While it is clear that there is a strong qualitative similarity between the phenomenon described by (1.5.8) and the HF effect for neutral superfluids,the precise correspondence is somewhat obscured by the presence of a quadratic term (in $\boldsymbol{A}(r)$) in (1.5.6) which has no analog in (1.5.3). We can clarify the situation by viewing it in the neutral case not from the laboratory frame but from the frame of reference rotating with the container: as shown in Appendix 1A, the expressions both for the effective Hamiltonian, apart from a relatively uninteresting "centrifugal" term, and for the mass current then exactly coincide for the charged and neutral cases, with the correspondence

$$e\boldsymbol{A}(\boldsymbol{r}) \rightleftarrows m\boldsymbol{\omega} \times \boldsymbol{r} \tag{1.5.10}$$

Thus, the HF effect as viewed from the ***rotating*** frame corresponds exactly (apart from the centrifugal effect) to the effect described by Eqn. (1.5.8) as viewed from the *laboratory* frame. I will return in Chapter 5 to the details of the behavior of a superconducting ring in an applied Aharonov–Bohm flux, and in footnote 22 of that chapter will briefly address the question of whether there is a direct analog, in neutral systems, of the Meissner effect as such.

Appendix

1A Statistical mechanics in a rotating container

In order to apply the standard prescriptions of equilibrium statistical mechanics, we need to work in a frame of reference in which the Hamiltonian is time-independent. In the case of a uniformly rotating container, the only such frame is that which rotates with the container (hereafter called simply the "rotating frame"). The question then arises, what is the appropriate form of the Hamiltonian to use in this frame?[23]

For pedagogical simplicity I shall attack this problem in three stages. First, consider a single classical particle of mass m in a container which for convenience I take to be approximately but not exactly cylindrically symmetric. If the container is stationary in the laboratory frame of reference,[24] then the (classical) Hamiltonian is

$$H = \boldsymbol{p}^2/2m + V(\boldsymbol{r}) \equiv H_0 \tag{1.A.1}$$

where $V(\boldsymbol{r})$ is the potential due to the container. Suppose now that the container rotates with angular velocity ω about some axis which we take as the z-axis of an appropriately chosen cylindrical polar coordinate system, and define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{z}}$. Then the form of the kinetic-energy term in (1.A.1) is unchanged, but the potential term becomes $V(\boldsymbol{r}'(t))$, where $\boldsymbol{r}'(t)$, the coordinate viewed from the rotating frame, is specified in terms of its cylindrical polar components by

$$r' = r, \quad z' = z, \quad \theta' = \theta - \omega t \tag{1.A.2}$$

[23] In the following, it is important to distinguish between the statement that the probability distribution is "stationary," that is, time-independent, as viewed from a given frame, and the statement that the system is at rest in that frame; the latter statement implies the former but not vice versa.

[24] Which for the purposes of this discussion I take to be an inertial frame, although strictly speaking, owing to the Earth's rotation, etc., it is not.

Note that the time derivatives of $\boldsymbol{r}$ and $\boldsymbol{r}'$ are related by

$$\dot{\boldsymbol{r}}(t) = \dot{\boldsymbol{r}'}(t) + \boldsymbol{\omega} \times \boldsymbol{r}' \tag{1.A.3}$$

How to find the form of Hamiltonian appropriate for use in the rotating frame? It is tempting but incorrect to simply take the lab-frame Hamiltonian [i.e. (1.A.1) with $V(\boldsymbol{r}) \to V(\boldsymbol{r}'(t))$] and express it in terms of the rotating-frame variables. Rather, we should follow the canonical prescription for doing Hamiltonian mechanics in an arbitrary coordinate system: see e.g. Goldstein (1980), Chapter 8. That is, we start by writing down the Lagrangian and expressing it in rotating-frame coordinates using (1.A.3):

$$\begin{aligned}\mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) &= \tfrac{1}{2} m \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}'[\boldsymbol{r}, t]) \\ &= \tfrac{1}{2} m (\dot{\boldsymbol{r}}' + \boldsymbol{\omega} \times \boldsymbol{r}')^2 - V(\boldsymbol{r}') \equiv \mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}')\end{aligned} \tag{1.A.4}$$

Next, we obtain the canonical momentum in the rotating frame by the standard prescription:

$$\boldsymbol{p}' \equiv \frac{\partial \mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}\prime)}{\partial \dot{\boldsymbol{r}}'} = m(\dot{\boldsymbol{r}}' + \boldsymbol{\omega} \times \boldsymbol{r}') \tag{1.A.5}$$

Note that $\boldsymbol{p}'$ is not the kinematic momentum $m\dot{\boldsymbol{r}}'$ as viewed from the rotating frame. The final step is to define the rotating-frame Hamiltonian $H'(\boldsymbol{r}', \boldsymbol{p}')$ in the standard way, by

$$H'(\boldsymbol{r}', \boldsymbol{p}') \cdot = \dot{\boldsymbol{r}}' \cdot \dot{\boldsymbol{p}}' - \mathcal{L} \tag{1.A.6}$$

Expressing $\mathcal{L}$ in terms of $\boldsymbol{r}'$ and $\boldsymbol{p}'$, and rearranging the terms in the triple product, we find

$$H'(\boldsymbol{r}', \boldsymbol{p}') = \boldsymbol{p}'/2m - \boldsymbol{\omega} \cdot \boldsymbol{r}' \times \boldsymbol{p}' + V(\boldsymbol{r}') \tag{1.A.7}$$

If now the particle is in equilibrium with a thermal bath which is itself stationary in the rotating frame (e.g. the phonons in the container walls) then we can go ahead and apply all the usual rules of equilibrium statistical mechanics: e.g. the probability of finding the particle with coordinate $\boldsymbol{r}'$ and $\boldsymbol{p}'$ will be proportional to the Gibbs factor $\exp[-\beta H'(\boldsymbol{r}', \boldsymbol{p}')]$, and will be (trivially) stationary in the rotating frame and thus in general not stationary in the lab frame. However, we are free to choose a "special" time $t = 2n\pi/\omega$ at which the rotating and lab frames coincide, and express H' as a function of the lab-frame coordinate $\boldsymbol{r}$ and momentum $\boldsymbol{p}(\equiv m\dot{\boldsymbol{r}})$: using (1.A.5) and (1.A.3) we find that at such times

$$H'(\boldsymbol{r}', \boldsymbol{p}') = \frac{p^2}{2m} - \boldsymbol{\omega} \cdot \boldsymbol{r} \times \boldsymbol{p} + V(r) \equiv H_0 - \boldsymbol{\omega} \cdot \boldsymbol{L} \equiv H_{\text{eff}}(\boldsymbol{r}, \boldsymbol{p}) \tag{1.A.8}$$

Thus we know that at such special times the distribution is determined in the lab frame by H_{eff} (and in general, if $V(\boldsymbol{r})$ lacks cylindrical symmetry, will itself not be cylindrically symmetric). At general times, since we know that the distribution is stationary in the rotating frame, its anisotropy will simply rotate, as viewed from the lab frame, with angular velocity ω.

Actually it is often the case that we wish to take the limit of a cylindrically symmetric $V(\boldsymbol{r})$.[25] In that case it is clear that we can simply use the effective Hamiltonian (1.A.8) at all times, and the distribution is then stationary (i.e. time-independent) also as viewed from the lab frame (but may of course correspond to a finite angular velocity as viewed from that frame).

Before proceeding to the many-body case, let's make the analogy with electromagnetism explicit. To do so we go back to (1.A.7) and rewrite it in the equivalent form

$$\begin{aligned} H'(\boldsymbol{r}', \boldsymbol{p}') &= (\boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}')^2/2m + \tilde{V}(\boldsymbol{r}') \\ \tilde{V}(\boldsymbol{r}') &\equiv V(\boldsymbol{r}') - \tfrac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{r}')^2 \end{aligned} \tag{1.A.9}$$

This is the Hamiltonian appropriate to the rotating frame. As viewed from this frame the mass current (kinematic momentum) is

$$\boldsymbol{j}' = m\boldsymbol{r}' = \boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}' \tag{1.A.10}$$

Let's now consider a charged system and write down the standard expressions, in the lab frame but in the presence of an electromagnetic vector potential $A(\boldsymbol{r})$, for the Hamiltonian and for the particle current:

$$H = (\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r}))^2/2m + V(\boldsymbol{r}) \tag{1.A.11}$$

$$\boldsymbol{j} = m\dot{\boldsymbol{r}} = \boldsymbol{p} - eA(\boldsymbol{r}) \tag{1.A.12}$$

From a comparison of (1.A.9) and (1.A.10) with (1.A.11) and (1.A.12) we see that the problem of the neutral system viewed from the rotating frame (with a potential which includes the "centrifugal" contribution $-\frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{r}')^2$) is formally identical to the problem of the charged system viewed from the rest frame, with the correspondence $e\boldsymbol{A}(\boldsymbol{r}) \rightleftharpoons m(\boldsymbol{\omega} \times \boldsymbol{r})$. In the case of a constant magnetic field $\boldsymbol{B}$ the vector potential $\boldsymbol{A}(\boldsymbol{r})$ can be written, by a suitable choice of gauge, in the form $\frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r})$, so we have the correspondence $\boldsymbol{\omega} \rightleftharpoons e\boldsymbol{B}/2m$.

It is now straightforward to generalize the above results to the many-body case of interest, *provided* the two-body interatomic-potential $U(\boldsymbol{r}_i - \boldsymbol{r}_j)$ is a function only of the distance $|\boldsymbol{r}_i - \boldsymbol{r}_j| \equiv| \boldsymbol{r}'_i - \boldsymbol{r}'_j|$ and hence invariant under the rotation. (If this condition is not fulfilled, as is likely to be the case (e.g.) in the case of appreciable spin–orbit interaction, then there exists in general no frame of reference in which the Hamiltonian is time-independent, so we cannot do equilibrium statistical mechanics and the problem becomes similar to that of Couette flow (see e.g. Chossat and Iooss 1994).) With this premise the argument is an obvious generalization of the above one for the single case: in the rotating frame the Hamiltonian is

$$H'\{\boldsymbol{r}'_i, \boldsymbol{p}'_i\} = \sum_i (\boldsymbol{p}'_i - m(\boldsymbol{\omega} \times \boldsymbol{r}'_i))^2/2m + \sum_i \tilde{V}(\boldsymbol{r}_i) + \frac{1}{2}\sum_{ij} U(|\boldsymbol{r}'_i - \boldsymbol{r}'_j|) \tag{1.A.13}$$

(where $\tilde{V}$ contains the centrifugal term), and in the lab frame we have

$$H'\{\boldsymbol{r}'_i . \boldsymbol{p}'_i\} = H_0\{\boldsymbol{r}_i, \boldsymbol{p}_i\} - \boldsymbol{\omega} \cdot \boldsymbol{L}, \quad \boldsymbol{L} \equiv \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i \tag{1.A.14}$$

[25] For consistency we must take this limit only *after* taking the limit of infinite time, in order to establish thermal equilibrium with the rotating container.