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The 4-loop β-function in the 2D non-Abelian Thirring model, and comparison with its conjectured "exact" form

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Abstract

Recently, B. Gerganov et al. [Phys. Rev. Lett. 86 (2001) 4753] have proposed an "exact" (all-orders) β-function for 2-dimensional conformal field theories with Kac–Moody current-algebra symmetry at any level k, based on a Lie group $\mathcal{G}$, which are perturbed by a current–current interaction. This theory is also known as the non-Abelian Thirring model. We check this conjecture with an explicit calculation of the β-function to 4-loop order, for the classical groups $\mathcal{G} = SU(N)$, $SO(N)$ and $SP(N)$ at level $k = 0$. We find a contribution at 4-loop order, proportional to a higher-order group-theoretical invariant, which is incompatible with the proposed β-function in *all* possible regularization schemes.

1. Introduction

Perturbations of conformal field theories (CFT) in two dimensions (2D) have been a very active topic of study for a long time. The focus of this paper are 2D conformal field theories possessing Kac–Moody current algebra (or: "affine Lie algebra") symmetry [1] associated with a Lie group $\mathcal{G}$, which are perturbed by a bilinear in the Noether-current (i.e., by a "left–right current–current bilinear interaction"). Non-Abelian Thirring models [2,3] and Gross–Neveu models [4] are much studied examples, see, e.g., [5–9]. Typically, such perturbations are (marginally) relevant and generate a mass scale; in these cases the long-distance (infrared) behavior of these theories is that of a massive field theory. However,

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generalizations of these theories may exhibit [10] massless long-distance fixed points in a certain zero species ("replica") limit, or, which can be seen to be equivalent [11], when the symmetry group $\mathcal{G}$ is replaced by a certain supergroup. Such theories are of great interest in condensed matter physics, since the mentioned "zero-species", or equivalently, supersymmetry generalizations describe (de-)localization transitions known to occur in dirty, i.e., disordered, non-interacting electronic systems in two spatial dimensions, subject to static random impurities. Indeed, the aim of [10] was to study the integer quantum Hall plateau transition,[1] and to provide an alternative to the formulation given by Levine, Libby and Pruisken [12–15] in terms of a strongly coupled non-linear sigma model with a topological term (see also [16]). More recently it was recognized [17–19] that disordered superconductors (and other systems) provide an entire new arena capable of exhibiting (de-)localization transitions of a similar kind (albeit in entirely new universality classes). Since then, the study of (de-)localization transitions in non-interacting quantum systems in 2D has seen an immense surge of research activity (see, e.g., [20–27] and references therein). A general understanding of the strong-coupling (long-distance) behavior of 2D Kac–Moody current algebras perturbed by current–current interactions would be a valuable tool to describe a number of such transitions. Indeed, a good understanding of such perturbations has been achieved in a few cases [10,28–30]. However, in general, this is not the case, to date.

An intriguing conjecture has recently been advanced by Gerganov, LeClair and Moriconi [31], who consider, as above, a general Kac–Moody current algebra conformal field theory with symmetry group $\mathcal{G}$ at any level k, perturbed by right–left current bilinears. (The perturbations they discuss may also be anisotropic, or involve a supergroup, but this will not be important for the arguments presented in this article, which focuses entirely on the isotropic situation and bosonic groups.) Their paper builds on earlier work by Kutasov [32], who computed the renormalization group (RG) β-function for the isotropic case[2] of a (bosonic) symmetry group $\mathcal{G}$ to leading order in the large level k of the current algebra. The authors of [31] argued that Kutasov's result be exact for *any* value of the level k, in a particular regularization scheme. Specifically, for the isotropic case, the authors of Ref. [31] conjecture that the *exact* β-function for the coupling g be given by[3]

$$\beta(g) := \frac{\mathrm{d}g}{\mathrm{d}l} = \frac{1}{2}\frac{C_2 g^2}{(1+kg/4)^2}, \tag{1.1}$$

where C_2 is the eigenvalue of the quadratic Casimir operator in the adjoint representation of the symmetry group $\mathcal{G}$. Clearly, the notion of an *exact* β-function is delicate due to its dependence on the regularization scheme. (The contributions to the β-function beyond 2-loop order are scheme dependent.) Ref. [31] appears to be working in some scheme related to the left–right factorization of the underlying CFT. However, an explicit cut-off procedure, within which Eq. (1.1) is to be valid, is not specified in more detail in [31]. The authors indicate that certain checks to 3-loop order were performed. Checks beyond three-loop order have never been performed, to our knowledge. The 3-loop β-function within dimensional regularization has also been discussed in Refs. [33,34].

[1] In the absence of long-range electron–electron interactions.

[2] This theory is renormalizable with a single coupling constant.

[3] $l := \ln(a/L)$ is the RG-flow parameter, and a and L are the UV and IR cutoffs, respectively.

For the case where a symmetry $\mathcal{G} = SU(2)$ is broken down to $U(1)$ by a purely imaginary easy-axis anisotropy, and for level $k = 1$, the above conjecture (more precisely, an appropriate generalization) reproduces [35] known exact results [36–38]. The massless RG flow of the resulting non-unitary theory interpolates between ultraviolet (UV) and massless infrared (IR) fixed points. (Both lie on the line of free scalar field theories with central charge $c = 1$ but with different compactification radii[4]). In this case, the conjectured β-function reproduces correctly the exactly known universal relationship between the exact scaling exponents (i.e., the slopes of the β-function) at the IR and the UV fixed points. This is the only universal information contained in any β-function describing this flow.

On the other hand, the conjectured form of the β-function, when appropriately generalized to supergroups, and to the anisotropic case, has recently been applied [40] to theories describing disordered systems, of the kind mentioned above. Here, however, certain problems were encountered: integration of the conjectured RG equations led to flows which reached a singularity after a finite scale transformation, which appears to be an unacceptable result.

Motivated by these inconclusive results concerning the validity of the conjecture, we were led to check the conjectured form of the β-function by explicit computation to high loop order. We consider the classical symmetry groups $\mathcal{G} = SU(N), SO(N)$, and $SP(N)$, and the special case of *isotropic* current–current interactions at level $k = 0$. Our results are summarized in the following section. For all the classical groups, we find a contribution to the β-function at 4-loop order which is incompatible with the conjecture in *all* possible regularization schemes. The discrepancy is caused by an extra logarithmic divergence in perturbation theory, proportional to an additional group theoretical invariant (besides the quadratic Casimir), which first appears at 4-loop order. This divergence is not accounted for by the conjectured form of the β-function. Implications of our results, obtained for level $k = 0$, for the k-dependence of the β-function in any scheme are discussed in the conclusion, Section 5. In this section, we also come back to, and comment on the special case of the anisotropic $SU(2)$ model mentioned above. The reader who wishes to skip the technical details of our calculation, which are presented in Sections 3 and 4, will find a self-contained exposition of our results in Sections 2 and 5.

2. Presentation of results

In order to check the conjecture, we consider, as mentioned above, the specific case of an isotropic perturbation with symmetry group $\mathcal{G}$, and level $k = 0$. The conjectured β-function (1.1) then becomes

$$\beta(g) = \frac{1}{2} C_2 g^2. \tag{2.1}$$

Here C_2 denotes the eigenvalue of the quadratic Casimir invariant in the adjoint representation of $\mathcal{G}$. For the classical groups, $\mathcal{G} = SU(N), SO(N)$, and $SP(N)$, these are listed, in our normalizations, in Fig. 1. In Sections 4.1–4.7 we present an explicit

[4] Due to the non-unitarity of the theory, this is not in conflict with Zamolodchikov's c-theorem [39].

$$SU(N): \quad C_2 = N, \qquad d_2 = \frac{3}{2}N^2,$$
$$SO(N): \quad C_2 = N-2, \qquad d_2 = 24 - \frac{45}{2}N + 6N^2 - \frac{3}{8}N^3,$$
$$SP(N): \quad C_2 = \frac{N+2}{2}, \qquad d_2 = \frac{3}{2} + \frac{45}{32}N + \frac{3}{8}N^2 + \frac{3}{128}N^3.$$

Fig. 1. Group theoretical numbers for the classical groups, used in the main text.

Fig. 2. Chain-diagrams up to 4-loop order. The number of bubbles is the number of loops.

perturbative calculation of the β-function up to 4-loop order. This calculation proceeds in three steps:

(i) use the current-algebra to calculate the diagrams;
(ii) simplify the diagrams using elementary algebra;
(iii) evaluate the integrals, which represent the ("Feynman") diagrams.

After step (i), we encounter a great number of (rather complicated looking) diagrams. However, after step (ii), we are left with only two classes of diagrams:

(1) chain diagrams ("bubble" diagrams);
(2) non-chain diagrams.

Chain-diagrams appear at loop-order 1, 2, 3 and 4. They have the form depicted in Fig. 2. Each bubble comes with a factor of g (the coupling-constant), with a group-theoretical factor of C_2, and the whole chain with an (N-independent) integral $\mathcal{I}_n$, at n-loop order (n bubbles in the chain). The integral $\mathcal{I}_n$ depends on the cutoffs L (infrared) and a (ultraviolet), and is a polynomial[5] of degree n in $[\ln(\frac{L}{a})]$, plus terms which are finite for $L/a \to \infty$. (The fact that an n-loop integral is bounded by $c[\ln(\frac{L}{a})]^n$ with some constant c is a necessity to ensure renormalizability.) Only the leading term in $\mathcal{I}_n$, with the highest power of $\ln(\frac{L}{a})$, is universal. This applies, e.g., to the diverging part of the 1-loop integral $\mathcal{I}_1$. Thus the contribution of the chain diagrams to the renormalization of the coupling g is, at n-loop order (up to a combinatorial factor)

$$g(C_2 g)^n \mathcal{I}_n. \tag{2.2}$$

Non-chain diagrams first appear at 4-loop order. They are proportional to d_2, which is an additional group-theoretical invariant (in the adjoint representation), independent of the quadratic Casimir C_2. Its value for the classical groups is given on Fig. 1. This invariant can be constructed by drawing a cube, where one puts a factor of f_c^{ab} on each corner, with one of its three indices on each adjoining edge, see Fig. 3. Finally, indices on the same edge are contracted.

[5] See (A.7) for concreteness.

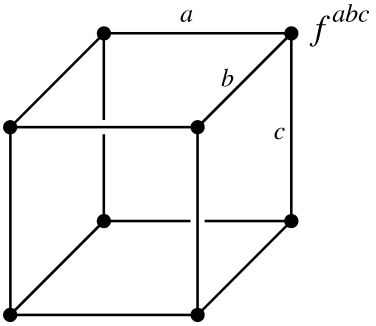


Fig. 3. The group-theoretical invariant d_2 and its graphical representation.

Each non-chain diagram has a global divergence proportional to $[\ln(\frac{L}{a})]$ ("single log"), and subdivergences (higher powers of $[\ln(\frac{L}{a})]$). However, it turns out that to the order considered here, one can always group non-chain diagrams together into classes, such that each class has only a global divergence, but *no subdivergence*. This means that the integral (over positions) is proportional to $\ln(\frac{L}{a})$ + finite, and that the prefactor in front of $[\ln(\frac{L}{a})]$ is again *universal*.

Let us recall here that a diagram proportional to $[\ln(\frac{L}{a})]$, i.e., a "single log", gives a finite contribution to the β-function. Using the above information, we thus find the following β-function at 4-loop order

$$\beta(g) = g\left[\frac{1}{2}C_2 g + a_2(C_2 g)^2 + a_3(C_2 g)^3 + a_4(C_2 g)^4 - d_2\frac{\pi}{240}\left(6+\pi^2\right)g^4\right] + O\left(g^6\right). \tag{2.3}$$

Here, the numbers a_3 and a_4 depend on the regularization scheme (but not on N or the value of the cutoffs). In contrast, the remaining three terms are universal. The first term comes from the "single log" of the 1-loop chain (giving the contribution $\frac{1}{2}C_2 g$) and the last term from that of the non-chain diagrams (giving $-d_2\frac{\pi}{240}(6+\pi^2)g^4$). Furthermore, we have $a_2 = 0$ since we consider level $k = 0$; this is a consequence of the universality of the β-function up to 2 loops, and will be checked for a specific scheme in Appendix A.1. Note that to arrive at the result in (2.3), *no specific form of the cut-off procedure has to be chosen*.

We are now in a position to answer the question, of whether the conjecture is compatible with our explicit calculation, in some given cut-off scheme. To see this, consider, for example, the group $\mathcal{G} = SU(N)$, where we have $C_2 = N$, and $d_2 = \frac{3}{2}N^2$ (see Fig. 1). Thus at 4-loop order, we have the following contributions

$$g\left[a_4 N^4 g^4 - \frac{3N^2}{2}\frac{\pi}{240}\left(6+\pi^2\right)g^4\right]. \tag{2.4}$$

Since C_2 and d_2 contain all the dependence on N, and since a_4 is independent of N, there is no possible choice of a_4, and thus no cutoff procedure, which cancels this term *for all* N. This proves, that the conjecture is incorrect for all possible cut-off procedures, for level $k = 0$. The same conclusion is arrived at for the other choices of groups.

Let us finally give the result in a specific scheme, namely, in the scheme in which an n-loop chain diagram is proportional to $[\ln(\frac{L}{a})]^n$, with no subleading term in $\ln(\frac{L}{a})$:

$$\beta(g) = \frac{1}{2}C_2 g^2 - d_2 \frac{\pi}{240}\big(6+\pi^2\big)g^5 + O\big(g^6\big). \tag{2.5}$$

Indeed, this is the scheme used in the large-N expansion of the Gross–Neveu model, where the β-function becomes quadratic [41], to leading order in $1/N$. Note that this is compatible with our result (2.5), since (upon rescaling g by N), the first (1-loop) term is order $O(1)$, whereas the second (4-loop) term is order $O(1/N^2)$.

In conclusion, we have found that the conjecture is violated at 4-loop order, and at order $1/N^2$ for $SU(N)$. For $SO(N)$ and $SP(N)$ corrections appear at order $1/N$, as can be seen from the table in Fig. 1.[6]

Let us now outline the organization of the article; the reader wishing to skip the technical details of our paper can proceed directly to Section 5: in Section 3 we introduce the model, the current-algebra and basic notations. Our calculations are presented in Section 4: we show in Section 4.1 how the Kac–Moody current-algebra is used to successively eliminate interaction vertices from expectation values, and how this can be used to evaluate OPE-coefficients. This is a non-trivial task. Indeed, the raw result of this reduction procedure depends on the order of the successive reductions and is highly asymmetric, whereas the OPE-coefficient should be symmetric. To obtain a more symmetric result, the raw result can be simplified by using algebraic relations which we have baptized "magic rules", for their efficiency. This will explicitly be demonstrated in Section 4.2 on the example of the 2-loop diagrams. In Section 4.3 we proceed to 3-loop order, and show again how the initial highly asymmetrically looking OPE-coefficient is simplified. As in 2-loop order, all resulting diagrams are chain-diagrams which in a suitable scheme factorize, and thus do not give a new contribution to the β-function. Proceeding to 4-loop order in Section 4.4, one finds diagrams which the magic rules are no longer able to simplify to chain-diagrams. Due to the sheer number of initial diagrams, namely 576, this approach is not very illuminating. In Section 4.5 we, therefore, pursue a different route: we first calculate OPE-coefficients for "adjoint" perturbations $\Phi^a = f^{abc} J^b \bar{J}^c$. We then show in Section 4.6 how an n-loop OPE-coefficient can be expressed as a simple algebraic function times an OPE-coefficient at order $n-1$, involving two adjoint perturbations. This allows us to identify at 4-loop order a

[6] One expects additional group theoretical invariants to appear in the β-function also at higher loop orders. We have found that the group theoretical invariant associated with the generalization of the cube to a "chain" of ℓ square plaquettes will appear at ℓ-loop order. For $SU(N)$ it scales like N^2 when ℓ is even, and like N^3 when ℓ is odd, as $N \to 0$. If the sum of the corresponding integrals is diverging, then this term cannot have a counter-term at even loop orders. It would thus represent, in this case, a new contribution to the β-function at ℓ-loop order. This may suggest that additional group theoretical invariants, beyond the one discussed in this paper, appear in the β-function at least at all even orders, except for $\ell = 2$. (As an example, consider 6-loop order: the above mentioned chain build out of 6 plaquettes scales like N^2 for small N. Suppose there are counter-terms to the corresponding diagrams, then these must be products of diagrams at lower order, e.g., a 4-loop diagram $\times$ a 1-loop diagram $\times$ a 1-loop diagram. This would scale for small N at least as $N^2 \times N \times N \sim N^4$ (and actually as N^6, if one were to use only 1-loop counter-terms), thus has a higher power in N than the chain of six plaquettes. Therefore, it cannot be a counter-term of the former diagram, and the former can—due to renormalizability—not have a subdivergence. Thus its integral is universal (see the discussion in Section 4 and Appendix B), and there is a priori no reason that it should vanish.

combination of eight diagrams, which cannot be factorized as chains (non-chain diagrams). Their contribution to the 4-loop β-function is calculated analytically in Appendix B. All these ingredients are collected in Section 4.7, where we obtain the 4-loop β-function. We have relegated some basic group-theoretical relations to Appendices C.1–C.3. Conclusions and further perspectives are offered in Section 5.

3. Model and method

We study the Non-Abelian Thirring Model (NATM) in two dimensions. The model may be defined as a perturbation of a 2D conformal field theory, with action S_0, which is invariant under a symmetry group $\mathcal{G}$ acting in the standard way [1,42]. The chiral components of the conserved Noether currents, $J^a(z)$ and $\bar{J}^a(\bar{z})$, depend (as indicated) only on $z = x + iy$ and $\bar{z} = x - iy$, and satisfy the defining operator product expansion (OPE) of the Affine Lie algebra (Kac–Moody algebra) at level k

$$J^a(z)J^b(0) = \frac{\frac{k}{2}\delta^{ab}}{z^2} + \frac{1}{z} f_c^{ab} J^c(0) + \cdots,$$
$$\bar{J}^a(\bar{z})\bar{J}^b(0) = \frac{\frac{k}{2}\delta^{ab}}{\bar{z}^2} + \frac{1}{\bar{z}} f_c^{ab} \bar{J}^c(0) + \cdots, \tag{3.1}$$

where f_c^{ab} are the structure constants of $\mathcal{G}$. (Repeated indices are summed throughout this paper, unless stated otherwise.) The model we study is defined by the action[7]

$$\mathcal{S} = \mathcal{S}_0 + g_0 \int_z \Phi(z,\bar{z}), \qquad \int_z := \int \frac{\mathrm{d}^2 z}{2\pi}, \tag{3.2}$$

where the perturbing operator

$$\Phi(z,\bar{z}) \equiv J^a(z)\bar{J}^a(\bar{z}) \tag{3.3}$$

is invariant under global transformations of the symmetry group $\mathcal{G}$. This theory is known to be renormalizable with a single coupling g. The conjectured form [31] of the β-function for the renormalized coupling g is quoted in (1.1).

We compute the β-function explicitly to 4-loop order. To this end, consider the perturbative evaluation of the expectation value in the fully interacting theory of some quantity $\mathcal{O}$, which may represent an operator, or a product of operators at different spatial positions,

$$\langle \mathcal{O} \rangle_{g_0} = \frac{\mathcal{Z}(0)}{\mathcal{Z}(g_0)} \left\langle \mathcal{O} \mathrm{e}^{-g_0 \int_z \Phi(z,\bar{z})} \right\rangle_0. \tag{3.4}$$

Here, the expectation value $\langle \cdots \rangle_0$ is taken in the unperturbed CFT with action $\mathcal{S}_0$, normalized such that $\langle 1 \rangle_0 = 1$. $\mathcal{Z}(g_0)$ is the fully interacting partition function.[7] A cut-off (regularization) procedure, depending on short- and large-distance cut-offs a and L, is required to render all terms in this expansion finite, and is specified below. The

[7] The partition function is $\mathcal{Z}(g_0) = \int \mathcal{D}[\text{fields}] \exp(-\mathcal{S})$.

renormalized coupling g (which depends on g_0 and a/L) is found by computing how the coefficient g_0 of the first order term in the expansion of the exponential

$$\langle\cdots 1\rangle_0 + \left\langle\cdots\left(-g_0\int\limits_{z_1}\Phi(z_1,\bar z_1)\right)\right\rangle_0 + \left\langle\cdots\left(\frac{g_0^2}{2!}\iint\limits_{z_1 z_2}\Phi(z_1,\bar z_1)\Phi(z_2,\bar z_2)\right)\right\rangle_0$$
$$+\left\langle\cdots\left(\frac{-g_0^3}{3!}\iiint\limits_{z_1 z_2 z_3}\Phi(z_1,\bar z_1)\Phi(z_2,\bar z_2)\Phi(z_3,\bar z_3)\right)\right\rangle_0 + \cdots \tag{3.5}$$

is modified by the higher order expansion terms. This modification is independent of the potential presence of any operator $\mathcal{O}$ in this expectation value, indicated by the ellipses.[8] The required calculation can be conveniently expressed in terms of multiple OPE-coefficients of the perturbing operator $\Phi(z,\bar z)$, evaluated in the unperturbed theory. The product of $(n+1)$ such operators at different positions may be expanded into a "complete set" of operators Φ^A sitting at the position of, say, the last operator. The expansion coefficients depend on the n relative coordinates,

$$\begin{aligned}&\Phi(z_1,\bar z_1)\cdots\Phi(z_n,\bar z_n)\Phi(z_{n+1},\bar z_{n+1})\\ &\quad=\sum_A \mathbf{C}_A\big[(z_1-z_{n+1}),(\bar z_1-\bar z_{n+1});\cdots;(z_n-z_{n+1}),(\bar z_n-\bar z_{n+1})\big]\\ &\qquad\times\Phi^A(z_{n+1},\bar z_{n+1}).\end{aligned} \tag{3.6}$$

The non-vanishing expansion coefficients are exactly known in any CFT. In the present case they are especially simple, and can be obtained by successive use of the OPE of the currents, (3.1). In particular, the perturbing operator Φ is the most relevant operator (besides the identity when $k\neq 0$) appearing amongst the Φ^A; all others are irrelevant. We find it convenient to denote the needed *multiple OPE-coefficient*, where $\Phi^A=\Phi$, by the symbol

$$\begin{aligned}&\big(\Phi(z_1,\bar z_1)\cdots\Phi(z_n,\bar z_n)\Phi(z_{n+1},\bar z_{n+1})|\Phi(z_{n+1},\bar z_{n+1})\big)\\ &\quad=\mathbf{C}_\Phi\big[(z_1-z_{n+1}),(\bar z_1-\bar z_{n+1});\cdots;(z_n-z_{n+1}),(\bar z_n-\bar z_{n+1})\big].\end{aligned} \tag{3.7}$$

The renormalization process is now easily understood by inserting (3.6) into (3.5). Explicitly, denote the relevant integrals over the multiple OPE-coefficients by $\mathcal{F}_n$ (for "Feynman"-diagram):

$$\begin{aligned}\mathcal{F}_n:=&\int\limits_{z_1,z_2,\ldots,z_n}\big(\Phi(z_1,\bar z_1)\cdots\Phi(z_{n+1},\bar z_{n+1})|\Phi(z_{n+1},\bar z_{n+1})\big)\\ &\times\mathcal{C}(z_1,\bar z_1,\ldots,z_{n+1},\bar z_{n+1}).\end{aligned} \tag{3.8}$$

These integrals are regularized by a cut-off prescription, which is achieved by inserting a cut-off function $\mathcal{C}(z_1,\bar z_1,\ldots,z_{n+1},\bar z_{n+1})$ in the integral, as indicated. There are many

[8] The operator itself requires an analogous treatment, which, however, can be discussed independently; this will not be needed here.

possible choices. In this article we choose a (circular) hard cut-off implemented by

$$\mathcal{C}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) := \prod_{i \neq j} \Theta\big(a < |z_i - z_j| < L\big), \tag{3.9}$$

where Θ is the usual step function. This cut-off procedure restricts the distances between any pair of integration variables to lie between the short- and the long-distance cut-offs a and L. All integrals $\mathcal{F}_n$ are thus finite functions of a/L. As usual, inserting (3.6) in (3.5), and using (3.8) gives:

$$\begin{aligned}
&\langle \cdots 1 \rangle_0 + \left\langle \cdots \left(-g_0 \int_z \Phi(z, \bar{z}) \right) \right\rangle_0 + \left\langle \cdots \left(\frac{g_0^2}{2!} \mathcal{F}_1 \int_z \Phi(z, \bar{z}) \right) \right\rangle_0 \\
&\quad + \left\langle \cdots \left(\frac{-g_0^3}{3!} \mathcal{F}_2 \int_z \Phi(z, \bar{z}) \right) \right\rangle_0 + \cdots \\
&\quad = \langle \cdots 1 \rangle_0 + \left\langle \cdots \left(-g \int_z \Phi(z, \bar{z}) \right) \right\rangle_0 + \cdots = \left\langle \cdots \left(\mathrm{e}^{-g \int_z \Phi(z, \bar{z})} \right) \right\rangle_0 .
\end{aligned}$$

Following standard reasoning we have re-exponentiated in the last line. One can now read off the renormalized coupling:

$$g\left(g_0, \frac{a}{L}\right) = g_0 \left[1 - \frac{g_0}{2!} \mathcal{F}_1\left(\frac{a}{L}\right) + \frac{g_0^2}{3!} \mathcal{F}_2\left(\frac{a}{L}\right) - \cdots \right]. \tag{3.10}$$

The β-function is obtained as the change of g in response to changing a (or $1/L$), while keeping the bare coupling g_0 fixed:

$$\beta(g) := a \frac{\partial}{\partial a}\bigg|_{g_0} g\left(g_0, \frac{a}{L}\right). \tag{3.11}$$

In the remaining sections of the paper we will obtain the integrals $\mathcal{F}_1, \ldots, \mathcal{F}_4$. This gives us the result for the 4-loop β-function written in (2.3) above.

4. Calculation

In this section we present in detail the evaluation of the integrals $\mathcal{F}_n$ defined in (3.8) ("Feynman"-diagrams), needed to obtain the β-function, as explained in Section 3. The core of this calculation consists in obtaining the OPE-coefficients defined in (3.7), by repeated use of the current-algebra OPE (3.1). We start with the simplest case, i.e., with the 1-loop integral $\mathcal{F}_1$, and proceed successively to the more involved cases, up to 4-loop order.

4.1. 1-loop order

At 1-loop order, we need the OPE-coefficient

$$\big(\Phi(z, \bar{z}) \Phi(w, \bar{w}) | \Phi(w, \bar{w})\big). \tag{4.1}$$

To evaluate it, we have to eliminate $\Phi(z,\bar{z})$ from $\Phi(z,\bar{z})\Phi(w,\bar{w})$, using (3.1). This is done as follows[9]

$$\begin{aligned}&\Phi(z,\bar{z})\Phi(w,\bar{w}) \equiv J^a(z)\bar{J}^a(\bar{z})J^b(w)\bar{J}^b(\bar{w})\\ &\quad\longrightarrow\quad f^{ab}_c f^{ab}_d J^c(w)\bar{J}^d(\bar{w})\frac{1}{|w-z|^2} = C_2 J^c(w)\bar{J}^c(\bar{w})\frac{1}{|w-z|^2}.\end{aligned} \tag{4.2}$$

We have used that $f^{abc}f^{abd} = C_2\delta^{cd}$, with the second Casimir C_2. (This and more group theoretical relations are derived in Appendix C.1.) We denote (4.1) in short by

$$\big(\Phi(z,\bar{z})\Phi(w,\bar{w})|\Phi(\bar{w},w)\big) = \quad z \overset{\curvearrowright}{\bullet} w = z \overset{\curvearrowright}{\bullet} w = \frac{C_2}{|w-z|^2}. \tag{4.3}$$

The arrows show the direction in which the elimination has been made. This defines the sign. To be specific, an arrow from z to w represents $1/(z-w)$. A dashed such arrow represents $1/(\bar{z}-\bar{w})$. When a solid and a dashed arrow (with the same direction) connect the same two points, one can drop the arrows for simplicity of notation; seeing a solid and a dashed line thus means that when adding the arrows, both arrows are pointing in the same direction.

The OPE-coefficient (4.3) yields the 1-loop diagram $\mathcal{F}_1$

$$\mathcal{F}_1 = \int_z \quad \bullet \overset{\curvearrowright}{\bullet} \quad = \int_z \frac{C_2}{|w-z|^2}\Theta\big(a<|w-z|<L\big) = C_2\ln\left(\frac{L}{a}\right). \tag{4.4}$$

4.2. 2-loop order and the magic rule

At 2-loop order, we have three Φ's. Denoting $\Phi_i := J^a(z_i)\bar{J}^a(\bar{z}_i)$, we need to calculate $(\Phi_1\Phi_2\Phi_3|\Phi_3)$. Straightforward use of the current-algebra (3.1) with $k=0$, eliminating the currents one by one, starting with point 1, and continuing with point 2, yields

$$\begin{aligned}&(\Phi_1\Phi_2\Phi_3|\Phi_3)\\ &\quad= \frac{1}{2}C_2^2\left[\frac{2}{|z_{12}|^2|z_{23}|^2} + \frac{2}{|z_{13}|^2|z_{23}|^2} - \frac{1}{|z_{23}|^2 z_{13}\bar{z}_{12}} - \frac{1}{|z_{23}|^2 z_{12}\bar{z}_{13}}\right],\end{aligned} \tag{4.5}$$

where we have abbreviated $z_{ij} := z_i - z_j$. Here, and throughout this article, we use the labeling of points as indicated in Fig. 4. The result is graphically presented in Fig. 5 (top). Eq. (4.5) apparently contains a new diagram, which renders the OPE-coefficient asymmetric upon exchange of point 1 with point 2, or of point 1 with point 3. However, there is a simple algebraic identity, the "magic" rule for the real part $\Re$ of $\frac{1}{z\bar{w}}$:

$$\Re\left[\frac{1}{z\bar{w}}\right] = \Re\left[\frac{\bar{z}w}{|z|^2|w|^2}\right] = \left[\frac{\vec{z}\vec{w}}{|\vec{z}|^2|\vec{w}|^2}\right] = \frac{1}{2}\left[\frac{1}{|\vec{z}|^2} + \frac{1}{|\vec{w}|^2} - \frac{|\vec{z}-\vec{w}|^2}{|\vec{z}|^2|\vec{w}|^2}\right]. \tag{4.6}$$

The most useful application is in the presence of an additional factor $1/|\vec{w}-\vec{z}|^2$, which cancels the numerator in the last term. This leads to the decomposition of the new diagrams

[9] Recall that summation over repeated indices is implied, and that we work at level $k=0$.

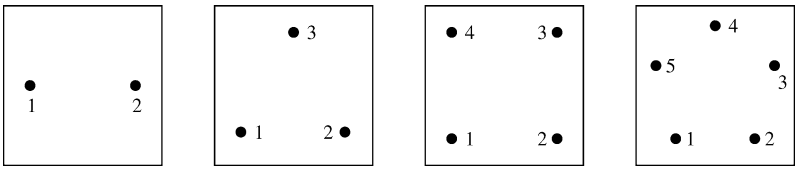


Fig. 4. Labeling of the points in Figs. 5–7, and diagrams in the main text.

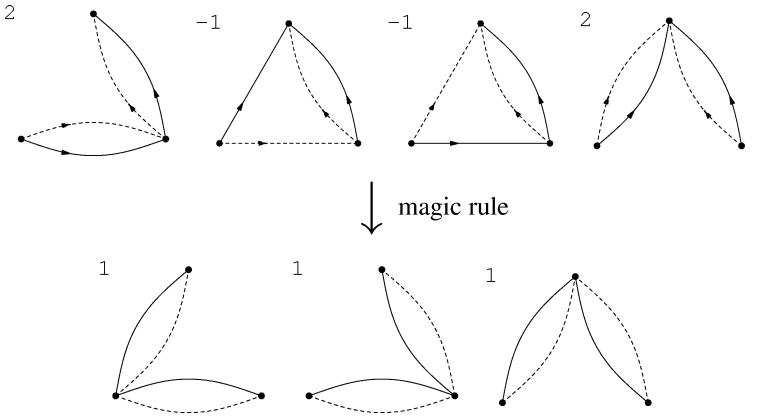
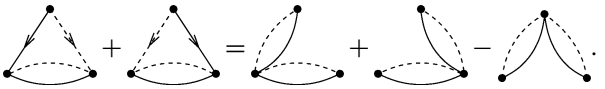


Fig. 5. 2-loop diagrams after reducing the structure-constants to numbers. To be multiplied by $\frac{1}{2}C_2^2$. The numbers given are the weight. The first line is the raw result, as obtained by using the reduction algorithm. Arrows indicate the direction of the reduction. The second line after using magic relations.

in (4.6) into chain-diagrams (drawn below rotated by -120^0 as compared to Fig. 5 (top))



(4.7)

The OPE-coefficient (4.5) simplifies to

$$(\Phi_1\Phi_2\Phi_3|\Phi_3) = \frac{1}{2}C_2^2\left[\frac{1}{|z_{12}|^2|z_{13}|^2} + \frac{1}{|z_{12}|^2|z_{23}|^2} + \frac{1}{|z_{13}|^2|z_{23}|^2}\right], \tag{4.8}$$

which is manifestly symmetric, as it should be. The resulting expression for the OPE-coefficient in (4.8) is graphically represented in Fig. 5 (bottom). One sees that after using the "magic" rule, the OPE-coefficient, and hence the integral $\mathcal{F}_2$, can be written in terms of chain diagrams. This suggests, that the corresponding diagrams (i.e., the "Feynman" integral $\mathcal{F}_2$) factorize, are of order $\ln(\frac{L}{a})^2$ without a pure $\ln(\frac{L}{a})$ and thus give no contribution to the β-function at 2-loop order. This is indeed correct, as checked in Appendix A.1 for the cut-off procedure introduced in Section 3.

For the model at hand, the cut-off procedure is subtle. The reason is that one cannot put a cut-off on the lines, as would be most convenient to immediately prove factorization of chain-diagrams: in constructing the diagram, we have used magic rules to move around the lines, and if we leave behind a cut-off function, then the resulting diagram will not be

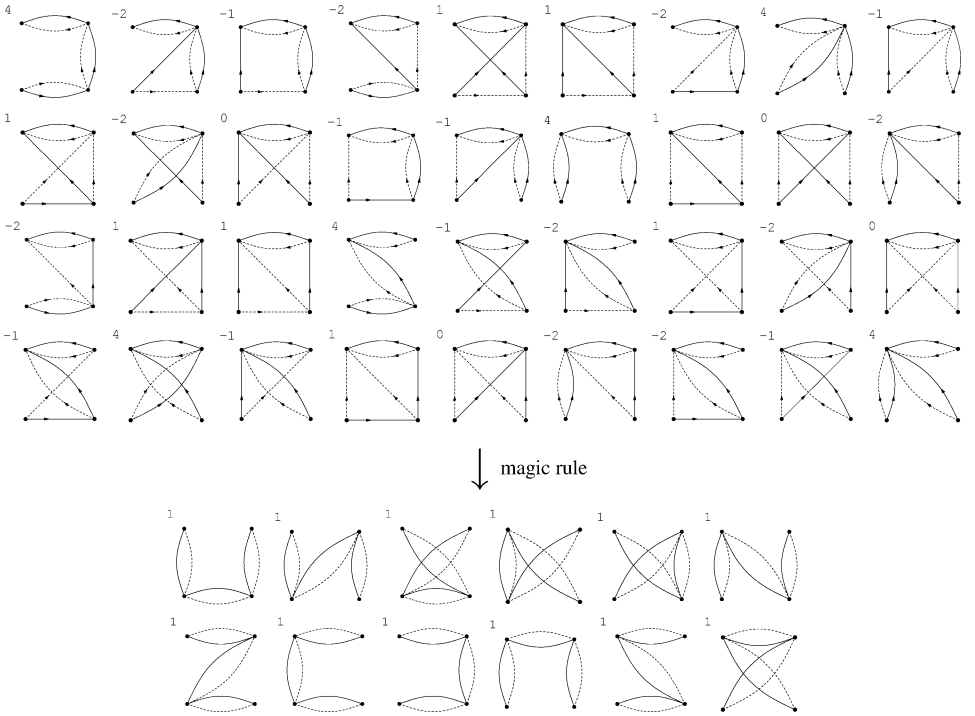


Fig. 6. 3-loop diagrams after reducing the structure-constants to numbers. To be multiplied by $\frac{1}{4}C_2^3$. The numbers given are the weight. The first 4 lines are the raw result, as obtained by using the reduction algorithm. Note that 4 diagrams have weight 0. Arrows indicate the direction of the reduction. The last two lines after using magic relations, dropping the redundant arrows.

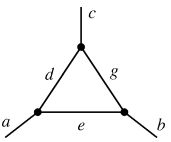
totally symmetric, as it should and as it is in our construction. The only way out of the above dilemma, is to put cut-offs between any pair of points, regardless of whether the two points are connected with a line or not (compare (3.8)). However, then the factorization is no longer a trivial statement, and has to be checked. This has been done for the 2-loop chains in Appendix A.1. As we have argued in Section 2, this is not essential for our arguments, and the conclusions remain valid in any scheme. Let us, however, mention, that in order to recover the large-N limit of $SU(N)$, factorization is needed, and is sufficient to *uniquely* fix the RG-procedure up to 4-loop order; but not necessarily beyond.

4.3. 3-loop order

At 3-loop order, 36 diagrams appear, presented on top of Fig. 6. These diagrams all contain six structure-constants, and have two free indices a and b, which are contracted with the remaining $\bar{J}^a J^b$. Since the only invariant object with two indices is δ^{ab}, one can contract the last lines to obtain the algebraic factor; the final result has of course to be divided by the dimension of the adjoint representation. One can then convince oneself by drawing pictures, that all objects which can be constructed, contain at least one loop made

out of two or three vertices[10], i.e., objects of the form

$$f^{acb}f^{bcd} = \quad = -C_2\delta^{ad}, \tag{4.9}$$



$$f^{aed}f^{bge}f^{cdg} = \quad = -\frac{C_2}{2}f^{abc}, \tag{4.10}$$

which we give together with a group-theoretical identity (derived in Appendix C.1), which is sufficient to reduce the number of f in any given diagram. Repeatedly using (4.9) and (4.10) thus allows to eliminate all f. This procedure is performed using a computer, and the reader would have a hard time verifying it by hand. We have thus shown that all 3-loop diagrams are proportional to C_2^3, thus no additional group theory invariants, besides the second Casimir, appear at this order.

The diagrams are given with their combinatorial factor on top of Fig. 6, to be multiplied by $\frac{1}{4}C_2^3$. Applying magic rules leads to chain-diagrams, presented graphically at the bottom of Fig. 6. Algebraically, the result is

$$\begin{aligned}
&(\Phi_1\Phi_2\Phi_3\Phi_4|\Phi_1)\\
&\quad = \frac{C_2^3}{4}\Bigg[\frac{1}{|z_{12}|^2|z_{14}|^2|z_{23}|^2} + \frac{1}{|z_{13}|^2|z_{14}|^2|z_{23}|^2} + \frac{1}{|z_{12}|^2|z_{13}|^2|z_{24}|^2}\\
&\qquad + \frac{1}{|z_{13}|^2|z_{14}|^2|z_{24}|^2} + \frac{1}{|z_{13}|^2|z_{23}|^2|z_{24}|^2} + \frac{1}{|z_{14}|^2|z_{23}|^2|z_{24}|^2}\\
&\qquad + \frac{1}{|z_{12}|^2|z_{13}|^2|z_{34}|^2} + \frac{1}{|z_{12}|^2|z_{14}|^2|z_{34}|^2} + \frac{1}{|z_{12}|^2|z_{23}|^2|z_{34}|^2}\\
&\qquad + \frac{1}{|z_{14}|^2|z_{23}|^2|z_{34}|^2} + \frac{1}{|z_{12}|^2|z_{24}|^2|z_{34}|^2} + \frac{1}{|z_{13}|^2|z_{24}|^2|z_{34}|^2}\Bigg].
\end{aligned} \tag{4.11}$$

4.4. 4-loop order, direct approach

Let us now continue to 4-loop order. After using the current-algebra, there are 576 diagrams, which again we generate computer-algebraically. The group theoretical factors appearing with these diagrams are much more involved. An example is a cube, where each corner represents a structure-factor f^{abc} and each link identifies a pair of common indices between two f's. This is drawn on Fig. 3 and detailed in Appendix C.1. It is at this loop-order that an additional group theoretical invariant besides the quadratic Casimir arises. After reducing the algebra, one finds that 380 terms are proportional to C_2^4. Using magic rules, these diagrams can be reduced to 60 chains; these are in fact all the chains which can be drawn through 5 points.[11] Each chain comes with a weight of $\frac{1}{8}C_2^4$. For the remaining

[10] The simplest object without such a loop would be a cube, which indeed appears at 4-loop order, see Fig. 3.

[11] In general, there is a total of $\frac{1}{2}n\cdot(n-1)!$ chains that can be drawn through n points.

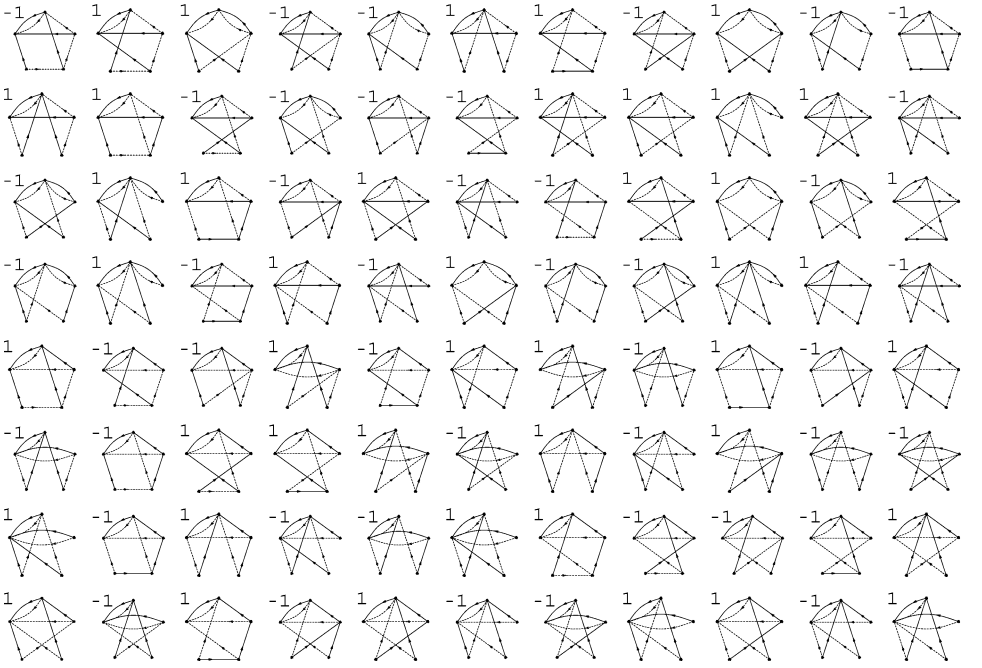


Fig. 7. 4-loop diagrams not proportional to C_2^4.

diagrams not proportional to C_2^4, presented in Fig. 7, our reduction-algorithm based on magic rules is incapable of further simplifying it. In Section 4.6 we will present a simple calculation, reducing the task to calculating a combination of eight diagrams. To this aim, we need correlation functions involving operators which we call "adjoint" perturbations, defined below.

4.5. OPE for adjoint perturbations

Define the "adjoint" perturbation at position $(z_i, \bar{z}_i)$ as

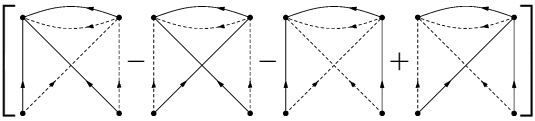
$$\Phi^a(z_i, \bar{z}_i) \equiv \Phi_i^a := f^{abc} J^b(z_i) \bar{J}^c(\bar{z}_i). \tag{4.12}$$

We now apply the same procedure as in the previous sections: eliminate currents one by one using (i) the current-algebra, (ii) evaluation of the group theoretical factors, and (iii) simplifications with the magic rule. After some lengthy calculations (done again computer-algebraically), we find up to 3-loop order:

$$\left(\Phi_1^a \Phi_2^a | \Phi_2\right) = \frac{C_2^2}{2} \frac{1}{|z_{12}|^2}, \tag{4.13}$$

$$\left(\Phi_1^a \Phi_2 \Phi_3^a | \Phi_3\right) = \frac{C_2^3}{4} \left[\frac{1}{|z_{12}|^2 |z_{13}|^2} + \frac{1}{|z_{13}|^2 |z_{23}|^2} \right], \tag{4.14}$$

$$
\begin{aligned}
&\big(\Phi_1^a \Phi_2^a \Phi_3 \Phi_4 \big| \Phi_4\big) \\
&\quad = \frac{C_2^4}{8}\Bigg[\frac{1}{|z_{12}|^2|z_{14}|^2|z_{23}|^2} + \frac{1}{|z_{12}|^2|z_{13}|^2|z_{24}|^2} + \frac{1}{|z_{12}|^2|z_{13}|^2|z_{34}|^2} \\
&\qquad + \frac{1}{|z_{12}|^2|z_{14}|^2|z_{34}|^2} + \frac{1}{|z_{12}|^2|z_{23}|^2|z_{34}|^2} + \frac{1}{|z_{12}|^2|z_{24}|^2|z_{34}|^2}\Bigg] \\
&\qquad + d_2 \Bigg[
\end{aligned}
$$



$$
\Bigg] . \qquad (4.15)
$$

The additional group-theoretical invariant d_2 is defined[12] in Appendix C.1. For $SU(N)$ this reads $d_2 = \frac{3}{2}N^2$ compared to the leading term $C_2^4 = N^2$. The results for $SO(N)$ and $SP(N)$ are listed in Fig. 1, see also Appendix C.3. For these groups, $C_2^4 \sim N^4$ and again d_2 is subdominant, with $d_2 \sim N^3$.

4.6. 4-loop order simplified

We have seen in Section 4.4 that a direct 4-loop calculation is quite cumbersome. Instead, we use here a different approach, inspired by the original work by Kutasov [32]. We start by eliminating Φ_n from the multiple OPE-coefficient (3.7). Let us first give the result and then explain how we have obtained it:

$$
\begin{aligned}
&(\Phi_n \Phi_1 \Phi_2 \cdots \Phi_{n-1} | \Phi_{n-1}) \\
&\quad = - \sum_{i,j=1,\ldots,n-1, i\neq j} \frac{1}{z_n - z_i} \frac{1}{\bar z_n - \bar z_j} \big(\Phi_1 \cdots \Phi_i^a \cdots \Phi_j^a \cdots \Phi_{n-1} | \Phi_{n-1}\big) \\
&\qquad + \sum_{i=1}^{n-1} \frac{C_2}{|z_n - z_i|^2} (\Phi_1 \cdots \Phi_{n-1} | \Phi_{n-1}). \qquad (4.16)
\end{aligned}
$$

We have eliminated all currents at point n. Using the current-algebra (3.1) again with $k = 0$, there is a contribution from each pair of points $\{i, j\}$ with $i, j \neq n$. The first line of (4.16) contains the contributions with $i \neq j$, for which we have listed below the corresponding current-algebra identities in (4.17) and (4.18). The last line of (4.16) is the case $i = j$, and is obtained by using the current-algebra both for the holomorphic and antiholomorphic current, as given in (4.19) below.

$$
J^a(z_n)\bar J^b(\bar z_i) J^b(z_i) \longrightarrow \frac{f^{abc}}{z_n - z_i} J^c(z_i)\bar J^b(\bar z_i) = -\frac{1}{z_n - z_i}\Phi_i^a, \qquad (4.17)
$$

$$
\bar J^a(\bar z_n)\bar J^b(\bar z_j) J^b(z_j) \longrightarrow \frac{f^{abc}}{\bar z_n - \bar z_j} \bar J^c(\bar z_j) J^b(z_j) = \frac{1}{\bar z_n - \bar z_j}\Phi_j^a, \qquad (4.18)
$$

$$
\bar J^a(\bar z_n) J^a(z_n)\bar J^b(\bar z_i) J^b(z_i) \longrightarrow \frac{f^{abc} f^{abd}}{|z_n - z_i|^2} J^c(z_i)\bar J^d(\bar z_i) = \frac{C_2}{|z_n - z_i|}\Phi_i. \qquad (4.19)
$$

[12] Constructing a symmetrized tensor d^{abcd} out of the trace of 4 structure-constants f_c^{ab}, d_2 is the non-dominant contribution (in N) of the square of d^{abcd} (as defined in (C.13)).

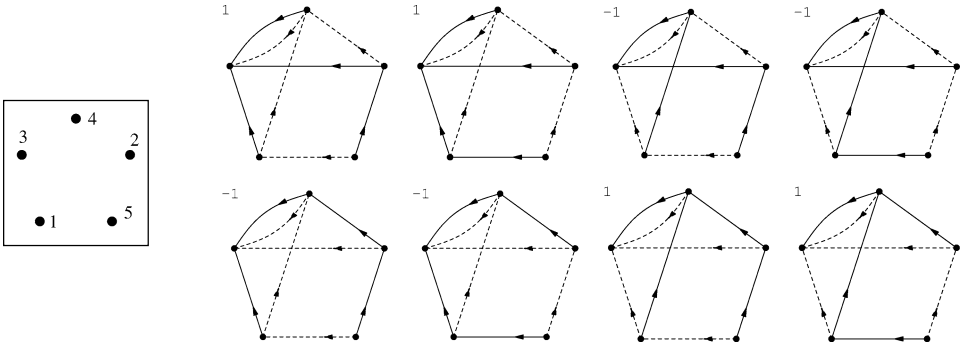


Fig. 8. The combination of 8 diagrams contributing at 4-loop order. Note the different labeling of points given in the inset, as compared to the labels of Fig. 4, used in Fig. 7.

Note that eliminating point n (instead of point 1 as we were used to do) is for later calculational (and representational) convenience only.

We now turn to the 4-loop calculation, i.e., set $n = 5$. One can check that starting from (4.16), using (4.11) and (4.15), one reconstructs all the 60 chains connecting 5 points, as found in Section 4.4. The remaining terms are obtained from the first term in (4.16) times the term proportional to d_2 in (4.15). There are $\frac{(n-1)(n-2)}{2} = 6$ such terms, each being a combination of 8 diagrams depicted in Fig. 8. Since each of the 6 terms gives the same contribution upon integration, we only have to calculate the integral over one of them. Analytically, this is most easily written as (we have chosen i, j to be the pair $1, 2$ and the starting point is 5)

$$\mathcal{I} := \int \left(\frac{1}{z_{14}\bar{z}_{13}} - \frac{1}{\bar{z}_{14}z_{13}}\right)\left(\frac{1}{z_{24}\bar{z}_{23}} - \frac{1}{\bar{z}_{24}z_{23}}\right)\frac{1}{|z_{34}|^2}\left(\frac{1}{z_{15}\bar{z}_{25}} + \frac{1}{\bar{z}_{15}z_{25}}\right) \times \mathcal{C}(z_1, \bar{z}_1, \ldots, z_5, \bar{z}_5), \tag{4.20}$$

where the integral is over all but one point, and the cut-off function $\mathcal{C}(\cdots)$ was introduced in (3.9). This integral is evaluated in Appendix B to be

$$\mathcal{I} = -\frac{\pi}{12}(6 + \pi^2)\ln\left(\frac{L}{a}\right). \tag{4.21}$$

4.7. The β-function up to 4-loop order

Now we are ready to put everything together to obtain the β-function. In a scheme in which the chains factorize, we obtain by collecting the results (4.4), (4.8), (4.11), and the paragraph below (4.16), and upon use of (3.10):

$$g = g_0\left\{1 - \frac{1}{2!}\left[g_0 C_2 \ln\left(\frac{L}{a}\right)\right] + \frac{1}{3!}\frac{3}{2}\left[g_0 C_2 \ln\left(\frac{L}{a}\right)\right]^2 - \frac{1}{4!}\frac{12}{4}\left[g_0 C_2 \ln\left(\frac{L}{a}\right)\right]^3 + \frac{1}{5!}\left(\frac{60}{8}\left[g_0 C_2 \ln\left(\frac{L}{a}\right)\right]^4 + 6\frac{\pi}{12}(6 + \pi^2)d_2 g_0^4 \ln\left(\frac{L}{a}\right)\right)\right\}. \tag{4.22}$$

Note that the factors $1/n!$ are from the exponential, then for the chains the next factor is the number of chains times their dependence on g_0, times the group-theoretical factor[13] $C_2^n/2^{n-1}$. The last term (which comes from the non-chain diagrams) has a factor of 6 from combinatorics as discussed in the previous section and the minus signs from the integral (4.21) and from (4.16) cancel.

Inserting (4.22) into (3.11) leads to the 4-loop β-function in terms of the renormalized coupling g:

$$\beta(g) = \frac{1}{2}C_2 g^2 - d_2\frac{\pi}{240}\big(6+\pi^2\big)g^5 + O\big(g^6\big), \tag{4.23}$$

with $d_2 = \frac{3}{2}N^2$ for $SU(N)$, $d_2 = 24 - \frac{45}{2}N + 6N^2 - \frac{3}{8}N^3$ for $SO(N)$ and $d_2 = \frac{3}{2} + \frac{45}{32}N + \frac{3}{8}N^2 + \frac{3}{128}N^3$ for $SP(N)$, as calculated in Appendices C.2 and C.3.

In schemes in which the chains do not factorize, there are additional terms, see (2.3) and the discussion below that equation.

Some comments on the procedure are in order. Readers used to the Wilson-scheme, will recover that procedure by studying the change of g in (4.22) under an infinitesimal change of a, corresponding to the integration over an infinitesimal shell from a to $a+\delta a$. The only difference is that this is a shell in position space, and not in momentum space.

Second, to our knowledge this is the first 4-loop calculation with a hard cutoff, or equivalently the first 4-loop calculation in a Wilson scheme.

5. Conclusion and further perspectives

In this article we have performed an explicit perturbative calculation of the β-function for the non-Abelian Thirring model at $k = 0$ up to 4-loop order. We have found that the conjectured form of the β-function [31], Eq. (1.1), is incompatible with our result in *all* regularization schemes. The discrepancy arises from an extra logarithmic divergence, which appears first at 4-loop order, and which is proportional to a higher group-theoretical invariant (evaluated in the adjoint representation of the symmetry group) which is different from the quadratic Casimir invariant. This divergence is not accounted for by the conjectured β-function.

It is worth pointing out that our explicit 4-loop result at level $k = 0$ does not only rule out the particular conjectured form of the (isotropic) β-function Eq. (1.1), but a more general class of conjectures for the β-function. This way of presenting our 4-loop result emphasizes the dependence on the level k, whereas in Section 2 only the special case $k = 0$ was discussed. Such forms which we can rule out arise by attempting to "scale" with the level k. Specifically, for any one of the classical groups $\mathcal{G} = SU(N), SO(N)$ and $SP(N)$, the β-function (of the isotropic theory) will in general be a function of three variables, the coupling constant g, the level k, as well as N, or equivalently $C_2 = C_2(N)$, the second Casimir invariant in the adjoint representation:

$$\frac{\mathrm{d}g}{\mathrm{d}l} = \beta(g, k, C_2). \tag{5.1}$$

[13] See (4.4), (4.8), (4.11), and the paragraph below (4.16).

It was argued in [35] that by rescaling the Kac–Moody currents $J^a \to \sqrt{k}\, J^a$ (as suggested by the large-k calculations done in [32]), the (isotropic) β-function should satisfy (perhaps in a suitable scheme) the "scaling form"

$$\beta(g,k,C_2) \stackrel{!}{=} \frac{1}{k} F\left(\frac{C_2}{k}, kg\right) = gH\left(gC_2, \frac{k}{C_2}\right). \tag{5.2}$$

The conjectured β-function of Ref. [31], i.e., Eq. (1.1), is a special case of this. The second equation above gives an equivalent way of writing the scaling form, useful when considering the limit $k \to 0$ for $g =$ fixed, whereas the form in the first equation is useful in the large k limit where $1/k \to 0$ for $kg =$ fixed. Since we know that the perturbative β-function must have a finite limit as $k \to 0$, the second equation in (5.2), when specialized to $k = 0$, leads to a form of the β-function, whose g-dependence is only through the combination gC_2 (apart from an overall factor of g). Comparison with Eq. (2.3) shows that this is incompatible with the explicit 4-loop result that we have found in any possible scheme. Hence, our result implies that the β-function must have an explicit dependence on the level k, and that the latter can in no scheme be "scaled out" in the way indicated in (5.2).

Finally, a point which deserves clarification is why in the case where a symmetry $\mathcal{G} = SU(2)$ is broken down to $U(1)$ by a purely imaginary easy-axis anisotropy, and for level $k = 1$, the conjecture reproduces [35] known exact results [36–38]. (This was mentioned in the introduction, Section 1.) Indeed, Ref. [35] proposes that this agreement should provide a strong check of the conjecture. Here we would like to point out, however, that this agreement is not surprising, because the theory is very special. It possesses a hidden quantum group symmetry (or, "fractional supersymmetry") [43], present for all values of the level k. This symmetry imposes strong constraints on the k-dependence of the relationship between the slopes of the β-function, i.e., the RG eigenvalues y of the perturbation, at the UV and IR fixed points. As a consequence of the symmetry, this relationship is [44]:

$$\frac{1}{k y_{\mathrm{IR}}} + \frac{1}{k y_{\mathrm{UV}}} = 1. \tag{5.3}$$

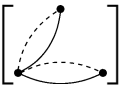
For the remainder of our argument, we only need the result in (5.3) about the exact relationship between y_{UV} and y_{IR}. First, we note that the same result can also be obtained by using the conjectured β-function of [31], see [35]. Now since, following Kutasov [32], the conjecture is just the leading term in a $1/k$-expansion of the beta function, it should yield a relation between y_{IR} and y_{UV}, which is valid at leading order in $1/k$, but will not contain information about corrections to this of order $1/k^2$ or higher. However, due to Eq. (5.3), the *exact* relation between y_{IR} and y_{UV} has no such higher order corrections at all. Thus the leading order term in $1/k$ happens to give already the whole, i.e., the exact result for these quantities. This explains why the conjecture reproduces the exact result even for level $k = 1$. We end our discussion by noting that it would be interesting to obtain, generalizing Kutasov's work, who computed (as mentioned) the β-function of the non-Abelian Thirring model to first order in $1/k$ in the large-k expansion, higher order terms in this large-k expansion. These will not in general be absent, as our work presented in this paper shows. Work along these lines is in progress.

Appendix A. Factorization of chain-diagrams

In this appendix, we show how chain-diagrams factor, restricting ourselves to 2-loop order. This is done in Appendix A.1. As a tool, we need the "conformal mapping technique", which was introduced in [45,46], reviewed in [47], and which we present here for completeness, and since in contrast to the cited references we here work exactly at the critical dimension, where we need both an ultraviolet and an infrared cutoff.

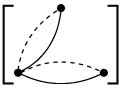
A.1. Factorization of chain-diagrams at 2-loop order

At 2-loop order, everything can with the help of the magic relation be reduced to the bubble-chain. The subtracted 2-loop diagram, i.e., the 2-loop diagram minus the square of the 1-loop diagram is (we denote by $\mathcal{S}$ this subtraction-operator, which also contains the integration and the cut-off functions)

$\mathcal{S}$ 

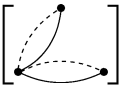
$$= \int\limits_{z,w} \frac{1}{|z|^2|w|^2}\big[\Theta\big(a<|z|,|w|,|z-w|<L\big)-\Theta\big(a<|z|,|w|<L\big)\big]. \tag{A.1}$$

The first term on the r.h.s. represents the 2-loop integral, the second term the subtracted 1-loop integrals (where integration over w and z factorizes). Applying $-a\frac{\partial}{\partial a}$ to the above gives

$-a\frac{\partial}{\partial a}\mathcal{S}$ 

$$\begin{aligned} &= a\int\limits_{z,w} \frac{1}{|z|^2|w|^2}\big[\Theta\big(a=|z|<|w|,|z-w|<L\big) \\ &\qquad + \Theta\big(a=|w|<|z|,|z-w|<L\big) \\ &\qquad + \Theta\big(a=|z-w|<|w|,|z|<L\big) \\ &\qquad - \Theta\big(a=|z|<|w|<L\big)-\Theta\big(a=|w|<|z|<L\big)\big]. \end{aligned} \tag{A.2}$$

Using the conformal mapping technique of [45–47], which is summarized in Appendix A.2, all terms can be mapped onto $|z|=a$; with the result (we have used that $a^2/|z|^2=1$):

$-a\frac{\partial}{\partial a}\mathcal{S}$ 

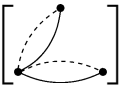
$$= \int\limits_{w} \frac{1}{|w|^2}\left[\Theta\left(\max_{\min}\big(|z|=a,|w|,|w-z|\big)<\frac{L}{a}\right) - \Theta\left(\max_{\min}\big(a,|w|\big)<\frac{L}{a}\right)\right]. \tag{A.3}$$

The function $\max\limits_{\min}(a_1,\ldots,a_n)$ is defined as

$$\max_{\min}(a_1,\ldots,a_n) := \frac{\max(a_1,\ldots,a_n)}{\min(a_1,\ldots,a_n)}. \tag{A.4}$$

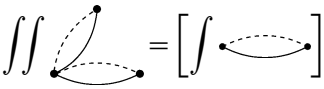
The above is a function of L/a, and can be bounded for $L/a > c_1$ by

$$\left| -a\frac{\partial}{\partial a}\mathcal{S}\left[\right] \right| < c_2 \frac{a}{L} \tag{A.5}$$



with $c_2 = c_2(c_1)$. Taking $c_1 > 3$ allows the bound $c_2 = 2$. The important thing is that the integral is not diverging: this means we have subtracted the right 1-loop counter-term. Moreover, the limit of large L/a can be taken; since it is zero, there is no single ln-contribution in the 2-loop integral. We can denote symbolically the result as

$$\iint \quad = \left[\int \quad \right]^2 . \tag{A.6}$$



A.2. Conformal mapping

As a tool to prove factorization of chains (see Appendix A.1), we need the "conformal mapping" technique, which was introduced in [45,46], reviewed in [47], and which we present here for completeness, and since in contrast to the cited references we here work exactly at the critical dimension, where we need both an ultraviolet and an infrared cutoff.

Note that a general N-loop integral $\mathcal{I}_N$ will behave as

$$\mathcal{I}_N(a, L) = a_0 + a_1 \ln\frac{L}{a} + a_2\left(\ln\frac{L}{a}\right)^2 + \cdots + a_N\left(\ln\frac{L}{a}\right)^N , \tag{A.7}$$

where we dropped terms which vanish in the limit of $L/a \to \infty$. Deriving w.r.t. a leads to

$$-a\frac{\partial}{\partial a}\mathcal{I}_n(a, L) = a_1 + 2a_2 \ln\frac{L}{a} + \cdots + Na_N\left(\ln\frac{L}{a}\right)^{N-1} . \tag{A.8}$$

On the level of the integral, this operation amounts to fixing the smallest distance to be a. Due to our normalizations, this is equivalent to fixing the both endpoints of this smallest distance. The integration over the remaining points has then to be done.

We now state a very important theorem for the integral over a function f at order $N-1$ loops: if $f(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N)$ is a homogeneous function of dimension $-2(N-1)$ (z and $\bar{z}$ have dimension 1), then the integral over $z_1, \ldots, z_{N-1}$ (the relative coordinates between points)

$$\mathcal{I}_N(a, L) := \int\limits_{z_1,\ldots,z_{N-1}} f(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N)\mathcal{C}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) \tag{A.9}$$

has dimension 0. Consider a sector $\mathcal{S}$ (ordering of the distances). Be $x_\alpha := |z_i - z_j|$, with $1 \leqslant \alpha \leqslant m := N(N-1)/2$. Then $\mathcal{S} := \{z_1, \ldots, \bar{z}_N\}$, s.t. $x_1 < x_2 < \cdots < x_m$. (Actually, we have chosen the labeling of the distances x_α to account for the ordering. This is not always the most practical thing to do.) Also define the characteristic function $\chi_{\mathcal{S}}(x_1, \ldots, x_m)$ of a sector $\mathcal{S}$ as being 1 if all distances satisfy the inequalities of the sector and 0 otherwise. The a-derivative of the integral restricted to the sector $\mathcal{S}$ is

$$\mathcal{J}^{\mathcal{S}} := -a\frac{\partial}{\partial a}\mathcal{I}_N^{\mathcal{S}}(a, L) = \int f(z_1, \ldots, \bar{z}_N)\big|_{x_1=a}\Theta(x_m < L)\chi_{\mathcal{S}}(x_1, \ldots, x_m). \tag{A.10}$$

The conformal mapping theorem [45–47], whose proof we reproduce below for completeness, now states that *if* the integral (A.10) is Riemann-integrable everywhere, then

$$\mathcal{J}^{\mathcal{S}} \equiv \int f(z_1, \ldots, \bar{z}_N)\big|_{x_i=a} \Theta(x_m/x_1 < L/a)\chi_S(x_1, \ldots, x_m). \tag{A.11}$$

In words: the above integral can be evaluated by fixing any of the distances to be a (or 1 equivalently). The constraint on the smallest and largest distances is captured by the condition that the ratio of largest to smallest distance is bounded by L/a, as it is in the original integral, which is thus just a special case of the expression (A.11).

Proof. First of all, since $x_1 = a$, and introducing a δ-function to enforce it, $\mathcal{J}^{\mathcal{S}}$ becomes

$$\mathcal{J}^{\mathcal{S}} = \int f(z_1, \ldots, \bar{z}_N)\delta(x_1 - a)\Theta(x_m/x_1 < L/a)\chi_{\mathcal{S}}(x_1, \ldots, x_m). \tag{A.12}$$

We now aim at integrating over distances $x_1, \ldots, x_m$ instead of coordinates with an arbitrary function g

$$\begin{aligned}&\int \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_{N-1}\, g(x_1, \ldots, x_m) \\ &\quad = \int \mathrm{d}x_1 \cdots \mathrm{d}x_m\, \mu(x_1, \ldots, x_m) g(x_1, \ldots, x_m).\end{aligned} \tag{A.13}$$

The measure is easily constructed as

$$\begin{aligned}&\mu(x_1, \ldots, x_m) \\ &\quad = \int \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_{N-1}\, \delta\big(x_1 - |z_1 - z_2|\big) \cdots \delta\big(x_m - |z_{N-1} - z_N|\big),\end{aligned} \tag{A.14}$$

where the δ-distributions enforce the x_i's to be the distances between the z_j's.

We now want to map onto $x_l = a$. To achieve this, we can always do the integration over x_l last. This gives for $\mathcal{J}^{\mathcal{S}}$

$$\begin{aligned}\mathcal{J}^{\mathcal{S}} = \int \mathrm{d}x_l \int \mathrm{d}x_1 \cdots \mathrm{d}x_{l-1}\, \mathrm{d}x_{l+1} \cdots \mathrm{d}x_m\, \mu(x_1, \ldots, x_m)\delta(x_1 - a) \\ \times f(x_1, \ldots, x_m)\Theta(x_m/x_1 < L/a)\chi_{\mathcal{S}}(x_1, \ldots, x_m).\end{aligned} \tag{A.15}$$

We now make a change of variables. For all i but l, set

$$x_i := \tilde{x}_i x_l/a. \tag{A.16}$$

We also define $\tilde{x}_l := a$, and introduce this into (A.15) as $1 = \int \mathrm{d}\tilde{x}_l\, \delta(\tilde{x}_l - a)$:

$$\begin{aligned}\mathcal{J}^{\mathcal{S}} = \int \mathrm{d}x_l \int \mathrm{d}\tilde{x}_1 \cdots \mathrm{d}\tilde{x}_m\, \mu(\tilde{x}_1, \ldots, \tilde{x}_m)\delta(\tilde{x}_l - a) \\ \times f(\tilde{x}_1, \ldots, \tilde{x}_m)\Theta(\tilde{x}_m/\tilde{x}_1 < L/a)\chi_{\mathcal{S}}(\tilde{x}_1, \ldots, \tilde{x}_m) \\ \times \delta(\tilde{x}_1 x_l - a)\frac{a}{x_l}.\end{aligned} \tag{A.17}$$

Note that the factor of a/x_l consists of $(x_l/a)^{N(N-1)/2-1}$ from the terms $\mathrm{d}\tilde{x}_i$ but $\mathrm{d}\tilde{x}_l$; a factor of $(x_l/a)^{(N-1)(2-N/2)}$ from the measure; and a factor of $(x_l/a)^{-2(N-1)}$ from f.

Using that

$$\int \mathrm{d}x_l\, \delta(\tilde{x}_1 x_l - a)\frac{a}{x_l} = 1, \tag{A.18}$$

we obtain

$$\mathcal{J}^{\mathcal{S}} = \int \mathrm{d}\tilde{x}_1 \cdots \mathrm{d}\tilde{x}_m\, \mu(\tilde{x}_1, \ldots, \tilde{x}_m)\delta(\tilde{x}_l - a) \times f(\tilde{x}_1, \ldots, \tilde{x}_m)\Theta(\tilde{x}_m/\tilde{x}_1 < L/a)\chi_{\mathcal{S}}(\tilde{x}_1, \ldots, \tilde{x}_m). \tag{A.19}$$

Dropping the tildes, this is nothing but (A.15) with x_1 replaced by x_l which completes the proof. □

Appendix B. The 4-loop integral

In this appendix we evaluate analytically the integral (4.20) needed in Section (4.6) to obtain the *universal* part of the 4-loop contribution to the β-function (i.e., the last term in (2.3)), with the result quoted in (4.21). The integrand of the integral (4.20) in question is

$$\mathcal{M} := \left(\frac{1}{z_{14}\bar{z}_{13}} - \frac{1}{\bar{z}_{14}z_{13}}\right)\left(\frac{1}{z_{24}\bar{z}_{23}} - \frac{1}{\bar{z}_{24}z_{23}}\right)\frac{1}{|z_{34}|^2}\left(\frac{1}{z_{15}\bar{z}_{25}} + \frac{1}{\bar{z}_{15}z_{25}}\right). \tag{B.1}$$

Graphically, this is depicted in Fig. 8. We observe that we can make the following simplification (due to the "second magic rule"):

$$\frac{1}{w\bar{u}} - \frac{1}{\bar{w}u} = \frac{\bar{w}u - w\bar{u}}{\bar{w}w\bar{u}u} = 2i\frac{\vec{w}\times\vec{u}}{|w|^2|u|^2} = 2i\frac{|\vec{w}-\vec{u}|h}{|w|^2|u|^2}, \tag{B.2}$$

where h is the height of the triangle spanned by $\vec{w}$ and $\vec{u}$; if the angle is larger than π, then h is negative (see Fig. 9).

Note that the first two factors of the integrand $\mathcal{M}$ both contribute a term $|z_{34}|$, thus canceling the third term $1/|z_{34}|^2$. This allows us to see that the integral has no subdivergences; it will contain only a "global divergence", i.e., it will be proportional to a single power of $\ln(L/a)$ (L and a are the IR and UV cutoffs, respectively). We now proceed to check this by explicit calculation and to compute the precise coefficient of the single logarithmic divergence. Let us now introduce distances as depicted in Fig. 10.

Here all distances are measured from 0 except for x' and z which are measured from their intersection point. In these conventions, x' and $-z$ in the figure are negative. The integrand can then be written as[14]

$$\mathcal{M} = (2i)^2\frac{y}{(y^2+x^2)(y^2+(x'+b)^2)}\frac{-z}{(z^2+x'^2)(z^2+(x+b)^2)} \times\left(\frac{1}{z_{15}\bar{z}_{25}} + \frac{1}{\bar{z}_{15}z_{25}}\right), \tag{B.3}$$

[14] There are four complex integration variables, equivalent to eight real integration variables. We make use of this equivalence whenever convenient.

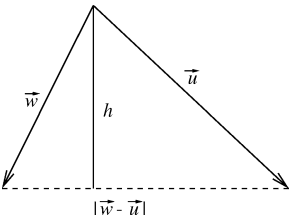


Fig. 9.

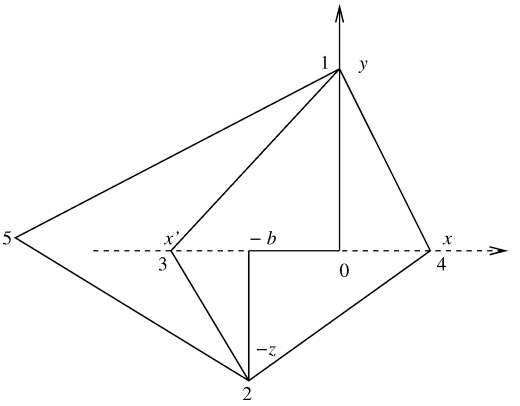


Fig. 10.

where all variables are to be integrated over. Let us first do the integrals over x, x', b, z and $z_5 = \Re(z_5) + i\Im(z_5)$, i.e., all distances except for y, which is kept *fixed and positive*. At the end, we integrate over the vector y, both over *its magnitude and direction*. (This fixes the coordinate system.) We note that choosing $a \ll |y| \ll L$, boundary terms can be neglected, since the integrals do not contain subdivergences, neither in the UV nor in the IR.

Doing first the integral over point 5, we obtain using (D.1) from Appendix D

$$\begin{aligned}\int \mathrm{d}^2 z_5 \left(\frac{1}{z_{15}\bar{z}_{25}} + \frac{1}{\bar{z}_{15} z_{25}} \right) &= -2\pi \ln\left(|z_{12}|^2\right) + \text{const.} \\ &= -2\pi \ln\left[(y+z)^2 + b^2\right] + \text{const}, \end{aligned} \tag{B.4}$$

where the constant depends on the IR-cutoff L. However, one easily sees that it drops from the above calculation due to the asymmetry of $z \to -z$ of the remaining terms in (B.3). We finally have to integrate:

$$-8\pi \frac{y}{(y^2+x^2)(y^2+(x'+b)^2)} \frac{z}{(z^2+x'^2)(z^2+(x+b)^2)} \ln\left[(y+z)^2+b^2\right]. \tag{B.5}$$

The simplest integrals are those over x and x'. We use

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{x^2+y^2} \frac{1}{(b+x)^2+z^2} = \frac{\pi(|y|+|z|)}{|yz|(b^2+(|y|+|z|)^2)}, \tag{B.6}$$

which can be done by residue-calculus. Integrating (B.5) over x and x' thus gives:

$$-8\pi^3 \frac{(|y|+|z|)^2}{yz(b^2+(|y|+|z|)^2)^2} \ln\big[(y+z)^2+b^2\big]. \tag{B.7}$$

To continue, we recall that by *construction* y (which is the module of a vector) is positive. (B.7) can thus be written as the integral over positive z only

$$-8\pi^3 \frac{(y+z)^2}{yz(b^2+(y+z)^2)^2} \big(\ln\big[(y+z)^2+b^2\big] - \ln\big[(y-z)^2+b^2\big]\big). \tag{B.8}$$

The easiest integral to do is that over b, which nevertheless is a little bit tricky. We need

$$\int_{-\infty}^{\infty} \frac{\ln(b^2+d^2)}{(b^2+s^2)^2}\,\mathrm{d}b = \frac{-\pi}{s^2(|d|+|s|)} + \frac{\pi \ln(|d|+|s|)}{|s|^3} \tag{B.9}$$

which can be verified with the help of the residue-theorem. To do so, one splits the $\ln(b^2+d^2) = \ln(b+i|d|) + \ln(b-i|d|)$ which both have branch-cuts. But the integral can be closed either in the upper or lower domain, and we close it in the domain where there is no branch-cut. This leaves us with

$$8\pi^4 \int_0^{\infty} \mathrm{d}z \frac{(y+z)^2}{yz} \left\{ \left[\frac{1}{(y+z)^2(2|y+z|)} - \frac{\ln(2|y+z|)}{(y+z)^3} \right] - \left[\frac{1}{(y+z)^2(|y-z|+|y+z|)} - \frac{\ln(|y-z|+|y+z|)}{|y+z|^3} \right] \right\}. \tag{B.10}$$

Scaling out y, and splitting the integral into domains where the absolute values have a definite sign gives

$$\begin{aligned}
&\frac{8\pi^4}{y^2} \int_0^{\infty} \mathrm{d}z \frac{1}{z} \left\{ \left[\frac{1}{2|1+z|} - \frac{\ln(2|1+z|)}{(1+z)} \right] - \left[\frac{1}{(|1-z|+|1+z|)} - \frac{\ln(|1-z|+|1+z|)}{|1+z|} \right] \right\} \\
&= \frac{8\pi^4}{y^2} \int_0^1 \mathrm{d}z \left\{ \left[\frac{1}{2z(1+z)} - \frac{\ln(2(1+z))}{z(1+z)} \right] - \left[\frac{1}{2z} - \frac{\ln(2)}{z(1+z)} \right] \right\} \\
&\quad + \frac{8\pi^4}{y^2} \int_1^{\infty} \mathrm{d}z \left\{ \left[\frac{1}{2z(1+z)} - \frac{\ln(2(1+z))}{z(1+z)} \right] - \left[\frac{1}{2z^2} - \frac{\ln(2z)}{z(1+z)} \right] \right\} \\
&= -\frac{2}{3}\pi^4 \big(6+\pi^2\big) \frac{1}{y^2}.
\end{aligned} \tag{B.11}$$

The final integral over y contains the integral over the modulus of y and its direction, which contributes a factor of 2π:

$$\int_a^L \mathrm{d}y\, 2\pi y \left[-\frac{2}{3}\pi^4 \left(6+\pi^2\right) \frac{1}{y^2} \right] = -\frac{4}{3}\pi^5 \left(6+\pi^2\right) \ln\left(\frac{L}{a}\right). \tag{B.12}$$

To conform to the normalizations used in the main text, see Eq. (3.2), this still has to be divided by $(2\pi)^4$, yielding the final result (with the integral running over all but one of the points, and normalizations according to Eq. (3.2))

$$\mathcal{I} := \int \mathcal{M} = -\frac{\pi}{12}\left(6+\pi^2\right) \ln\left(\frac{L}{a}\right) + \text{finite}. \tag{B.13}$$

We have indicated an additional finite term in the result, which depends on the specific regularization prescription, and which is either a constant or decays to 0 in the limit of $L/a \to \infty$.

Appendix C. Some remarks on group theory

In this appendix, we collect a number of useful group-theoretical identities, first in C.1 for a general Lie-group $\mathcal{G}$, then in C.2 for $SU(N)$, and finally in C.3 for the other classical groups, $SO(N)$ and $SP(N)$.

C.1. Group theoretical invariants

In this appendix we discuss the additional group theoretical invariant, referred to in the main text. Since we are using the current-algebra, only the adjoint representation of the symmetry group $\mathcal{G}$ appears in our calculations. Therefore, all group-theoretical invariants that can possibly appear, can all be constructed out the structure constants. The simplest such invariant is of course the eigenvalue of the quadratic Casimir invariant C_2 in the adjoint representation, which is of second order in structure constants f^{abc}. Here we consider invariants which are of higher order in the structure constants.

Notation: The zero modes $j^a := J_0^a = \oint (\mathrm{d}z/2\pi i) J^a(z)$ of the Kac–Moody currents [1] are the generators of the Lie-group $\mathcal{G}$, satisfying the commutation relations

$$\left[j^a, j^b\right] = f_c{}^{ab} j^c, \tag{C.1}$$

which are represented in the adjoint representation by matrices

$$\left(T^a\right)_c{}^b := f_c{}^{ab}. \tag{C.2}$$

We work with *antihermitean* generators j^a, so that the structure constants $f_c{}^{ab}$ are real.

The $\mathcal{G}$-invariant Killing form η^{ab}, and its inverse η_{bc}, defined by

$$\eta^{ab} := \frac{-1}{\mathcal{N}} \mathrm{tr}\left(T^a T^b\right), \qquad \eta^{ab}\eta_{bc} = \delta_c^a \tag{C.3}$$

may be used to raise and lower adjoint indices $a, b, \ldots$. Here $\mathcal{N}$ is a suitable normalization constant. Then, (C.1) and (C.3) imply that the structure constants $f^{cab} = f^{abc}$ are totally antisymmetric. Throughout this subsection, we choose a basis of the Lie algebra for which $\eta^{ab} = \delta^{ab}$. Hence, no distinction between upper and lower adjoint indices has to be made. (The matrices $(T^a)_c{}^b$ in (C.2) are then antihermitean.)

We now proceed to discuss various group-theoretical invariants, needed in the main text, which can be constructed out of products of structure constants. Our discussion is organized according to the number of factors f^{abc} appearing.

Quadratic Casimir: The eigenvalue C_2 of the quadratic Casimir invariant in the adjoint representation,[15]

$$\left(T^a T^a\right)_c{}^d = f_c{}^{ab} f_b{}^{ad} = -C_2 \delta_c{}^d \quad \Longleftrightarrow \quad f^{abc} f^{abd} = C_2 \delta^{cd} \tag{C.4}$$

is of 2nd order in the structure constants. Eq. (C.4) is graphically depicted in (4.9).

"Triangle rule": The Jacobi-identity implies the following relation for the structure constants:

$$f_e{}^{ad} f_d{}^{bc} + f_e{}^{bd} f_d{}^{ca} + f_e{}^{cd} f_d{}^{ab} = 0 \tag{C.5}$$

which is just (C.1):

$$\left(\left[T^a, T^b\right]\right)_e{}^c = f_d{}^{ab} \left(T^d\right)_e{}^c. \tag{C.6}$$

Multiplying (C.5) with $f^g{}_{ab}$, yields

$$\begin{aligned} 0 &= f^g{}_{ab} \left[f^{ead} f_d{}^{bc} + f^{ebd} f_d{}^{ca} + f^{ecd} f_d{}^{ab} \right] \\ &= -\mathrm{tr}\left(T^g T^c T^e\right) - \mathrm{tr}\left(T^c T^e T^g\right) + C_2 f^{ecg}. \end{aligned} \tag{C.7}$$

Using the cyclic invariance of the trace, this yields the "triangle rule"

$$\mathrm{tr}\left(T^g T^c T^e\right) = \frac{-1}{2} C_2 f^{gce}. \tag{C.8}$$

Eq. (C.8) is graphically depicted in (4.10).

Invariant 4-index tensor d^{abcd}: Next we consider the following totally symmetrized trace of four (adjoint) representation matrices

$$d^{abcd} := \mathrm{tr}\left(T^{\{a} T^b T^c T^{d\}}\right), \tag{C.9}$$

which is $\mathcal{G}$-invariant by construction. This invariant arises when considering traces of four matrices T, Eq. (C.2). The result is given in (C.12) below. To derive it, observe that for traces of more than three generators T, which cannot be reduced using (C.8), one can permute two T's, with the aim of creating a loop of 3 with the remaining T's, which, in turn, can then be reduced using (C.8). For a trace of four T's, this reads

$$\mathrm{tr}\left(T^a T^b T^c T^d\right) = -\frac{1}{2} C_2 f^{abh} f^{cdh} + \mathrm{tr}\left(T^b T^a T^c T^d\right). \tag{C.10}$$

[15] As usual, all repeated indices are summed.

This also tells us that

$$\mathrm{tr}\big(T^a T^b T^c T^d\big) = \mathrm{tr}\big(T^b T^a T^d T^c\big) = \mathrm{tr}\big(T^d T^c T^b T^a\big), \tag{C.11}$$

where the second relation is obtained using the cyclic invariance of the trace. We now want to calculate a general trace of four T's. First, by using (C.11), and the cyclic invariance of the trace, we find that of the 6 possible permutations, which leave the first index unchanged only 3 are independent. These are $K_1 = \mathrm{tr}(T^a T^b T^c T^d) = \mathrm{tr}(T^a T^d T^c T^b)$, $K_2 = \mathrm{tr}(T^a T^c T^d T^b) = \mathrm{tr}(T^a T^b T^d T^c)$, and $K_3 = \mathrm{tr}(T^a T^d T^b T^c) = \mathrm{tr}(T^a T^c T^b T^d)$. The totally symmetrized trace, defined in (C.9), can now be expressed in terms of the K_i as: $d^{abcd} = \frac{1}{3}(K_1 + K_2 + K_3)$. Writing $K_1 = d^{abcd} + \frac{1}{3}[(K_1 - K_2) + (K_1 - K_3)]$, and using (C.6), we can rewrite each of these terms with the help of d^{abcd} and f's as

$$\mathrm{tr}\big(T^a T^b T^c T^d\big) = d^{abcd} + \frac{C_2}{6}\big[f^{adh} f^{bch} - f^{abh} f^{cdh}\big]. \tag{C.12}$$

The invariant d_2 is now defined by

$$\frac{1}{\mathcal{N}_{\mathrm{ad}}} d^{abcd}\, d^{abcd} = \frac{C_2^4}{24} + d_2, \tag{C.13}$$

where $\mathcal{N}_{\mathrm{ad}}$ is the dimension of the adjoint representation. Note that (C.12), (C.13), (C.8) imply

$$\mathrm{tr}\big(T^a T^b T^c T^d\big)\mathrm{tr}\big(T^a T^b T^c T^d\big) = \mathcal{N}_{\mathrm{ad}}\left(\frac{C_2^4}{8} + d_2\right). \tag{C.14}$$

The l.h.s. can graphically be viewed as the "cube"-invariant, discussed in Section 2, and depicted in Fig. 3 of the same section (recall (C.2)).

In Section C.2, we show that for $SU(N)$

$$d_2 = \frac{3}{2}N^2. \tag{C.15}$$

(The quadratic Casimir is $C_2 = N$ in our conventions.) This is in agreement with the results of Ref. [34]. Hence, for $SU(N)$, the term d_2 in (C.13) is subleading in N, as compared to the first term. This subleading N-dependence of d_2 is also true for all the remaining classical groups, which follows from (C.13) and (C.27).

C.2. $SU(N)$

In this section we present a derivation of the value of the invariant d_2 for $\mathcal{G} = SU(N)$, i.e., of (C.15), which provides an independent check of this result given in Ref. [34].[16]

We start by recalling the generators in the (complexified) Lie algebra of $SU(N)$ in the *fundamental* representation

$$X^a \equiv X^{\alpha}_{\bar{\alpha}} := \{\text{matrix with 1 in column } \alpha,\ \text{row } \bar{\alpha};\ 0 \text{ elsewhere}\}$$
$$(\alpha, \bar{\alpha} = 1, \ldots, N), \tag{C.16}$$

[16] This method can also be used to calculate higher invariants [48].

where the adjoint index $a = \{^{\alpha}_{\bar{\alpha}}\}$. These satisfy

$$\left[X^{\alpha}_{\bar{\alpha}}, X^{\beta}_{\bar{\beta}}\right] = \delta^{\alpha}_{\bar{\beta}} X^{\beta}_{\bar{\alpha}} - \delta^{\beta}_{\bar{\alpha}} X^{\alpha}_{\bar{\beta}}, \tag{C.17}$$

which yields the structure constants in this basis

$$f_c{}^{ab} \equiv f^{\bar{\gamma}\alpha\beta}_{\gamma\bar{\alpha}\bar{\beta}} = \delta^{\bar{\gamma}}_{\bar{\alpha}}\delta^{\beta}_{\gamma}\delta^{\alpha}_{\bar{\beta}} - \delta^{\alpha}_{\gamma}\delta^{\bar{\gamma}}_{\bar{\beta}}\delta^{\beta}_{\bar{\alpha}}. \tag{C.18}$$

The Killing form is given by

$$\begin{aligned}\eta^{ab} \equiv \eta^{\alpha\beta}_{\bar{\alpha}\bar{\beta}} &:= \frac{-1}{N}\mathrm{tr}\left(T^a T^b\right) = \frac{-1}{N}\left(f_d{}^{ac} f_c{}^{bd}\right) = -2\left(\delta^{\alpha}_{\bar{\beta}}\delta^{\beta}_{\bar{\alpha}} - \frac{1}{N}\delta^{\alpha}_{\bar{\alpha}}\delta^{\beta}_{\bar{\beta}}\right) \\ &= -2(\text{ projector onto the adjoint})\end{aligned} \tag{C.19}$$

and its inverse

$$\eta_{ab} = \eta^{\bar{\alpha}\bar{\beta}}_{\alpha\beta} = \frac{-1}{2}\left(\delta^{\bar{\beta}}_{\alpha}\delta^{\bar{\alpha}}_{\beta} - \frac{1}{N}\delta^{\bar{\alpha}}_{\alpha}\delta^{\bar{\beta}}_{\beta}\right). \tag{C.20}$$

One easily finds

$$\eta_{ab}\eta^{ba} = (N-1)(N+1) = \mathcal{N}_{\mathrm{ad}} = \text{dimension of adjoint representation.} \tag{C.21}$$

Since we use η to raise and lower indices $a = \{^{\alpha}_{\bar{\alpha}}\}$ of structure constants which are traceless (see (C.18)), one can also use the simplified form

$$\eta_{ab} \to \eta^{\mathrm{simp}}_{ab} = \frac{-1}{2}\delta^{\bar{\beta}}_{\alpha}\delta^{\bar{\alpha}}_{\beta} \tag{C.22}$$

instead of η_{ab}, for calculational convenience. Writing $(T^a)_c{}^b = f_c{}^{ab}$ we obtain:[17]

$$\frac{1}{\mathcal{N}_{\mathrm{ad}}}\mathrm{tr}\left(T^a T^{a'}\right)\eta_{aa'} = -N, \tag{C.23}$$

$$\frac{1}{\mathcal{N}_{\mathrm{ad}}}\mathrm{tr}\left(T^a T^b\right)\mathrm{tr}\left(T^{a'} T^{b'}\right)\eta_{aa'}\eta_{bb'} = N^2, \tag{C.24}$$

$$\frac{1}{\mathcal{N}_{\mathrm{ad}}}\mathrm{tr}\left(T^a T^b T^c\right)\mathrm{tr}\left(T^{a'} T^{b'} T^{c'}\right)\eta_{aa'}\eta_{bb'}\eta_{cc'} = \frac{1}{4}N^3, \tag{C.25}$$

$$\frac{1}{\mathcal{N}_{\mathrm{ad}}}\mathrm{tr}\left(T^a T^b T^c T^d\right)\mathrm{tr}\left(T^{a'} T^{b'} T^{c'} T^{d'}\right)\eta_{aa'}\eta_{bb'}\eta_{cc'}\eta_{dd'} = \frac{1}{8}N^4 + \frac{3}{2}N^2. \tag{C.26}$$

Comparison of (C.26) with (C.14) yields $d_2 = \frac{3}{2}N^2$, in agreement with (C.14), and (C.15).

C.3. Other groups

Besides $SU(N)$ we also consider $SO(N)$ and $SP(N)$. The results of Ref. [49] yield:

$$SU(N): \quad C_2 = N, \qquad d_2 = \frac{3}{2}N^2,$$

[17] Again we use a computer to do the algebra.

$$SO(N): \quad C_2 = N-2, \qquad d_2 = 24 - \frac{45}{2}N + 6N^2 - \frac{3}{8}N^3,$$
$$SP(N): \quad C_2 = \frac{N+2}{2}, \qquad d_2 = \frac{3}{2} + \frac{45}{32}N + \frac{3}{8}N^2 + \frac{3}{128}N^3. \tag{C.27}$$

We have already quoted these values for the group theoretical invariants on Fig. 1, but repeat them here for the convenience of the reader. Note that one can always normalize the 1-loop coefficient in the β-function for g (the term $\propto g^2$ in (2.3)) to $1/2$, by rescaling g by a constant. This means that the normalization-invariant quantity which enters at 4-loop order is d_2/C_2^4. This allows us to perform the following checks on (C.27), by using well known isomorphism between the corresponding Lie algebras:

$$\left.\frac{d_2}{C_2^4}\right|_{SU(2)} = \left.\frac{d_2}{C_2^4}\right|_{SO(3)} = \frac{3}{8}, \tag{C.28}$$
$$\left.\frac{d_2}{C_2^4}\right|_{SO(5)} = \left.\frac{d_2}{C_2^4}\right|_{SP(4)} = \frac{13}{72}, \tag{C.29}$$
$$\left.\frac{d_2}{C_2^4}\right|_{SU(4)} = \left.\frac{d_2}{C_2^4}\right|_{SO(6)} = \frac{3}{32}, \tag{C.30}$$
$$\left.\frac{d_2}{C_2^4}\right|_{SO(-N)} = \left.\frac{d_2}{C_2^4}\right|_{SP(N)} = \frac{3[32+N(14+N)]}{8(2+N)^3}. \tag{C.31}$$

Appendix D. Some elementary integrals

In this appendix, we consider some elementary integrals, quoted in the main text. Consider two points z_a and z_b in the complex plane, which are well inside a circle of (large) radius R centered at the origin. It is then elementary to establish the following result:

$$\int\limits_{|\mathfrak{z}|\leqslant R} \mathrm{d}^2\mathfrak{z}\, \frac{1}{(\mathfrak{z}-z_a)(\mathfrak{z}^*-z_b^*)} = -\pi \ln|z_b - z_a|^2 + \pi \ln R^2 + \pi \ln\left[1 - \frac{z_a z_b^*}{R^2}\right]. \tag{D.1}$$

Furthermore, for $|\mathfrak{z}-z_a| \leqslant a \leqslant |z_a - z_b|$

$$\int\limits_{|\mathfrak{z}-z_a|\leqslant a} \mathrm{d}^2\mathfrak{z}\, \frac{1}{(\mathfrak{z}-z_a)(\mathfrak{z}^*-z_b^*)} = 0. \tag{D.2}$$

Finally, this implies upon taking the limit of $R \to \infty$,

$$\int\limits_{|\mathfrak{z}-z_a|,|\mathfrak{z}-z_b|\geqslant a} \mathrm{d}^2\mathfrak{z}\left[\frac{1}{(\mathfrak{z}-z_a)(\mathfrak{z}^*-z_b^*)} - \frac{1}{|\mathfrak{z}-z_a|^2}\right] = -2\pi \ln\left|\frac{z_a - z_b}{a}\right| \tag{D.3}$$

as long as $|z_a - z_b| \geqslant 2a$ (up to terms of order a^2 which are neglected).

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