INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **17** (2005) S1889–S1898 doi:10.1088/0953-8984/17/20/016

Supersymmetry breaking in disordered systems and relation to functional renormalization and replica-symmetry breaking

Kay Jörg Wiese

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France

Received 12 November 2004, in final form 1 January 2005
Published 6 May 2005
Online at stacks.iop.org/JPhysCM/17/S1889

Abstract

In this article, we study an elastic manifold in quenched disorder in the limit of zero temperature. Naively it is equivalent to a free theory with elasticity in Fourier space proportional to k^4 instead of k^2, i.e. a model without disorder in two space dimensions less. This phenomenon, called dimensional reduction, is most elegantly obtained using supersymmetry. However, scaling arguments suggest, and functional renormalization shows, that dimensional reduction breaks down beyond the Larkin length. Thus one equivalently expects a breakdown of supersymmetry. Using methods of functional renormalization, we show how supersymmetry is broken. We also discuss the relation to replica-symmetry breaking, and how our formulation can be put into work to lift apparent ambiguities in standard functional renormalization group calculations.

1. Introduction

The statistical mechanics of even well understood physical systems subjected to quenched disorder still poses major challenges. For a large class of these systems, e.g. random-field models or elastic manifolds in quenched disorder, an apparent simplification appears: supposing that all moments of the disorder are finite, one can show that all correlation functions in the disordered model, in the limit of zero temperature, are equivalent to those of the pure system at finite temperature in two space dimensions less, at a temperature proportional to the second moment of the quenched disorder. This phenomenon is called dimensional reduction (DR) [1]. The most elegant way to prove it is to use the supersymmetry approach [2], as we will detail below. However, one also knows that dimensional reduction gives the wrong result at large scales, more precisely at scales larger than the Larkin length. The latter is obtained from an Imry–Ma type argument due to Larkin, balancing elastic energy and disorder,

as we detail below. For a d-dimensional elastic manifold in quenched disorder, the elastic and disorder energy are

$$E_{\rm el}[u] = \int {\rm d}^d x \, \tfrac{1}{2}(\nabla u(x))^2, \qquad E_{\rm DO}[u] = \int {\rm d}^d x \, V(x, u(x)). \tag{1}$$

For $d = 1$, these are polymers, for which a lot is known [3]; for $d = 2$ membranes; and for $d = 3$ elastic crystals, e.g. charge–density waves. For simplicity we consider disorder which at the microscopic scale is Gaussian and short ranged with second moment

$$\overline{V(x,u)V(x',u')} = \delta^d(x - x')R(u - u'). \tag{2}$$

Long-range correlated disorder $R(u)$ is possible, and leads in general to a different universality class. This will play no role in the following. The most important observable is the roughness exponent ζ, which describes the scaling of the two-point function

$$\overline{[u(x) - u(x')]^2} \sim |x - x'|^{2\zeta}. \tag{3}$$

The Larkin argument compares, as a function of system size L, elastic energy $E_{\rm el} \sim L^{d-2}$ and disorder energy $E_{\rm DO} \sim L^{d/2}$ to conclude that in dimensions smaller than four disorder always wins at large scales, leading to an RG-flow to strong coupling (in a way to be made more precise below). This suggests that the dimensional reduction result, derived below via the supersymmetry method,

$$\overline{u_k u_{-k}} = \frac{-R''(0)}{(k^2)^2} \qquad \Longrightarrow \qquad \zeta_{\rm DR} = \frac{4-d}{2} \tag{4}$$

will become incorrect below four dimensions.

2. The functional RG treatment

In this section we review some important points of the functional RG treatment, which will facilitate the derivation of the corresponding formulae in the supersymmetric treatment. Functional RG was first introduced in [4, 5], and pioneered for the problem at hand in [6–9], to cite the earliest contributions. Important improvements [10–30] have been obtained by several authors; see [31] for a more detailed introduction and review.

The Larkin argument suggests that four is the upper critical dimension and that an ϵ-expansion [3] with

$$\epsilon = 4 - d \tag{5}$$

about dimension four is possible. Taking the dimensional reduction result (4) in $d = 4$ dimensions tells us that the field u is dimensionless. Thus, the width $\sigma = -R''(0)$ of the disorder is not the only relevant coupling at small ϵ, but any function of u has the same scaling dimension in the limit of $\epsilon = 0$, and might thus equivalently contribute. The natural consequence is that one has to follow the full function $R(u)$ under renormalization, instead of just its second moment $R''(0)$. Such an RG-treatment is most easily implemented in the replica approach: The n times replicated partition function becomes after averaging over disorder

$$\overline{\exp\Big(-\frac{1}{T}\sum_{a=1}^{n} E_{\rm el}[u_a] - \frac{1}{T}\sum_{a=1}^{n} E_{\rm DO}[u_a]\Big)} = \exp\Big(-\frac{1}{T}\sum_{a=1}^{n} E_{\rm el}[u_a] + \frac{1}{2T^2}\sum_{a,b=1}^{n}\int {\rm d}^d x \, R(u_a(x) - u_b(x))\Big). \tag{6}$$

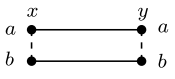
Perturbation theory is constructed along the following lines (see [11, 26] for more details). The bare correlation function, graphically depicted as a solid line, is with momentum k flowing through and replicas a and b

$$a \quad \text{———} \quad b \;=\; \frac{T\delta_{ab}}{k^2}\ . \tag{7}$$

The disorder vertex is

$$\begin{matrix} x & a \\ \bullet & \vdots \\ \bullet & b \end{matrix} \;=\; \int_x \sum_{a,b} R\Big(u_a(x) - u_b(x)\Big)\ . \tag{8}$$

The rules of the game are to find all contributions which correct R, and which survive in the limit of $T \to 0$. At leading order, i.e. order R^2, counting of factors of T shows that only the terms with one or two correlators contribute. On the other hand, $\sum_{a,b} R(u_a - u_b)$ has two independent sums over replicas. Thus at order R^2 four independent sums over replicas appear, and in order to reduce them to two one needs at least two correlators (each contributing a δ_{ab}). Thus, at leading order, only diagrams with two propagators survive. These are the following (denoting by $C(x-y)$ the Fourier transform of $1/k^2$):



$$= \int_x R''(u_a(x) - u_b(x))R''(u_a(y) - u_b(y))C(x-y)^2 \tag{9}$$

$$= -\int_x R''(u_a(x) - u_a(x))R''(u_a(y) - u_b(y))C(x-y)^2\ . \tag{10}$$

In a renormalization program, we are looking for the divergences of these diagrams. These divergences are localized at $x = y$, which allows us to approximate $R''(u_a(y) - u_b(y))$ by $R''(u_a(x) - u_b(x))$. The integral $\int_{x-y} C(x-y)^2 = \int_k \frac{1}{(k^2+m^2)^2} = \frac{m^{-\epsilon}}{\epsilon}$ (using the most convenient normalization for $\int_k$) is the standard one-loop diagram, which we have chosen to regulate in the infrared by a mass, i.e. physically by a harmonic well which is seen by the manifold.

Note that the following diagram also contains two correlators (correct counting in powers of temperature), but is not a two-replica but a three-replica sum:

. (11)

Taking into account the combinatorial factors, and a rescaling of R (which remember has dimension ϵ for a dimensionless field u) as well as of the field u (its dimension being the roughness exponent ζ), we arrive at

$$-m\frac{\partial}{\partial m}R(u) = (\epsilon - 4\zeta)R(u) + \zeta R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0). \tag{12}$$

Note that the elasticity is not renormalized due to the statistical tilt symmetry $u(x) \to u(x) + \alpha x$.

The crucial observation is that, when starting with smooth microscopic disorder, integration of the RG-equation leads to a cusp in the second derivative of the renormalized disorder at the Larkin length, as depicted in figure 1. This can easily be seen from the flow-equation of the fourth derivative (supposing analyticity), which from (12) is obtained as

$$-m\frac{\partial}{\partial m}R''''(0) = \epsilon R''''(0) + 3R''''(0)^2. \tag{13}$$

(Note that this explains also the appearance of the combination $\epsilon - 4\zeta$ in (12).) This equation has a singularity $R''''(0) = \infty$ after a finite renormalization time, equivalent to the appearance

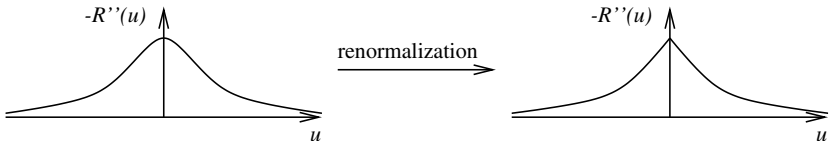


Figure 1. Change of $-R''(u)$ under renormalization and formation of the cusp.

of the cusp, as depicted in figure 1. After this dimensional reduction (4) is no longer valid. This can most easily be seen from the flow of $R''(0)$: Deriving (12) twice wrt u, and then taking the limit of $u \to 0$, leads to

$$-\frac{\partial}{\partial m} R''(0) = (\epsilon - 2\zeta) R''(0) + R'''(0^+)^2. \tag{14}$$

In the analytic regime $R'''(0^+) = 0$, such that the fixed-point condition $-\frac{\partial}{\partial m} R''(0) = 0$ implies $\zeta = \frac{\epsilon}{2} \equiv \frac{4-d}{2}$; after appearance of the cusp, $R'''(0^+) \neq 0$, thus ζ has to change.

This analysis can be continued to higher orders. Let us cite some key results at two-loop order [14, 26], for which the RG-equation reads

$$\begin{aligned} -m\frac{\partial}{\partial m} R(u) = {} & (\epsilon - 4\zeta) R(u) + \zeta u R'(u) + \frac{1}{2} R''(u)^2 - R''(u) R''(0) \\ & + \tfrac{1}{2}(R''(u) - R''(0)) R'''(u)^2 - \tfrac{1}{2} R'''(0^+)^2 R''(u). \end{aligned} \tag{15}$$

Different microscopic disorder leads to different RG fixed points. The latter are solutions of equation (15), with $-m\frac{\partial}{\partial m} R(u) = 0$; it is important to note that given a microscopic disorder, the exponent ζ, solution of (15) is unique. For random-bond disorder (short-ranged potential–potential correlation function) the result is $\zeta = 0.208\,298\,04\epsilon + 0.006\,858\epsilon^2$. In the case of random field disorder (short-ranged force–force correlations) $\zeta = \frac{\epsilon}{3}$. Both results compare well with numerical simulations.

One should also note that (15) contains a rather peculiar 'anomalous term', namely $R'''(0^+)^2$, which only appears after the occurrence of the cusp. This term is in general hard to get, since the calculation naturally gives factors of $R'''(0)$, which are zero by parity, and not $R'''(0^+) = -R'''(0^-)$. Several procedures to overcome these apparent ambiguities have been developed [26]. Supersymmetry will allow for another prescription, as will be discussed below.

3. Supersymmetry and its breaking

Another way to average over disorder is to use additional fermionic degrees of freedom. It is more commonly referred to as the supersymmetric method. Supersymmetry is manifest using one copy of the system, where it immediately leads to dimensional reduction, as we show below. However, it cannot account for the non-trivial physics due to the appearance of the cusp and the corresponding breakdown of dimensional reduction, and supersymmetry. This is possible when considering $n \neq 1$ copies of the system, with $n = 2$ being completely legitimate. Here we give a general formulation, in which one can either discard the fermionic degrees of freedom, thus reconstructing a replica formulation at $n = 0$, or set n to e.g. $n = 1$, thus exploring supersymmetry.

Define

$$\mathcal{H}[u_a, j_a, V] = \sum_{a=1}^{n} \int_x \tfrac{1}{2}(\nabla u_a(x))^2 + V(x, u_a(x)) + j_a(x) u_a(x). \tag{16}$$

Then the normalized generating function of correlation functions for a given disorder V is

$$\mathcal{Z}[j] := \frac{\int \prod_a \mathcal{D}[u_a] \mathrm{e}^{-\frac{1}{T}\mathcal{H}[u_a, j_a, V]}}{\int \prod_a \mathcal{D}[u_a] \mathrm{e}^{-\frac{1}{T}\mathcal{H}[u_a, 0, V]}}. \tag{17}$$

In the limit of $T \to 0$ only configurations which minimize the energy survive; these configurations satisfy $\frac{\delta \mathcal{H}[u_a, j_a, V]}{\delta u_a(x)} = 0$, of which we want to insert a δ-distribution in the path-integral. This has to be accompanied by a factor of $\det[\frac{\delta^2 \mathcal{H}[u_a, j_a, V]}{\delta u_a(x) \delta u_a(y)}]$, such that the integral over this configuration is normalized to unity, and supposing only a single configuration the denominator can be dropped, leading to

$$\mathcal{Z}[j] = \int \prod_a \mathcal{D}[u_a] \, \delta\left[\frac{\delta \mathcal{H}[u_a, j_a, V]}{\delta u_a(x)}\right] \det\left[\frac{\delta^2 \mathcal{H}[u_a, 0, V]}{\delta u_a(x) \delta u_a(y)}\right]. \tag{18}$$

Note that we neglect problems due to multiple minima, maxima, or saddle points. These configurations are incorrectly contained in (18), and are usually blamed for the failure of the supersymmetry approach. We will comment on this point later. For the moment, we continue with (18) and see how far we can get. Using an imaginary auxiliary field $\tilde{u}(x)$ and two anticommuting Grassmann fields $\bar{\psi}(x)$ and $\psi(x)$ (per replica), this can be written as

$$\begin{aligned}\mathcal{Z}[j] = \int \prod_a \mathcal{D}[u_a] \mathcal{D}[\tilde{u}_a] \mathcal{D}[\bar{\psi}_a] \mathcal{D}[\psi_a] \\ \exp\left[-\int_x \tilde{u}_a(x) \frac{\delta \mathcal{H}[u_a, j_a, V]}{\delta u_a(x)} + \bar{\psi}_a(x) \frac{\delta^2 \mathcal{H}[u_a, j_a, V]}{\delta u_a(x) \delta u_a(y)} \psi_a(y)\right].\end{aligned} \tag{19}$$

Averaging over disorder yields with the force–force correlator $\Delta(u) := -R''(u)$

$$\begin{aligned}\overline{\mathcal{Z}[j]} = \int \prod_a \mathcal{D}[u_a] \mathcal{D}[\tilde{u}_a] \mathcal{D}[\bar{\psi}_a] \mathcal{D}[\psi_a] \exp(-\mathcal{S}[u_a, \tilde{u}_a, \bar{\psi}_a, \psi_a, j_a]) \\ \mathcal{S}[u_a, \tilde{u}_a, \bar{\psi}_a, \psi_a, j_a] = \sum_a \int_x \tilde{u}_a(x)(-\nabla^2 u_a(x) + j_a(x)) + \bar{\psi}_a(x)(-\nabla^2)\psi_a(x) \\ - \sum_{a,b} \int_x \big[\tfrac{1}{2} \tilde{u}_a(x) \Delta(u_a(x) - u_b(x)) \tilde{u}_b(x) \\ + \tfrac{1}{2} \bar{\psi}_a(x) \psi_a(x) \Delta''(u_a(x) - u_b(x)) \bar{\psi}_b(x) \psi_b(x) \\ - \tilde{u}_a(x) \Delta'(u_a(x) - u_b(x)) \bar{\psi}_b(x) \psi_b(x)\big].\end{aligned} \tag{20}$$

We first analyse $n = 1$. Suppose that $\Delta(u)$ is even and analytic to start with, then only the following terms survive from (20):

$$\mathcal{S}_{\mathrm{Susy}}[u, \tilde{u}, \bar{\psi}, \psi, j] = \int_x \tilde{u}(x)(-\nabla^2 u(x) + j(x)) + \bar{\psi}(x)(-\nabla^2)\psi(x) - \tfrac{1}{2}\tilde{u}(x)\Delta(0)\tilde{u}(x). \tag{21}$$

(We have used the fact that $\bar{\psi}_a^2 = \psi_a^2 = 0$ to get rid of the four-fermion-term.) This action possesses a supersymmetry, which is most manifest when grouping terms together into a superfield

$$U(x, \bar{\Theta}, \Theta) = u(x) + \bar{\Theta}\psi(x) + \bar{\psi}(x)\Theta + \Theta\bar{\Theta}\tilde{u}(x). \tag{22}$$

The action (21) can then be written with the super-Laplacian Δ_{s} as

$$\mathcal{S}_{\mathrm{Susy}} = \int \mathrm{d}\Theta \mathrm{d}\bar{\Theta} \int_x U(x, \bar{\Theta}, \Theta)(\Delta_{\mathrm{s}}) U(x, \bar{\Theta}, \Theta), \qquad \Delta_{\mathrm{s}} := \nabla^2 - \Delta(0) \frac{\partial}{\partial \bar{\Theta}} \frac{\partial}{\partial \Theta} \tag{23}$$

and is invariant under the action of the supergenerators

$$Q := x\frac{\partial}{\partial\Theta} - \frac{2}{\Delta(0)}\bar{\Theta}\nabla, \qquad \bar{Q} := x\frac{\partial}{\partial\bar{\Theta}} + \frac{2}{\Delta(0)}\Theta\nabla. \tag{24}$$

Since 'bosons' u and $\tilde{u}$, and 'fermions' $\bar{\psi}$ and ψ, only appear to quadratic order, all expectation values are trivially Gaussian. In particular, we have

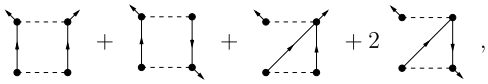
$$\overline{u_k u_{-k}} = \frac{\Delta(0)}{(k^2)^2}, \tag{25}$$

which is the result cited in (4), recalling that $\Delta(u) = -R''(u)$. Thus $\overline{[u(x)-u(y)]^2} \sim \Delta(0)|x-y|^{4-d}$, which should be compared to the thermal average $\langle[u(x)-u(y)]^2\rangle \sim T|x-y|^{2-d}$. Since both theories are Gaussian, the only difference is an apparent shift in the dimension of the system from d to $d-2$. This is usually referred to as dimensional reduction.

For more than $n = 1$ replicas, the theory is richer, and we will recover the renormalization of $\Delta(u)$ itself. With this purpose, write

$$\begin{aligned}\mathcal{S}[u_a, \tilde{u}_a, \bar{\psi}_a, \psi_a, j_a] = \sum_a \int_x \big[&\tilde{u}_a(x)(-\nabla^2 u_a(x) + j_a(x)) + \bar{\psi}_a(x)(-\nabla^2)\psi_a(x) \\ &- \tfrac{1}{2}\tilde{u}_a(x)\Delta(0)\tilde{u}_a(x)\big] - \sum_{a\neq b}\int_x \big[\tfrac{1}{2}\tilde{u}_a(x)\Delta(u_a(x)-u_b(x))\tilde{u}_b(x) \\ &+ \tfrac{1}{2}\bar{\psi}_a(x)\psi_a(x)\Delta''(u_a(x)-u_b(x))\bar{\psi}_b(x)\psi_b(x) \\ &- \tilde{u}_a(x)\Delta'(u_a(x)-u_b(x))\bar{\psi}_b(x)\psi_b(x)\big].\end{aligned} \tag{26}$$

Corrections to $\Delta(u)$ are easily constructed by remarking that the interaction term quadratic in $\tilde{u}$ is almost identical to the treatment of the dynamics in the static limit (i.e. after integration over times)



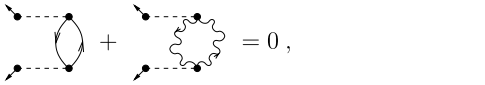
(27)

where an arrow indicates the correlation-function, $x \longrightarrow y = \langle\tilde{u}(x)u(y)\rangle = C(x-y)$. This leads to (in the order given above)

$$\delta\Delta(u) = \big[-\Delta(u)\Delta''(u) - \Delta'(u)^2 + \Delta''(u)\Delta(0)\big]\int_{x-y} C(x-y)^2 \tag{28}$$

where the last term (being odd in u) vanishes. Note that this reproduces the non-linear terms in (12).

A non-trivial ingredient is the cancellation of the acausal loop in the dynamics (the 'sloop', or three-replica term in the replica formulation) [26]. This is provided by taking two terms proportional to $\tilde{u}_a\Delta'(u_a-u_b)\bar{\psi}_b\psi_b$, and contracting all fermions:



(29)

since the fermionic loop (oriented wiggly line in the second diagram) contributes a factor of -1.

One can treat the interacting theory completely in a superspace formulation. The action is

$$\begin{aligned}\mathcal{S}[U_a] = \sum_a \int_{\Theta,\bar{\Theta}}\int_x U_a(x,\bar{\Theta},\Theta)(\Delta_s)U_a(x,\bar{\Theta},\Theta) \\ - \tfrac{1}{2}\sum_{a\neq b}\int_x\int_{\bar{\Theta},\Theta}\int_{\bar{\Theta}',\Theta'} R(U_a(x,\bar{\Theta},\Theta) - U_b(x,\bar{\Theta}',\Theta')).\end{aligned} \tag{30}$$

Thus non-locality in replica-space or in time is replaced by non-locality in superspace, or more precisely in its anticommuting component. Corrections to $R(u)$ all stem from 'superdiagrams', which result in bilobal interactions in superspace, not trilocal, or higher. The latter find their equivalent in three-local terms in replica-space in the replica formulation, and three-local terms in time in the dynamic formulation.

Supersymmetry is broken once $\Delta(0)$ changes. However, a new, let us call it 'effective supersymmetry' or 'scale-dependent supersymmetry' appears, in which the parameter $\Delta(0)$, which appears in the Susy-transformation, changes with scale, according to equation (14).

An interesting question is whether anomalous terms, proportional to $R'''(0^+)$, can be recovered from the supersymmetric formulation. We show now that this can indeed be done, in a very elegant way. The trick is to shift the disorder $V(u)$ which appears in (19) for the bosonic part, by a small amount δ for the fermionic part, and to take the limit of $\delta \to 0$ at the end. This modifies (20) to

$$\begin{aligned}
&\mathcal{S}[u_a, j_a, \tilde{u}_a, \bar{\psi}_a, \psi_a] \\
&\quad = \sum_a \int_x \tilde{u}_a(x)(-\nabla^2 u_a(x) + j_a(x)) + \bar{\psi}_a(x)(-\nabla^2)\psi_a(x) \\
&\qquad + \sum_{a,b} \int_x \big[\tfrac{1}{2}\tilde{u}_a(x)\Delta(u_a(x) - u_b(x))\tilde{u}_b(x) + \tfrac{1}{2}\bar{\psi}_a(x)\psi_a(x)\Delta''(u_a(x) - u_b(x)) \\
&\qquad \times \bar{\psi}_b(x)\psi_b(x) - \tilde{u}_a(x)\Delta'(\delta + u_a(x) - u_b(x))\bar{\psi}_b(x)\psi_b(x)\big] \qquad (31) \\
&\quad = \sum_a \int_x \big[\tilde{u}_a(x)(-\nabla^2 u_a(x) + j_a(x)) + \bar{\psi}_a(x)(-\nabla^2)\psi_a(x) + \tfrac{1}{2}\tilde{u}_a(x)^2\Delta(0) \\
&\qquad - \tilde{u}_a(x)\Delta'(\delta)\bar{\psi}_a(x)\psi_a(x)\big] \\
&\qquad + \sum_{a\neq b} \int_x \big[\tfrac{1}{2}\tilde{u}_a(x)\Delta(u_a(x) - u_b(x))\tilde{u}_b(x) + \tfrac{1}{2}\bar{\psi}_a(x)\psi_a(x)\Delta''(u_a(x) \\
&\qquad - u_b(x))\bar{\psi}_b(x)\psi_b(x) - \tilde{u}_a(x)\Delta'(\delta + u_a(x) - u_b(x))\bar{\psi}_b(x)\psi_b(x)\big]. \qquad (32)
\end{aligned}$$

Now the term with $a = b$ is also well defined; the last term of (31), as made explicit in (32), is of the form

$$\int_x \sum_a -\tilde{u}_a(x)\Delta'(\delta)\bar{\psi}_a(x)\psi_a(x) - \sum_{a\neq b} \tilde{u}_a(x)\Delta'(\delta + u_a(x) - u_b(x))\bar{\psi}_b(x)\psi_b(x). \qquad (33)$$

To demonstrate how this can be put into use, let us calculate the contribution to the two-point function at one-loop order, which is naively ambiguous [14, 19, 26]:

$$\begin{aligned}
\delta_{1\text{loop}} \langle u_a(q)u_a(-q)\rangle \;&=\; \\
&= -\frac{1}{(q^2+m^2)^2}\,\Delta'(\delta)^2 \int_p \frac{1}{(p+q/2)^2+m^2}\,\frac{1}{(p-q/2)^2+m^2}\,. \qquad (34)
\end{aligned}$$

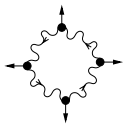
Note that the minus-sign comes from the closed fermion loop. In the limit of $\delta \to 0$, this gives

$$\delta_{1\text{loop}}\langle u_a(q)u_a(-q)\rangle = -\frac{1}{(q^2+m^2)^2}\Delta'(0^+)^2 \int_p \frac{1}{(p+q/2)^2+m^2}\,\frac{1}{(p-q/2)^2+m^2}. \qquad (35)$$

This may also prove useful for an N-component field $\vec{u} = \{u_a\}, a = 1 \ldots N$, a case which poses additional difficulties, since derivatives have to be taken in a given direction [11, 32].

Even higher correlation functions are immediate. For example, we have

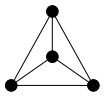
$$\Gamma_{\tilde{u}_a\tilde{u}_a\tilde{u}_a\tilde{u}_a} = \quad = -\frac{\Delta'(0^+)^4}{64}\int_q \frac{1}{(q^2+m^2)^4}\,. \tag{36}$$



This has first been obtained, using straightforward dynamical perturbation theory in [24]. While the principle is simple, the calculations are actually very cumbersome. The sloop method [26] is another efficient approach. (See also [30].)

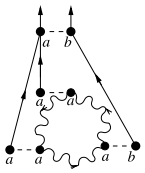
Let us finally mention that the method correctly calculates the anomalous term of the 'Mercedes-star' diagram at three-loop order [33],

$$\delta R(u) = \tfrac{1}{2}\left[R'''(u)^4 - R'''(u)^2 R'''(0^+)^2\right] \quad , \tag{37}$$



where the icon stands for the momentum integral only. The correction to $\Delta(u) = -R''(u)$, and picking the term proportional to $\Delta'''(u) = -R^{(5)}(u)$ comes from

$$\delta\Delta(u) = \cdots + \quad = \left[\cdots + \Delta'(0^+)^2\Delta'(u)\Delta'''(u)\right] \quad . \tag{38}$$



Note that two of the vertices are at argument $\Delta'(0^+)$. This can otherwise only be calculated within the sloop method.

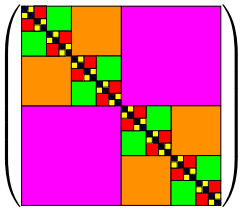
4. Relation between supersymmetry breaking, functional RG, and replica-symmetry breaking

Another popular approach to disordered systems is the replica-variational method, invoking replica-symmetry breaking (RSB). This method has for the problem at hand been developed in [34]. It consists in making the replacement

$$\sum_{a,b} \tilde{u}_a(x)\Delta(u_a(x) - u_b(x))\tilde{u}_b(x) \longrightarrow \sum_{a,b} \tilde{u}_a(x)\tilde{u}_b(x)\sigma_{ab}. \tag{39}$$

This approximation is valid in the limit of a field u with an infinite number N of components, so for the following discussion we have to restrain ourselves to that limit. The variational replica approach then makes an ansatz for σ_{ab}, with different correlations σ_{ab} between different pairs of replicas. Finally, a variational scheme is used to find an optimal σ_{ab}. The result (in the case of long-range correlated disorder, where the comparison can be made [27]) is a hierarchic matrix with an infinite number of different parameters, of the form

$$\sigma = \quad . \tag{40}$$



The exact form, and how it can be parametrized by a continuous function $[\sigma](z), 0 \leqslant z \leqslant 1$ is not of importance for the following.

What is important is that RSB appears exactly at the same moment as the cusp appears in the functional RG [35, 18, 27]. Moreover, there is a precise relation between the two-point function calculated by the RSB and FRG methods [18, 27]. The latter do not coincide, since the calculations are implicitly done in zero external field (RSB) and vanishing external field (FRG), leading to physically different situations (as for a standard ferromagnet).

In section 3 we have shown that this is also the moment when supersymmetry is broken. While the treatment there was for a one-component field, the conclusion is the same for an N-component field, and persists in the limit of $N \to \infty$. We can thus conclude that the breaking of supersymmetry and of replica symmetry and the appearance of the cusp are all but different manifestations of the same underlying physical principle: the appearance of multiple minima.

Acknowledgments

It is a pleasure to thank Andreas Ludwig for stimulating discussions on the supersymmetric method.

The work on the functional RG is part of a series of inspiring ongoing collaborations with Pierre Le Doussal, for which I would like to express my gratitude.

I am most grateful to Lothar Schäfer for years of continuous support, for always being available for discussion and for persisting until each problem had been clarified. His enthusiasm even at the oddest hours was an enormous encouragement. I will always be in his debt.

References

[1] Efetov K B and Larkin A I 1977 *Sov. Phys.—JETP* **45** 1236
[2] Parisi G and Sourlas N 1979 Random magnetic fields, supersymmetry, and negative dimensions *Phys. Rev. Lett.* **43** 744–5
[3] Schäfer L 1999 *Excluded Volume Effects in Polymer Solutions* (Berlin: Springer)
[4] Wilson K and Kogut J 1974 The renormalization group and the ε-expansion *Phys. Rep.* **12** 75–200
[5] Wegner F J and Houghton A 1973 Renormalization group equation for critical phenomena *Phys. Rev.* A **8** 401–12
[6] Fisher D S 1985 Random fields, random anisotropies, nonlinear sigma models and dimensional reduction *Phys. Rev.* B **31** 7233–51
[7] Fisher D S 1986 Interface fluctuations in disordered systems: $5 - \epsilon$ expansion *Phys. Rev. Lett.* **56** 1964–97
[8] Nattermann T, Stepanow S, Tang L H and Leschhorn H 1992 Dynamics of interface depinning in a disordered medium *J. Physique* II **2** 1483–8
[9] Narayan O and Fisher D S 1992 Dynamics of sliding charge–density waves in 4-epsilon dimensions *Phys. Rev. Lett.* **68** 3615–8
[10] Narayan O and Fisher D S 1993 Threshold critical dynamics of driven interfaces in random media *Phys. Rev.* B **48** 7030–42
[11] Balents L and Fisher D S 1993 Large-N expansion of $4 - \varepsilon$-dimensional oriented manifolds in random media *Phys. Rev.* B **48** 5949–63
[12] Leschhorn H, Nattermann T, Stepanow S and Tang L-H 1997 Driven interface depinning in a disordered medium *Ann. Phys., Lpz.* **6** 1–34
[13] Bucheli H, Wagner O S, Geshkenbein V B, Larkin A I and Blatter G 1998 $(4+N)$-dimensional elastic manifolds in random media: a renormalization-group analysis *Phys. Rev.* B **57** 7642–52
[14] Chauve P, Le Doussal P and Wiese K J 2001 Renormalization of pinned elastic systems: How does it work beyond one loop? *Phys. Rev. Lett.* **86** 1785–8 (*Preprint* cond-mat/0006056)
[15] Scheidl S and Dincer Y 2000 Interface fluctuations in disordered systems: Universality and non-Gaussian statistics *Preprint* cond-mat/0006048
[16] Scheidl S 2000–2004 Private communication about 2-loop calculations for the random manifold problem
[17] Chauve P and Le Doussal P 2001 Exact multilocal renormalization group and applications to disordered problems *Phys. Rev.* E **64** 051102 (*Preprint* cond-mat/9602023)

[18] Le Doussal P and Wiese K J 2002 Functional renormalization group at large N for random manifolds *Phys. Rev. Lett.* **89** 125702 (*Preprint* cond-mat/0109204v1)
[19] Le Doussal P, Wiese K J and Chauve P 2002 2-loop functional renormalization group analysis of the depinning transition *Phys. Rev.* B **66** 174201 (*Preprint* cond-mat/0205108)
[20] Le Doussal P and Wiese K J 2003 Functional renormalization group for anisotropic depinning and relation to branching processes *Phys. Rev.* E **67** 016121 (*Preprint* cond-mat/0208204)
[21] Gorokhov D A, Fisher D S and Blatter G 2002 Quantum collective creep: a quasiclassical Langevin equation approach *Phys. Rev.* B **66** 214203
[22] Glatz A, Nattermann T and Pokrovsky V 2000 Domain wall depinning in random media by ac fields *Phys. Rev. Lett.* **90** 047201
[23] Fedorenko A A and Stepanow S 2002 Depinning transition at the upper critical dimension *Preprint* cond-mat/0209171
[24] Le Doussal P and Wiese K J 2003 Higher correlations, universal distributions and finite size scaling in the field theory of depinning *Phys. Rev.* E **68** 046118 (*Preprint* cond-mat/0301465)
[25] Rosso A, Krauth W, Le Doussal P, Vannimenus J and Wiese K J 2003 Universal interface width distributions at the depinning threshold *Phys. Rev.* E **68** 036128 (*Preprint* cond-mat/0301464)
[26] Le Doussal P, Wiese K J and Chauve P 2004 Functional renormalization group and the field theory of disordered elastic systems *Phys. Rev.* E **69** 026112 (*Preprint* cont-mat/0304614)
[27] Le Doussal P and Wiese K J 2003 Functional renormalization group at large N for disordered elastic systems, and relation to replica symmetry breaking *Phys. Rev.* B **68** 17402 (*Preprint* cond-mat/0305634)
[28] Balents L and Le Doussal P 2003 Broad relaxation spectrum and the field theory of glassy dynamics for pinned elastic systems *Preprint* cond-mat/0312338
[29] Le Doussal P and Wiese K J 2004 Derivation of the functional renormalization group β-function at order $1/N$ for manifolds pinned by disorder *Preprint* cond-mat/0406297
[30] Balents L and Le Doussal P 2004 Thermal fluctuations in pinned elastic systems: field theory of rare events and droplets *Preprint* cond-mat/0408048
[31] Wiese K J 2003 The functional renormalization group treatment of disordered systems: a review *Ann. Henri Poincaré* **4** 473–96 (*Preprint* cond-mat/0302322)
[32] Le Doussal P and Wiese K J 2005 2-loop functional renormalization group treatment of pinned elastic manifolds in N dimensions, in preparation
[33] Wiese K J and Le Doussal P 2005 3-loop FRG study of pinned manifolds, in preparation
[34] Mézard M and Parisi G 1991 Replica field theory for random manifolds *J. Physique* I **1** 809–37
[35] Balents L, Bouchaud J P and Mézard M 1996 The large scale energy landscape of randomly pinned objects *J. Physique* I **6** 1007–20