order. In our generalized model, this is just the borderline between cases (i) and (iii). However, we now have the option of searching for a stable fixed point by expanding about any $D \neq 1$. Indeed, for $N = 0$ and $1 < D < 1.29$, the cubic fixed point lies in the upper left sector ($u > 0$ and $b < 0$) and is completely stable, as in flow pattern (iii).

The extrapolation for ν at the cubic fixed point is plotted in Figure 5, where it is compared to the results for the Heisenberg and Ising fixed points. The divergence of ν on approaching $D = 1$ from above, is due to the cubic fixed point going to infinity as mentioned earlier. Upon increasing D, the Ising and cubic fixed points approach, and merge for $D = 1.29$. For larger values of D, the cubic fixed point is to the right of the Ising one ($b^*_c > 0$), and only the latter is stable. Given this structure, there is no plateau for a numerical estimate of the random bond exponent ν_{DO}, and we can only posit the inequality $\nu_{DO} > \nu_{Ising}$. While this is derived at 1-loop order, it should also hold at higher orders since it merely depends on the general structure of the RG flows. One may compare this to four loop calculations of the random bond Ising model [16], which are consistent with $\alpha = 0$, *i.e.* at the border-line of the Harris criterion [15], with $\nu = 2/3$.

It is a pleasure to thank F. David, H.W. Diehl, and L. Schäfer for useful discussions. Part of this work was done during a visit of K.J.W. to MIT, with the financial support of Deutsche Forschungsgemeinschaft, through the Leibniz-Programm. The work at MIT is supported by the NSF Grant No. DMR-93-03667.

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