Stability of Random-Field and Random-Anisotropy Fixed Points at large N

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In this note, we clarify the stability of the large-N functional RG fixed points of the order/disorder transition in the random-field (RF) and random-anisotropy (RA) $O(N)$ models. We carefully distinguish between infinite N, and large but finite N. For infinite N, the Schwarz-Soffer inequality does not give a useful bound, and all fixed points found in Phys. Rev. Lett. 96, 197202 (2006) (cond-mat/0510344) correspond to physical disorder. For large but finite N (i.e. to first order in $1/N$) the non-analytic RF fixed point becomes unstable, and the disorder flows to an analytic fixed point characterized by dimensional reduction. However, for random anisotropy the fixed point remains non-analytic (i.e. exhibits a cusp) and is stable in the $1/N$ expansion, while the corresponding dimensional-reduction fixed point is unstable. In this case the Schwarz-Soffer inequality does not constrain the 2-point spin correlation. We compute the critical exponents of this new fixed point in a series in $1/N$ and to 2-loop order.

The random field (RF) and random anisotropy (RA) N-vector model is studied by expanding around the 4-dimensional non-linear σ-model [1]. To this aim consider $O(N)$ classical spins $\vec{n}(x)$ with N components and of unit norm $\vec{n}^2 = 1$. To describe disorder-averaged correlations one introduces replicas $\vec{n}_a(x)$, $a = 1, \ldots, k$, the limit $k = 0$ being implicit everywhere. This gives a non-linear sigma model, of partition function $\mathcal{Z} = \int \mathcal{D}[\pi]\, \mathrm{e}^{-\mathcal{S}[\pi]}$ and action:

$$\mathcal{S}[\pi] = \int \mathrm{d}^d x \,\Big[\frac{1}{2T_0} \sum_a [(\nabla \vec{\pi}_a)^2 + (\nabla \sigma_a)^2] - \frac{1}{T_0} \sum_a M_0 \sigma_a - \frac{1}{2T_0^2} \sum_{ab} \hat{R}_0(\vec{n}_a \vec{n}_b) \Big] \,, \tag{1}$$

where $\vec{n}_a = (\sigma_a, \vec{\pi}_a)$ with $\sigma_a(x) = \sqrt{1 - \vec{\pi}_a(x)^2}$. A small uniform external field $\sim M_0(1, \vec{0})$ acts as an infrared cut-off. The ferromagnetic exchange produces the 1-replica part, while the random field yields the 2-replica term $\hat{R}_0(z) = z$ for a bare Gaussian RF. Random anisotropy corresponds to $\hat{R}_0(z) = z^2$. As shown in [1] one must include a full function $\hat{R}_0(z)$, as it is generated under RG. It is marginal in $d = 4$.

Recently, we have obtained results at 2-loop order [3], and large N for the ferromagnetic to disorder transition. In Ref. [2] the authors argue that the large-N fixed points obtained by us (given after Eq. (10) in [3]) are unstable. Here we reply to their argument.

The authors of Ref. [2] correctly point out that the Schwartz-Soffer (SS) inequalities [4] put useful constraints on the phase diagram of the *random-field* $O(N)$ model and its (subtle) dependence in N. In our Letter [3] we have studied the Functional RG at large N and obtained a series of fixed points indexed by $n = 2, 3 \ldots$ where the disorder correlator $\hat{R}(z)$ (notations of [3]) has a non-analyticity at $z = 1$. The $n = 2$ fixed point (FP) has random field symmetry (RF) and $n = 3$ has random anisotropy (RA) symmetry ($\hat{R}(z)$ even in z). In addition we found two infinite-N analytic fixed points which obey dimensional reduction. One of them ($\hat{R}(z) = z - 1/2$) is the large-N limit of the Tarjus-Tissier (TT) FP [5] which exists for $N > N^*$ (at two loop we found $N^* = 18 - \frac{49}{5}\bar{\epsilon}$, $\bar{\epsilon} = d - 4 \geq 0$) and has a weaker and weaker "subcusp" non-analyticity as N increases. The question is which of these FPs describes the ferromagnetic/disordered (FD) transition at large N for $d \geq 4$.

First one should carefully distinguish: *(i)* strictly infinite $N = \infty$ from large but finite N, *(ii)* RF symmetry vs. RA. We have shown [3, 6] that for RF at $N = \infty$ physical initial conditions on the critical FD manifold converge to the $n = 2$ FP if the bare disorder is strong enough ($r_4 > 4$ in [3]). Hence for $N = \infty$ all these non-analytic (NA) FPs are consistent. One can indeed check that they correspond to a positive probability distribution of the disorder since all $\hat{R}^{(n)}(0)$, the variances of the corresponding random fields and anisotropies, are positive – a condition hereby referred to as physical. Furthermore the SS inequality does not yield any useful constraint at $N = \infty$ because it contains an amplitude itself proportional to $\sqrt{N}$.

Next, each of the above FPs can be followed down to finite N, within an $1/N$ expansion performed to a high order in Ref. [6, 7]. It yields (to first order in $\bar{\epsilon} = d - 4$) the critical exponents $\bar{\eta}(n, N)$ and $\eta(n, N)$ to high orders in $1/N$. One finds that the $n = 2$ FP acquires a *negative* $\hat{R}'(0)$ at order $1/N$, $\hat{R}'(0) = -\frac{3}{4}\frac{\bar{\epsilon}}{N^2} + O(\frac{1}{N^3})$; hence it becomes unphysical at finite N, a fact consistent with the violation of the SS inequality $\bar{\eta} \leq 2\eta$ correctly pointed out in [2]. A natural scenario for RF symmetry, as we indicated in our Letter [3], is that the FRG flows to the TT FP for any *finite* $N > N^*$. However, as we discussed there, if bare disorder is strong enough, it may approach the TT FP along a NA direction, since these arguments relied only on blowing up of $R''''(0)$ ($R(\phi) = \hat{R}(z = \cos(\phi))$).

A very interesting point, missed in Ref. [2], is that the SS inequalities do not constrain the 2-point function of the spin $S^i(x)$ for *random anisotropy* disorder (it only constrains the 2-point function of $\chi_{ij}(x) = S^i(x) S^j(x)$ as disorder couples to the latter). Furthermore we find [6, 7] that the $n = 3$ random anisotropy FP (which reads $NR(\phi)/|\epsilon| = \frac{9}{8}\big(2\cos(\phi)\cos(\frac{\phi+\pi}{3}) + \cos(\frac{\pi - 2\phi}{3}) - 1\big)$ in the $N = \infty$ limit) *remains physical* for finite N. Denoting $\hat{R}(z) = \bar{\epsilon}\mu\tilde{R}(z)$ with $\mu = \frac{1}{N-2}$ and $y_0 = \tilde{R}'(1)$, we obtain the following expansion to $O(\bar\epsilon)$ for the exponents $\eta = y_0\bar\epsilon/(N-2)$, $\bar\eta = (\frac{N-1}{N-2}y_0 - 1)\bar\epsilon$, where

$$y_0 = \frac{3}{2} + 23\mu - \frac{1750\mu^2}{3} + \frac{2129692\mu^3}{27} - \frac{13386562376\mu^4}{1215} + \frac{2004388412086052\mu^5}{1148175} - \frac{1074239336335145945598\mu^6}{361675125} + \frac{66496428379374257425781597\mu^7}{1253204308125} + O\left(\mu^9\right) \quad (2)$$

and all coefficients in the expansion of $\hat R^{(n)}(0)$ near $z=0$ remain indeed positive, e.g.:

$$\tilde R'(z) = \Big[\frac{70\mu}{9}+1\Big]z + \Big[\frac{1192\mu}{243}+\frac{4}{27}\Big]z^3 + \Big[\frac{4384\mu}{2187}+\frac{16}{243}\Big]z^5 + \Big[\frac{68608\mu}{59049}+\frac{256}{6561}\Big]z^7 + \Big[\frac{3735040\mu}{4782969}+\frac{14080}{531441}\Big]z^9 + O(z^{11})$$

Finally, for the $1/N$ expansion of the *analytic* (DR) FP corresponding to RA we obtain (with $y_0 = 1$):

$$\tilde R(z) = \frac{z^2}{2} + \left(-\frac{3}{2} + 4z^2 - 2z^4\right)\mu + \dots\,, \quad (3)$$

hence it becomes *unphysical* at finite N [8]. The scenario is thus the opposite of the RF case: The NA FP $n=3$ is the only one physical at large N (it exists for $N > N_c = 9.44121$) and has precisely one unstable eigenvector (within the RA symmetry) as expected for the FD transition. Using our 2-loop result [3] we further obtained, up to $O\big(\mu^2\big)$: $y_0 = \frac{3}{2} + 23\mu + \big(9\gamma_a - \frac{97}{4}\big)\mu\bar\epsilon$, $\eta = \mu(\frac{3}{2}\bar\epsilon + \bar\epsilon^2(3\gamma_a - \frac{27}{8}))$ and $\bar\eta = \frac{\epsilon}{2} + \mu(\frac{49}{2}\bar\epsilon + \bar\epsilon^2(9\gamma_a - \frac{203}{8}))$, where γ_a was defined in [3].

Our conclusion is thus that the random anisotropy FP smoothly matches to our solution $n=3$ at $N=\infty$ and remains non-analytic for all N, breaking dimensional reduction. It does not exhibit the TT phenomenon which seems a peculiarity of the RF class. It is further studied in [6, 7].

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[8] Computing $R''''(0)$ one finds that it exists for $N > N^*$ and corresponds to the *unstable* line in Fig. 3 of [3].
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