An exact mapping of the stochastic field theory for Manna sandpiles to interfaces in random media

Pierre Le Doussal and Kay Jörg Wiese
CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France.

We show that the stochastic field theory for directed percolation in presence of an additional conservation law (the C-DP class) can be mapped *exactly* to the continuum theory for the depinning of an elastic interface in short-range correlated quenched disorder. On one line of parameters commonly studied, this mapping leads to the simplest overdamped dynamics. Away from this line, an additional memory term arises in the interface dynamics; we argue that it does not change the universality class. Since C-DP is believed to describe the Manna class of self-organized criticality, this shows that Manna stochastic sandpiles and disordered elastic interfaces (i.e. the quenched Edwards-Wilkinson model) share the same universal large-scale behavior.

Self-organized criticality (SOC) and scale-free avalanches arise in a variety of models: deterministic and stochastic sandpiles [1–6], propagation of epidemics [7], and elastic objects slowly driven in random media [8–14]. In the last decade several authors found evidence that most of these models belong to a small number of common universality classes. A unifying framework was proposed based on the theory of *absorbing phase transitions* (APT) [15, 16]. These are non-equilibrium phase transitions, which occur in a vast number of systems between an active state and one –or many– absorbing states. The generic universality class in the absence of additional symmetries or conservation laws is the *directed-percolation class* (DP) [18, 19]. The spreading exponents of the critical DP clusters are interpreted as avalanche exponents in the corresponding SOC system [16]. In presence of additional conservation laws, other classes may arise: An important one is the *conserved directed percolation* class (C-DP), with an infinite number of absorbing states. Indeed, it is now accepted, though unproven, that the continuum fluctuation theory for the C-DP class provides the effective field theory for the activity in Manna sandpiles [17].

Stochastic sandpiles are cellular automata where the toppling rule contains randomness which is renewed at each toppling. A notable example is the Manna model [2], a stochastic variant of the deterministic Bak-Tang-Wiesenfeld (BTW) sandpile [1]: *Randomly throw grains on a lattice. If the height at one point is* ≥ 2*, then move two grains from this site to randomly chosen neighboring sites.* This model is not Abelian, i.e. the order in which the sites are updated matters. An Abelian version was proposed by Dhar [4, 20]. Careful numerical studies [21–24], starting with Manna himself [2], showed that the Manna model and the BTW belong to different universality classes (see [5, 21, 25] for reviews). Coarse-grained evolution equations for the Manna class were proposed in [26, 27]. If $\rho(x,t)$ denotes the local activity of the sandpile, i.e. the local density of unstable sites, and $\phi(x,t)$ the local density of grains, then they obey the stochastic continuum equations for the C-DP class:

$$\begin{aligned}\partial_t \rho(x,t) =& a\rho(x,t) - b\rho(x,t)^2 + D_\rho \nabla^2 \rho(x,t) \\ & + \sigma\eta(x,t)\sqrt{\rho(x,t)} + \gamma\rho(x,t)\phi(x,t) \ , \qquad (1)\\ \partial_t \phi(x,t) =& (D_\phi \nabla^2 - m^2)\rho(x,t) \ . \qquad (2)\end{aligned}$$

The parameters b, D_ρ, D_ϕ are positive; $\eta(x,t)$ is a (centered) spacio-temporal white noise,

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta^d(x-x')\delta(t-t') \ . \qquad (3)$$

Clearly, $\rho(x) = 0$ with arbitrary "background" field $\phi(x,t)$ forms an infinite set of (time-independent) absorbing states. The field $\phi(x,t)$ encodes the likeliness of absorbing configurations to propagate activity when perturbed. From (2) ϕ is a conserved field for $m = 0$, reflecting conservation of the total number of grains. The derivation of (1)-(2) was made more precise in [28, 29], where it is claimed that all "stochastic models with an infinite number of absorbing states, in which the order-parameter evolution is coupled to a nondiffusive conserved field, define a unique universality class", the C-DP. This is further supported in [30, 31]. The C-DP class is believed to contain conserved lattice-gas models, conserved threshold-transfer processes, and others [16, 29, 42].

On the other hand, there were early conjectures that sandpile models and disordered elastic manifolds belong to the same universality classes: The first claim relates the BTW model and elastic manifolds driven in a *periodic* disorder, i.e. charge-density waves [9], reexamined recently [32]. It was soon followed by a conjecture [33] on the equivalence of the Oslo model [3] to an elastic string driven by its endpoint in a *non-periodic* quenched random field. The random field emerges from the stochastic noise in the rule. Finally, it was conjectured that Manna sandpiles are equivalent to interfaces in random media [34]. These claims are based on exact, or almost exact, mappings, onto elastic manifolds with highly discretized evolution rules, and it is not clear what such discretization does to the model (see e.g. [22] for discussion).

Quite naturally, it was then proposed that C-DP and depinning of an interface in a quenched random medium belong to the same universality class [22, 27, 30, 35, 36]. Until now, however, this remarkable claim is mainly based on the numerical coincidence of critical exponents in simulations of discrete models, believed to belong to the respective universality classes [17, 22]. Ideally, one wants a direct connection at the level of the continuum theories. The field theory of interfaces subject to disorder is well known, described by the functional RG (FRG) method, involving an infinite number (a function) of relevant couplings near its upper critical dimension $d_{\rm c} = 4$ [37]. It describes depinning [38–40] and, more recently, avalanches [12, 13, 41]. Hence one would like to relate it to the field theory of the C-DP class. Although it was realized that renormalization of the C-DP class is more complex than that of standard DP which requires only a few couplings, the attempts to handle it were unsuccessful [42, 43]. More intriguingly, the full renormalized disorder correlator was measured numerically [44], and found indistinguishable from that of random interfaces obtained in [45].

The aim of this Letter is to provide an exact mapping in the continuum, between the C-DP class defined by Eqs. (1) and (2), and an interface driven in quenched disorder, with a specific, exponentially decaying, microscopic disorder correlator. Along a line in parameter space it maps C-DP to the simplest overdamped dynamics of the interface, thereby providing the long-sought proof of equivalence of the two systems. Away from this line, the dynamics of the interface is more complex, and involves a memory kernel. As we show, it nevertheless falls into the same universality class as the simplest overdamped model, i.e. quenched Edward-Wilkinson (QEW).

Let us start by considering the two coupled equations of motion (1) and (2). For convenience we added a parameter m^2, since it appears in the interface model as an infrared regulator. Although we are interested in the limit $m \to 0$, it is useful to define the theory with $m > 0$, since this insures that the activity $\rho(x,t)$ will stop, even without grains leaving the system, which therefore can be taken infinitely large. To simplify the identification, note that by rescaling of space we can set $D_\rho \to 1$. By rescaling $\phi(x,t)$, we can then set $D_\phi \to 1$. Finally rescaling both $\rho(x,t)$ and $\phi(x,t)$, we can set $\sigma \to 1$. This simplifies the model to

$$\begin{aligned}\partial_t \rho(x,t) =& a\rho(x,t) - b\rho(x,t)^2 + \nabla^2 \rho(x,t)\\ &+\eta(x,t)\sqrt{\rho(x,t)} + \gamma\rho(x,t)\phi(x,t) \qquad (4)\\ \partial_t \phi(x,t) =& (\nabla^2 - m^2)\rho(x,t)\ . \qquad (5)\end{aligned}$$

The activity variable $\rho(x,t) \geq 0$ for all times [57]. Note that the case $\gamma = 0$, with $b > 0$, corresponds to the field theory of directed percolation: In the absence of noise, i.e. in mean field, it exhibits a transition between $\rho > 0$ for $a > 0$ and $\rho = 0$ for $a \leq 0$. This transition exists in any d. The noise $\eta(x,t)$ becomes relevant for $d \leq d_{\rm c} = 4$, a property shared by DP and C-DP; the latter has $\gamma > 0$ which we now examine.

As we will see below, the case $\gamma = b$ is special. We therefore set $b := \gamma + \kappa$. We define new variables, a *force* $\mathcal{F}(x,t)$ and a *velocity* $\dot{u}(x,t)$ (denoting ∂_t or a dot time derivatives):

$$\mathcal{F}(x,t) := \rho(x,t) - \phi(x,t) - \frac{a+m^2}{\gamma}\ , \qquad (6)$$

$$\rho(x,t) := \dot{u}(x,t)\ . \qquad (7)$$

The total number of topplings at position x since $t = 0$ is $u(x,t) - u(x,t=0) = \int_0^t {\rm d}t' \rho(x,t)$. The identification of u as a height for the associated elastic interface is standard [44], while the identification of $\mathcal{F}$ as a "force" is new. Clearly, the initial value of the field $u(x,t=0)$ does not carry any information for the C-DP problem, while it does for the interface problem [58]. For notational simplicity we will set $u(x,t=0) = 0$. All our results can be extended to the general case by replacing $u(x,t) \to u(x,t) - u(x,t=0)$. The equations of motion for $\mathcal{F}(x,t)$ and $\dot{u}(x,t)$ then are

$$\begin{aligned}\partial_t \mathcal{F}(x,t) =& -\gamma\mathcal{F}(x,t)\dot{u}(x,t) - \kappa\dot{u}(x,t)^2\\ &+\eta(x,t)\sqrt{\dot{u}(x,t)}\ , \qquad (8)\\ \partial_t \dot{u}(x,t) =& [\nabla^2 - m^2]\dot{u}(x,t) + \partial_t \mathcal{F}(x,t)\ . \qquad (9)\end{aligned}$$

The problem is defined with initial data $\dot{u}(x,t=0)$ and $\mathcal{F}(x,t=0)$. The second equation (9) can be integrated into

$$\partial_t u(x,t) = [\nabla^2 - m^2]u(x,t) + \mathcal{F}(x,t) + f(x)\ , \qquad (10)$$

$$f(x) = \dot{u}(x,0) - \mathcal{F}(x,0) = \phi(x,0) + \frac{a+m^2}{\gamma}\ . \qquad (11)$$

Eq. (10) describes the motion of an elastic interface submitted to a known time-independent external force $f(x)$, and a space-time dependent force $\mathcal{F}(x,t)$. Because of the term m^2, the interface also sees a quadratic well. Integration of Eq. (5) shows that the change in the background field, $\phi(x,t) - \phi(x,0)$, can be interpreted as the sum of the elastic force plus the force from the quadratic well, acting on the interface. Eq. (8) determines $\mathcal{F}(x,t)$ *as a functional* of the field $u(x,t)$. It is a stochastic functional, depending on the noise, and is formally written as $\mathcal{F}(x,t) \equiv \mathcal{F}[u,\eta](x,t)$. Once $\mathcal{F}(x,t)$ is known, substituting it into Eq. (10) defines a problem of an elastic manifold in a random medium. As we show now, $\mathcal{F}(x,t)$ can be written *explicitly*. Eq. (8) is linear in $\mathcal{F}$ with two source terms, hence its solution is

$$\mathcal{F}(x,t) = e^{-\gamma u(x,t)}\mathcal{F}(x,t=0) + \mathcal{F}_{\rm dis}(x,t) + \mathcal{F}_{\rm ret}(x,t)\ . \qquad (12)$$

The first term depends on the initial condition, and decays to zero if the interface moves by more than $1/\gamma$; it can thus be ignored in the steady state. As we now show, the second term can be interpreted as a quenched random pinning force. It arises from the noise in Eq. (8), and is independent of κ. It is the only term when $\kappa = 0$ (then $\mathcal{F}_{\rm ret} = 0$) i.e. for $\gamma = b$. This term can be written as $\mathcal{F}_{\rm dis}(x,t) = F\big(u(x,t),x\big)$, where for each x, $F(u,x)$ is an Orstein-Uhlenbeck process [46], solution of the stochastic equation

$$\partial_u F(u,x) = -\gamma F(u,x) + \tilde{\eta}(x,u)\ , \qquad (13)$$

with initial data $F(0,x) = 0$, and $\tilde{\eta}(x,u)$ a white noise, uncorrelated in x and u. A pedestrian way to derive Eq. (13) is to write the white noise $\eta(x,t) = {\rm d}B_x(t)/{\rm d}t$ in Eq. (8) in terms of independent one-sided Brownians $B_x(t)$ indexed by x, with $B_x(0) = 0$, and integrate the linear equation as

$$\begin{aligned}\mathcal{F}_{\rm dis}(x,t) =& \int_0^t {\rm d}t' \frac{{\rm d}B_x(t')}{{\rm d}t'}\sqrt{\dot{u}(x,t')}e^{-\gamma[u(x,t)-u(x,t')]}\\ =& e^{-\gamma u(x,t)}\int_0^{u(x,t)} e^{\gamma u}{\rm d}\tilde{B}_x(u) = F\big(u(x,t),x\big)\ . \qquad (14)\end{aligned}$$

The force $F(u,x)$ is the solution of the Orstein-Uhlenbeck process (13) in terms of the white noises $\tilde{\eta}(x,u) =$

$\mathrm{d}\tilde{B}_x(u)/\mathrm{d}x$. It can be written as a (time-changed) Brownian,

$$F(u,x) = \frac{e^{-\gamma u}}{\sqrt{2\gamma}} \tilde{B}_x\big(e^{2\gamma u} - 1\big) \ . \qquad (15)$$

The second line in (14) is obtained by noting that under a time change $\mathrm{d}u = \dot{u}(x,t)\mathrm{d}t$ each Brownian $B_x(t)$ is changed into another Brownian $\tilde{B}_x(u)$ with $\tilde{B}_x(0) = 0$, as $\sqrt{\dot{u}(x,t')}\mathrm{d}B_x(t') = \mathrm{d}\tilde{B}_x\big(u(x,t')\big)$. Eq. (15) is obtained using the identity $\int_0^v f(u)\mathrm{d}B_x(u) = \tilde{B}_x(\int_0^v f(u)^2\mathrm{d}u)$ for test functions $f(u)$, resulting from the scale invariance of Brownian motion.

Hence, neglecting the first (decaying) term in Eq. (12), we showed that along the line $\gamma = b$ the C-DP system maps onto

$$\partial_t u(x,t) = [\nabla^2 - m^2]u(x,t) + F(u(x,t),x) + f(x) \ . \quad (16)$$

This is an interface driven in a quenched random force field $F(u,x)$. This random field is Gaussian, specified by its correlator, which can be calculated from (15), using $\overline{B_x(u)B_{x'}(u')} = \delta(x-x')\min(u,u')$. As expected, one finds that the Orstein-Uhlenbeck process becomes stationary when the interface has been driven on distances larger than $1/\gamma$:

$$\overline{F(u,x)F(u',x')} = \delta^d(x-x')\frac{\mathrm{e}^{-\gamma|u-u'|} - \mathrm{e}^{-\gamma(u+u')}}{2\gamma} \rightarrow_{\gamma u, \gamma u' \gg 1} \delta^d(x-x')\Delta_0(u-u') \quad (17)$$

with $\overline{F(u,x)} = 0$. The *bare disorder correlator* of the random pinning force thus is

$$\Delta_0(u) = \frac{\mathrm{e}^{-\gamma|u|}}{2\gamma} \ . \qquad (18)$$

It is clearly short-ranged, and as a peculiarity has a linear cusp. Usually one considers smooth microscopic disorder, i.e. an analytic $\Delta_0(u)$, which under RG (i.e. coarse graining) develops a cusp linked to the existence of many metastable states and avalanches beyond the Larkin scale $L_\mathrm{c} \sim 1/m_\mathrm{c}$ [59]. A cusp in the microscopic disorder means that there are avalanches of arbitrarily small sizes. On the other hand, it is known from the universality of the interface problem that any short-ranged force-force correlator flows at large scale, under coarse-graining, to the same renormalized disorder correlator, the universal *depinning* fixed point [59]. Its upper critical dimension is $d_\mathrm{c} = 4$, implying that C-DP also has $d_\mathrm{c} = 4$. The fixed-point function has been calculated analytically in an $\varepsilon = d_\mathrm{c} - d$ expansion [40] and measured numerically [45]. It determines the two independent exponents of the depinning transition, the roughness exponent ζ of the field $u \sim L^\zeta$, $\zeta > 0$ for $d < d_\mathrm{c}$, and the dynamic exponent z, $t \sim L^z$, $z < 2$ for $d < d_\mathrm{c}$, and their ε-expansions [40].

Let us now discuss the correspondence between the active-absorbing phase transitions for C-DP and depinning. For simplicity consider a spatially uniform initial condition $\phi(x,t=0) = \phi$, s.t. the initial driving force acting on the interface in Eq. (11) is uniform, $f(x) = f$. We now set the control parameter $m \to 0$ so that there is a globally active phase corresponding to an interface moving at constant steady-state mean velocity $\overline{\dot{u}(x,t)} = v \sim (f - f_\mathrm{c})^\beta > 0$, if $f > f_\mathrm{c}$. Here f_c is the depinning threshold force, which is, at least in principle, calculable once the correlator Δ_0 is known. Translating to the C-DP system it implies an active phase with $\rho > 0$, when $a + \gamma\phi > \gamma f_\mathrm{c}$, and a phase transition where ρ vanishes with the same exponent β as a function of the distance to criticality. Due to a symmetry of the interface problem, $\beta = \nu(z-\zeta) = \frac{z-\zeta}{2-\zeta}$. By scaling this gives $\rho = \dot{u} \sim t^{-\theta}$ at criticality with $\theta = 1 - \frac{\zeta}{z}$, e.g. as response to a (large) specially uniform perturbation at $t = 0^+$, in the limit of $v \to 0^+$. In the language of APT [16] this is a *steady-state exponent*.

Let us now consider the protocol for avalanches in the absorbing phase, near criticality. In the sandpile model (e.g. in numerical simulations of the Manna model) one usually starts from an initial condition with a non-vanishing activity $\rho(x,0) = \dot{u}(x,0) \geq 0$, either by adding a single grain, or adding grains in an extended region. This generates an avalanche which stops when $\rho(x,t) = 0$ for all x. For the elastic manifold it is equivalent to having the interface at rest up to time $t = 0$, and then to increase the force by $\dot{u}(x,0)$. This procedure is then repeated until one reaches the steady state (for $u(x,t) \gg 1/\gamma$), where the avalanche statistics becomes stationary. It is known for interfaces that under this procedure the system reaches the *Middleton attractor*, a sequence of well-characterized metastable states between successive avalanches [47]. Avalanches with this statistics have well-defined exponents, which were discussed e.g. in [13, 48, 49].

To summarize, along the line $\gamma = b$, i.e. $\kappa = 0$, we presented an exact and direct mapping (valid in any dimension) between the continuum C-DP Eqs. (1)-(2) and the simplest model of a driven interface with overdamped dynamics, subject to a *quenched* random force $F(u(x,t),x)$ with (microscopic) correlations given by Eq. (18), and confined in a parabolic well of curvature m^2. This confirms, and makes precise, the beautiful numerical study of Ref. [44]; there the authors observe that Manna sandpiles, the Oslo model, C-DP as given by Eqs. (1)-(2), and disordered elastic manifolds have the same renormalized (effective) disorder correlator. If one accepts that the Manna class coincides with C-DP, it establishes the long sought exact mapping to disordered elastic manifolds [60]. This agreement is valid in the stationary state, but our exact mapping establishes the complete correspondence, and allows to study the evolution from any given initial state.

Some remarks are in order. The interface equation (16) with the choice of correlator (17) possesses a special Markovian property, which it inherits from the force evolution equation (8) (for $\kappa = 0$), and which allows it to be solved without storing the full random-force landscape. The latter is constructed as the avalanche proceeds, hence is determined only for $u \leq u(x,t)$. This property was noted in [49, 50] and can be used for efficient numerics [49, 51].

The limit $\gamma \to 0$ is also of interest. If one keeps $\kappa = 0$, i.e. $b \to 0$, one sees from (11) and (12) that in that limit

$$\dot{u}(x,t)-\dot{u}(x,t=0) = [\nabla^2 - m^2]u(x,t) + \tilde{B}_x\big(u(x,t)\big) \quad (19)$$

This is the so-called Brownian force model (BFM), the mean-field theory for avalanches of the interface model [13, 52, 53]. If we keep $b > 0$, the limit instead is towards the DP class.

Let us finally discuss the C-DP system for $\kappa \neq 0$, i.e. away from the special line $\gamma = b$ in Eq. (4). If the new source term $\kappa \dot{u}^2$, which appears in Eq. (8) for $\partial_t \mathcal{F}$, were directly inserted into Eq. (9) for $\dot{u}$, the mapping to the interface would fail, as such a term is relevant [61]. Fortunately, this term is *screened* by the short-range disorder, and instead of being relevant is only marginal. To show this, let us come back to Eq. (12), which now has a second contribution,

$$\mathcal{F}_{\rm ret}(x,t) = -\kappa \int_0^t {\rm d}t'\, \dot{u}(x,t')^2\, e^{-\gamma[u(x,t)-u(x,t')]} \ . \quad (20)$$

This term can be rewritten using integration by parts as

$$\mathcal{F}_{\rm ret}(x,t) = \frac{\kappa}{\gamma} e^{-\gamma u(x,t)} \left[\dot{u}(x,0) + \int_0^t {\rm d}t'\, \ddot{u}(x,t')\, e^{\gamma u(x,t')} \right] - \frac{\kappa}{\gamma} \dot{u}(x,t) \ . \quad (21)$$

Inserting into Eq. (10) we finally obtain the equation of motion

$$\begin{aligned} \frac{b}{\gamma}\partial_t u(x,t) =& [\nabla^2 - m^2]u(x,t) + F(u(x,t),x) + f(x) \\ &+ \frac{\kappa}{\gamma} \int_0^t {\rm d}t' \ddot{u}(x,t') e^{-\gamma[u(x,t)-u(x,t')]} \\ &+ \left[\frac{b}{\gamma}\dot{u}(x,0) - f(x)\right] e^{-\gamma u(x,t)} \ . \end{aligned} \quad (22)$$

We recall $f(x) = \phi(x,0) + \frac{a+m^2}{\gamma}$. This equation is equivalent to the C-DP system (1)-(2) for $\rho(x,t) = \dot{u}(x,t)$ with specified initial data $\dot{u}(x,0)$, $\phi(x,0)$, and is a salient result of our letter. Note that it results from a simple change of variables, which maps a system with *annealed noise*, the C-DP, to a system with *quenched noise*, the interface; as such it bears some analogy to the Cole-Hopf transformation used to solve the Kardar-Parisi-Zhang (KPZ) equation.

Let us now discuss Eq. (22). The first line describes the standard overdamped equation of motion of the interface, with the same random force $F(u,x)$ as before, but a new friction coefficient b/γ. The third line depends on the initial condition and decays to zero when the interface has moved by more than $1/\gamma$. As we now focus on the stationary regime we can neglect it. The second line is a new memory term. To estimate its relevance at large scales, consider the large-γ limit, and replace $e^{-\gamma z} \to \frac{1}{\gamma}\delta(z)$, hence $e^{-\gamma[u(x,t)-u(x,t')]} \to \frac{1}{\gamma \dot{u}(x,t)}\delta(t-t')$. The second line of (22) then becomes

$$\frac{\kappa}{\gamma^2}\partial_t \ln \dot{u}(x,t) + \mathcal{O}(\gamma^{-3}) \quad (23)$$

where each power of $1/\gamma$ in the expansion yields terms which are more and more irrelevant by power counting, since each power of $1/\gamma$ comes with a power of $1/u \sim L^{-\zeta}$. This argument indicates that the new term is marginally irrelevant, and only shifts the numerical value of the friction. Hence we conclude that the universality class of C-DPand of the QEW model should be the same, even for $b \neq \gamma$.

The present work calls for further studies: First, Eq. (22) can be analyzed using FRG to confirm our conclusions and explore this unusual interface dynamics. Our work opens the way to study, within a common RG framework, a variety of models ranging from interfaces to absorbing phase transitions. It can be extended to long-range elasticity (long-range toppling), or to a variety of perturbations. The simplest one is to add $m^2 \dot{w}(x,t)$ to each of the Eqs. (1)-(2) in order to reproduce the standard driving for the interface [12]. Another extension is the crossover to DP as both γ and b are small.

Second, Eq. (22) permits to study initial conditions, hence to disentangle effects dominated by transients from those of the Middleton attractor. That allows to treat avalanches with localized seeds in the context of APTs, used to define *spreading exponents*. E.g. the survival probability in C-DP, $P^{\rm surv}_{\text{C-DP}}(t) \sim t^{-\delta}$ is related to the avalanche-duration distribution at depinning, $P_{\rm dep}(T) \sim T^{-\alpha}$, via $\delta = \alpha - 1 = (d-2+\zeta)/z$. We checked that indeed $\delta = 0.17$ and 0.48 in $d = 1$ and 2, both for depinning, see table 2 of [48], and Manna sandpiles [17, 62].

Third, since our mapping is local in space, it can be extended to finite-size systems with prescribed boundary conditions, in order to study the case $m = 0$. Imposing $\rho(x,t) = \phi(x,t) = 0$ at the boundary corresponds to the common choice to let grains "fall off" from the boundary. In our variables it implies $u(x,t) = \dot{\mathcal{F}}(x,t) = 0$ at the boundary.

Finally one should go back to Ref. [44], and understand cusps in a more general setting. A challenging questions is whether the quenched KPZ class can be treated in a similar setting, especially since some of its exponents in $d = 1$ are described by DP. Other unsolved problems, such as DP with quenched disorder [16] may now be studied.

In conclusion, we have provided an *exact mapping* from the field theory of a reaction-diffusion system with an additional conservation law, the C-DP system of Eqs. (1)-(2), to a specific continuum model of an interface driven in a random landscape. Using universality we show that the C-DP class, Manna stochastic sandpiles and the quenched Edwards-Wilkinson model belong to a single, and hence *very large* universality class which spans self-organized criticality, avalanches in disordered systems, and reaction-diffusion models. This raises the prospect of a unified field theory for all these systems using functional RG methods. It also defines a framework, in which probabilists could put this claim on *rigorous* grounds, as was recently done for the KPZ class [64, 65].

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[57] If the activity vanishes at a point, i.e. $\rho(x_0,t) = 0$, then $\nabla^2\rho(x,t) \geq 0$, which in the interface language corresponds to the Middleton theorem (no-crossing rule) [47].
[58] The arbitrariness in this choice leads to a symmetry, the statistical tilt symmetry (STS), which constrains the exponents. The elasticity is not renormalized and $\nu = 1/(2-\zeta)$.
[59] One defines a renormalized disorder correlator $\Delta_m(u)$ (as in [54–56]), which obeys an FRG equation with initial condition $\Delta_{m=\infty}(u) = \Delta_0(u)$. In the limit $m \to 0$ it takes the fixed-point scaling form $\Delta_m(u) \to m^{\varepsilon-2\zeta}\tilde{\Delta}^*(um^\zeta)$ with a linear cusp. The RG scale L is related to m via $L = L_m = 1/m$.
[60] Note that for the interface model (8)-(11) it is well known that the Middleton property [47] applies. This implies in particular that the order in which the interface is kicked does not matter, and it stops in the same Middleton state. The same property holds true for abelian sandpiles. While it is not exact at the discrete level in the original Manna model, it seems to hold in the continuum effective C-DP model, at least for $\kappa = 0$.
[61] $\kappa\dot{u}^2 \sim L^{2\zeta-2z}$ is more relevant than $\nabla^2\dot{u} \sim L^{\zeta-z-2}$ at large scales, since $\zeta + 2 - z > 0$ for $d < d_c$.
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