Mean field theory of the yielding transition

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An overview of a series of theoretical works with:
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Yielding of amorphous solids
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Physique : Atomes Lumière Matière

Outline

Introduction

How to solve structural glass models in infinite dimension: main ingredients

Part 1: Phase diagrams

Phase diagram of structural glasses in infinite dimensions. Emerge of two glass phases separated by the Gardner transition.

Part 2: A brief primer on the Gardner transition

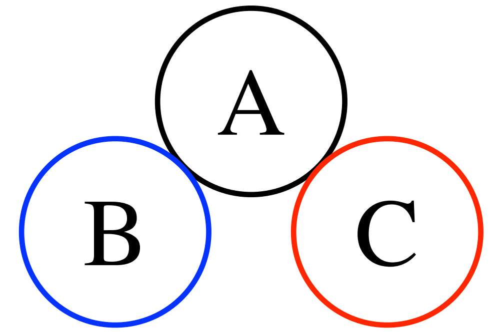
What is the Gardner transition and what are the properties of the two glass phases.

Part 3: Adding the strain

Infinite d stress strain curves. Shear jamming-Shear yielding phase diagram. Breakdown of elasticity. Criticality of the yielding transition.

Mean field theory of structural glasses. 1

First ingredient: large dimension



$$S[\rho(x)] = \int \mathrm{d}x \rho(x)(1 - \log \rho(x)) + \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \rho(x)\rho(y) f(x,y) + \text{three body terms}$$

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \qquad f(x,y) = -1 + e^{-\beta v(|x-y|)}$$ Mayer Function

The same can be done in the Ising model: write down the free energy as a function of the local magnetizations (Thouless, Anderson, Palmer, Georges, Yedidia, Plefka...)

$$-\beta F[\{m_i\}] = \sum_{i=1}^{N} S(m_i) + \frac{\beta}{2} \sum_{i,j=1}^{N} m_i J_{ij} m_j$$

$$\sim \int \mathrm{d}x \, S(m(x)) + \frac{\beta}{2} \int \mathrm{d}x \mathrm{d}y \, m(x) J(x-y) m(y)$$

Landau, Ginzburg

Mean field theory of structural glasses. 2

We want to study glassy metastable states

Recipe (Franz - Parisi):

- Consider a reference system of spheres (the master)
- Consider a slave system whose configurations are constrained to be close to the ones of the master system (the slave)
- Compute the free-energy of the slave system averaged over the configurations of the master systems

Introduce replicas

Mean field theory of structural glasses. 3

The order parameter

$$\Delta_{ab} = \frac{d}{N\mathcal{D}^2} \sum_{i=1}^{N} \left| x_i^{(a)} - x_i^{(b)} \right|^2$$

Mean square displacement matrix

Physical interpretation

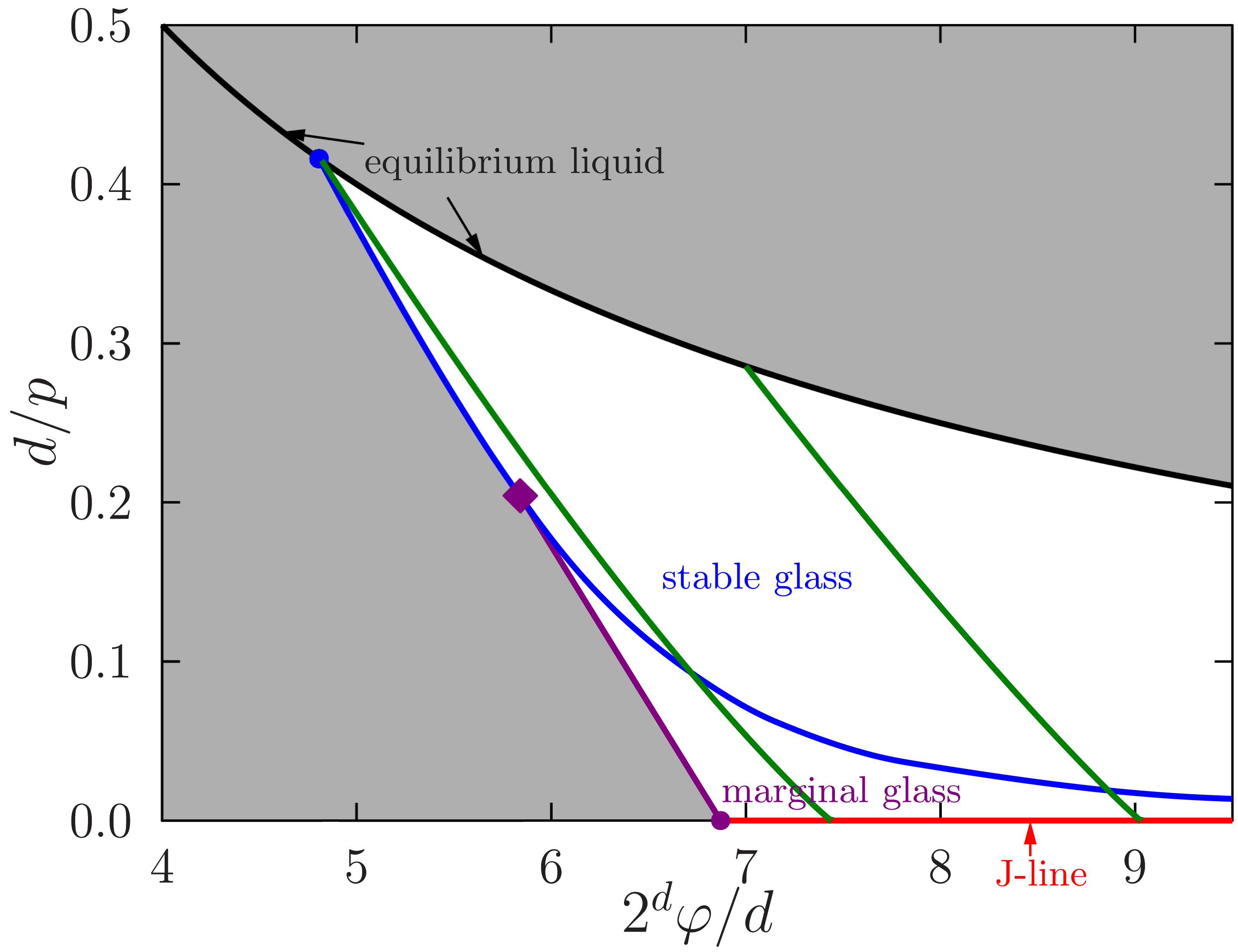
Dynamical MSD $\Delta(t) = \dfrac{d}{\mathcal{D}^2 N} \displaystyle\sum_{i=1}^{N} |x_i(t) - x_i(0)|^2$

$$\lim_{t\to\infty} \Delta(t) = \Delta_{a\neq b}$$

The liquid phase	The glass phase
$\Delta_{ab} \to \infty$	$\Delta_{ab} < \infty$
Particles diffuse	Particles are caged

MFT of Hard Sphere glasses

Compression



Kurchan, Parisi, Zamponi,
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J. Phys. Chem. B 117 (42), 12979 (2013)

Rainone, Urbani, Yoshino, Zamponi
PRL **114** (1) 015701 (2015)

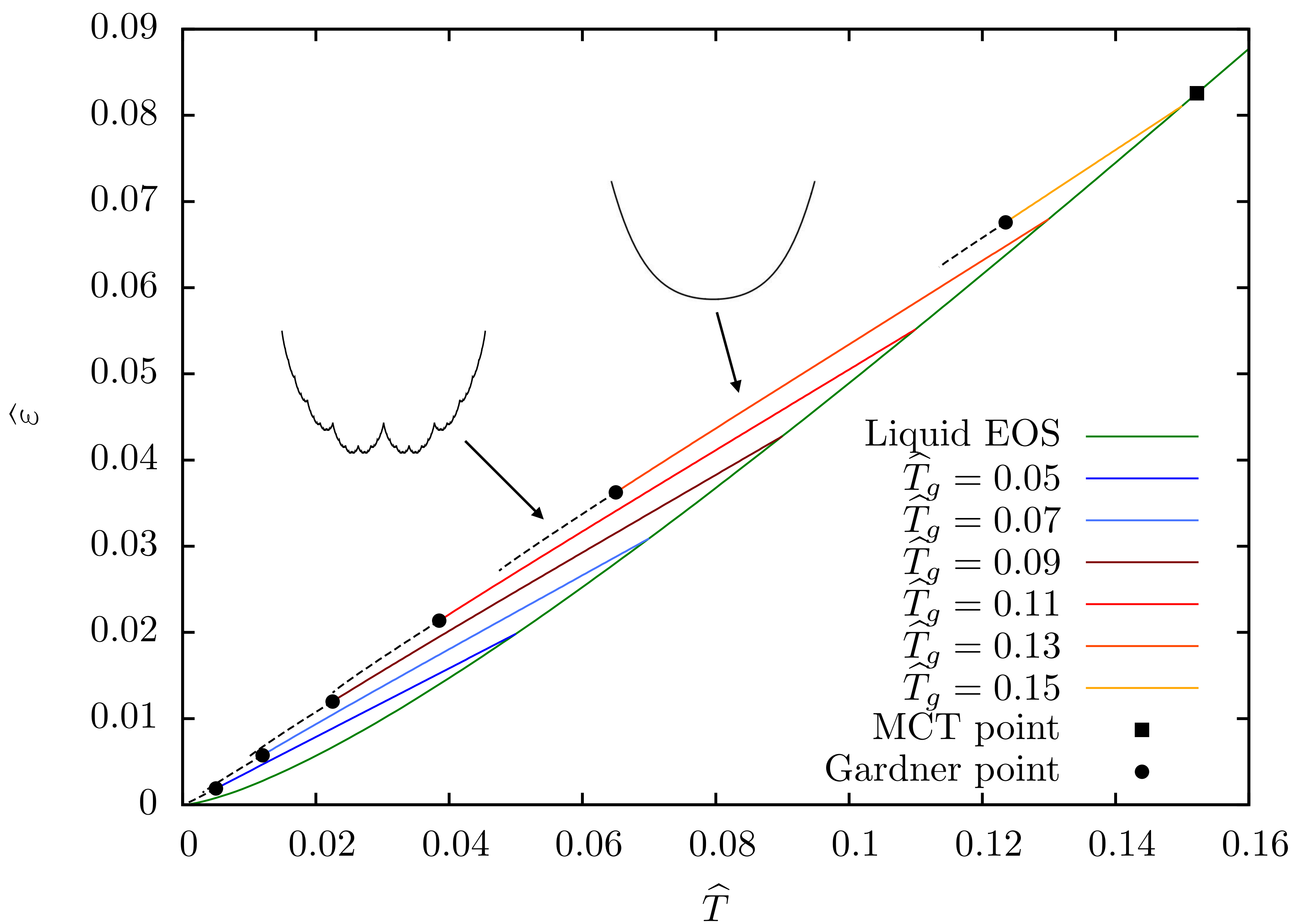
Charbonneau, Kurchan, Parisi, Urbani, Zamponi
Nature Communications 5, 3725 (2014)

MFT of Soft Sphere glasses

Harmonic Soft Spheres

$$h_{ij} = |x_i - x_j| - D$$

$$H[\{x_i\}] = \frac{\epsilon}{2} \sum_{\langle i,j \rangle} h_{ij}^2 \theta(-h_{ij})$$

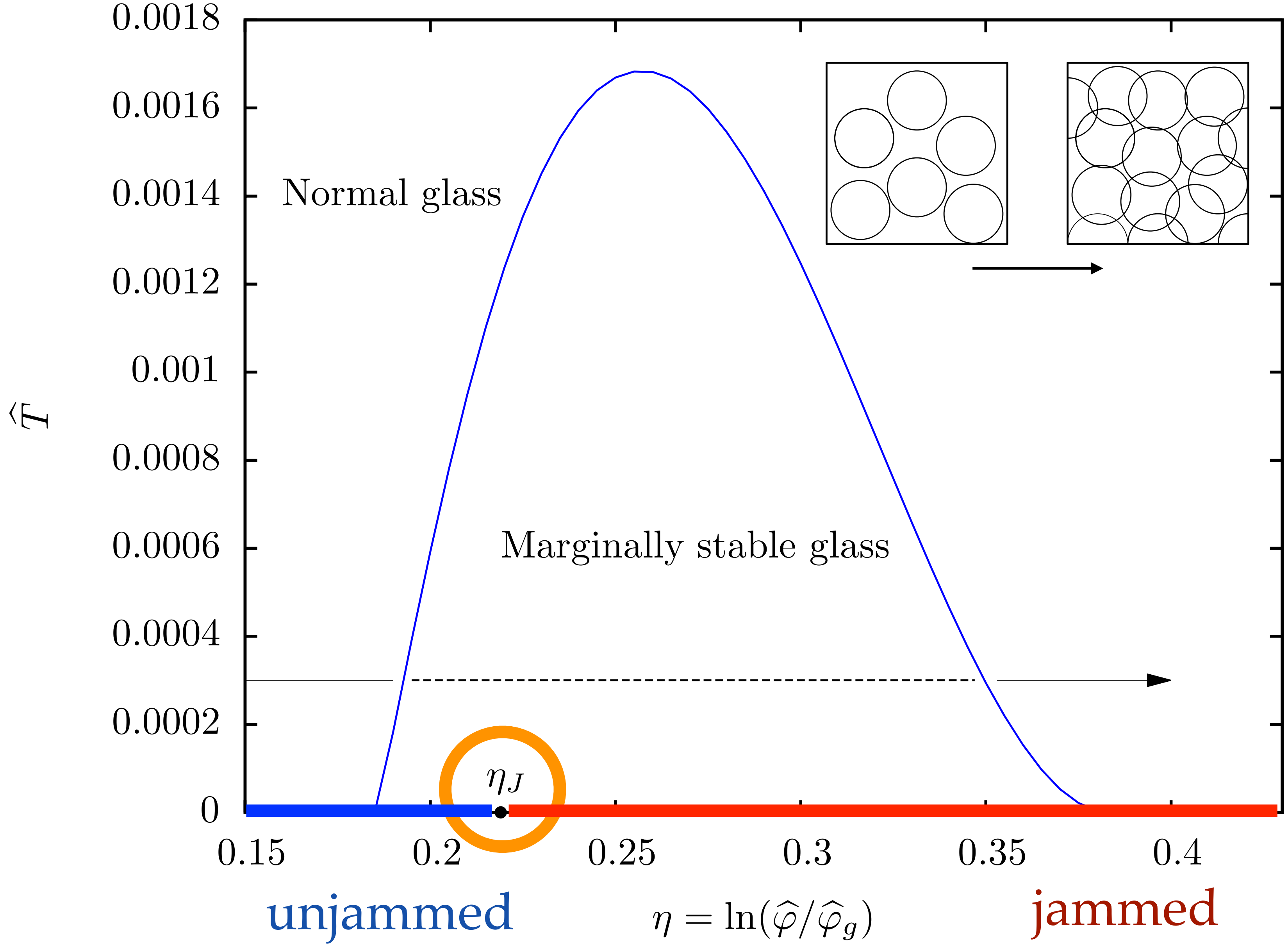


Cooling

Biroli, Urbani, Nat. Phys. **12**, 1130–1133 (16)

MFT of Soft Sphere glasses

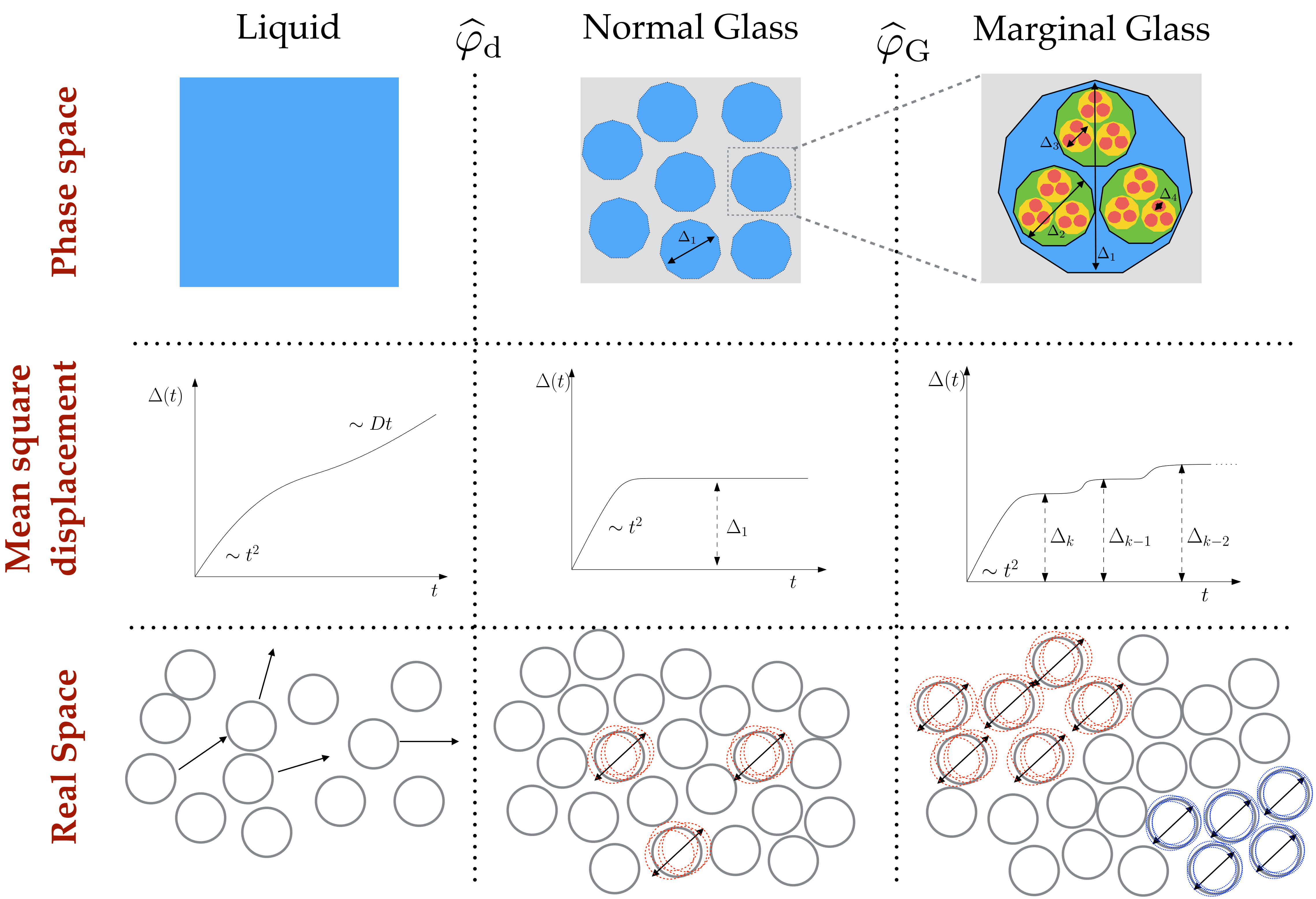
Compression: the importance of jamming



Biroli, Urbani, Nat. Phys. **12**, 1130–1133 (16)

A dome of marginally stable glass surrounds the jamming point

Three different phases



Charbonneau, Kurchan, Parisi, Urbani, Zamponi, Ann. Rev. Cond. Mat. 8, (2017)

The Gardner transition

- It is a continuous phase transition: diverging correlation length, diverging relaxation time, diverging susceptibility (the spin glass susceptibility)

- It is analogous to the spin glass transition in a field

- Beyond the Gardner point aging effects set in

- The Gardner phase is marginal: emergence of soft modes

However remember that this is a transition inside a metastable state (unless you cool or compress the ideal glass where an equilibrium Gardner transition can also appear)

Adding a small strain: elastic responses

Biroli, Urbani, Nat. Phys. **12**, 1130–1133 (16)

Study the elastic behavior of glasses

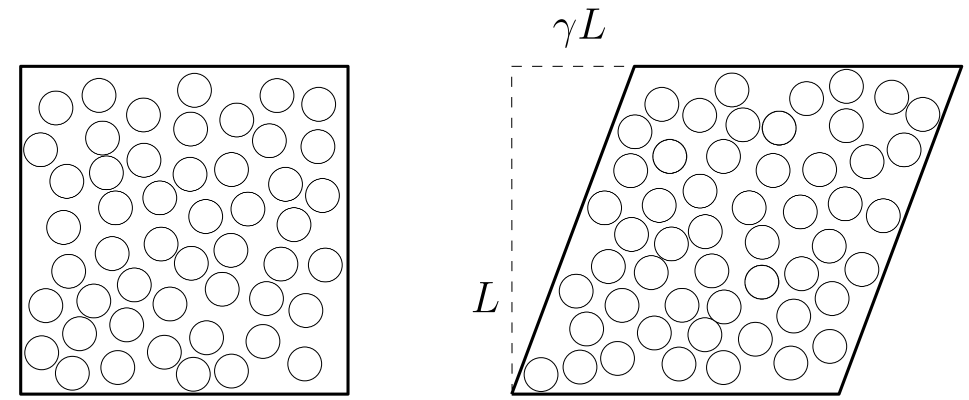
$$\frac{F_\alpha(\gamma)}{V} = \frac{F_\alpha(0)}{V} + \frac{1}{2}\mu_2^{(\alpha)}\gamma^2 + \frac{1}{4!}\mu_4^{(\alpha)}\gamma^4 + \dots$$

Shear Modulus ($\mu_2^{(\alpha)}$) — Nonlinear elastic susceptibility ($\mu_4^{(\alpha)}$)

Normal glass phase	At the Gardner critical point
$\overline{\mu_k} \sim \mathcal{O}(1)$	$\overline{\mu_k} \sim \mathcal{O}(1)$
$\overline{\left(\delta\mu_k\sqrt{V}\right)^2} \sim \mathcal{O}(1)$	$\overline{\left(\delta\mu_k\sqrt{V}\right)^2} \sim \frac{1}{\lvert T - T_G\rvert^{2k-3}}$
Normal fluctuations	**Anomalous (divergent!) fluctuations**

Adding a large strain deformation

Strain deformation



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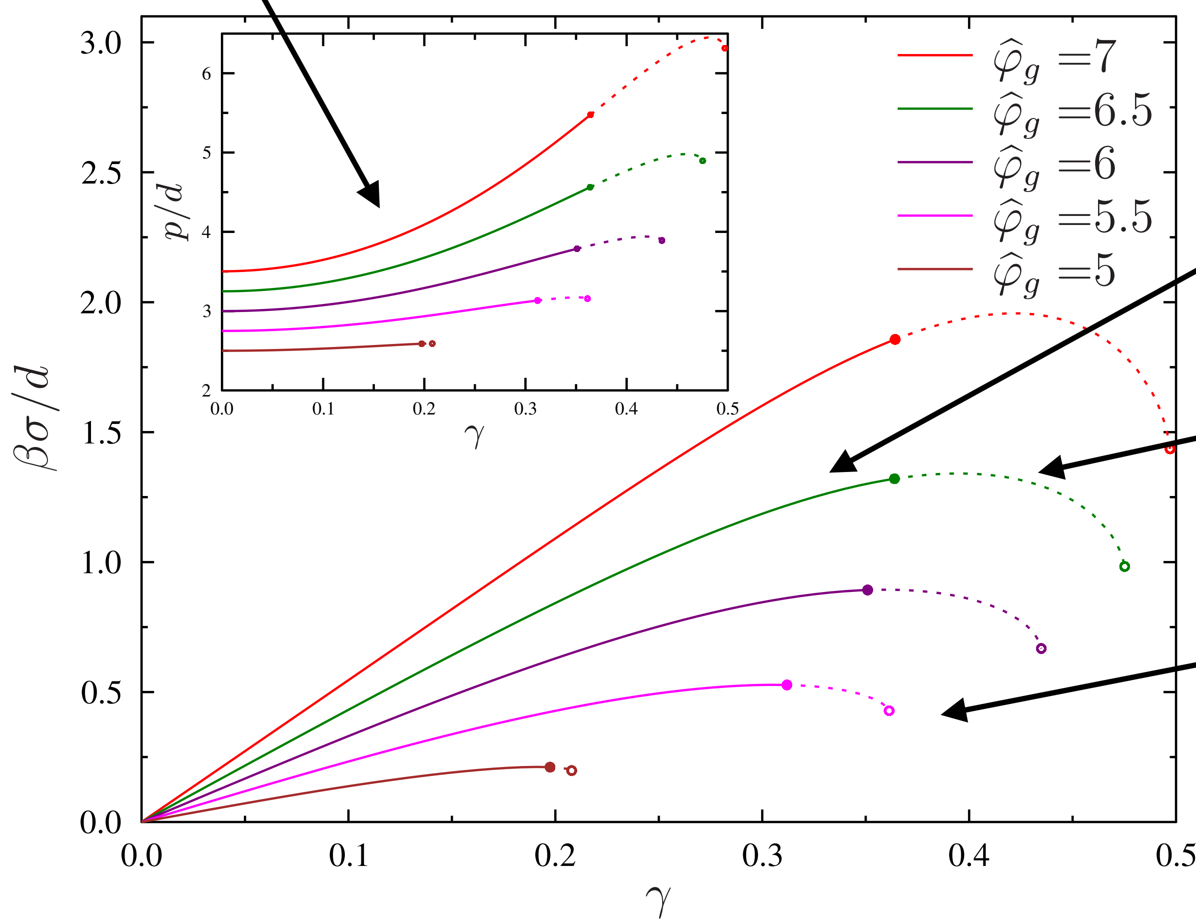
Stress-strain curves



Elastic phase

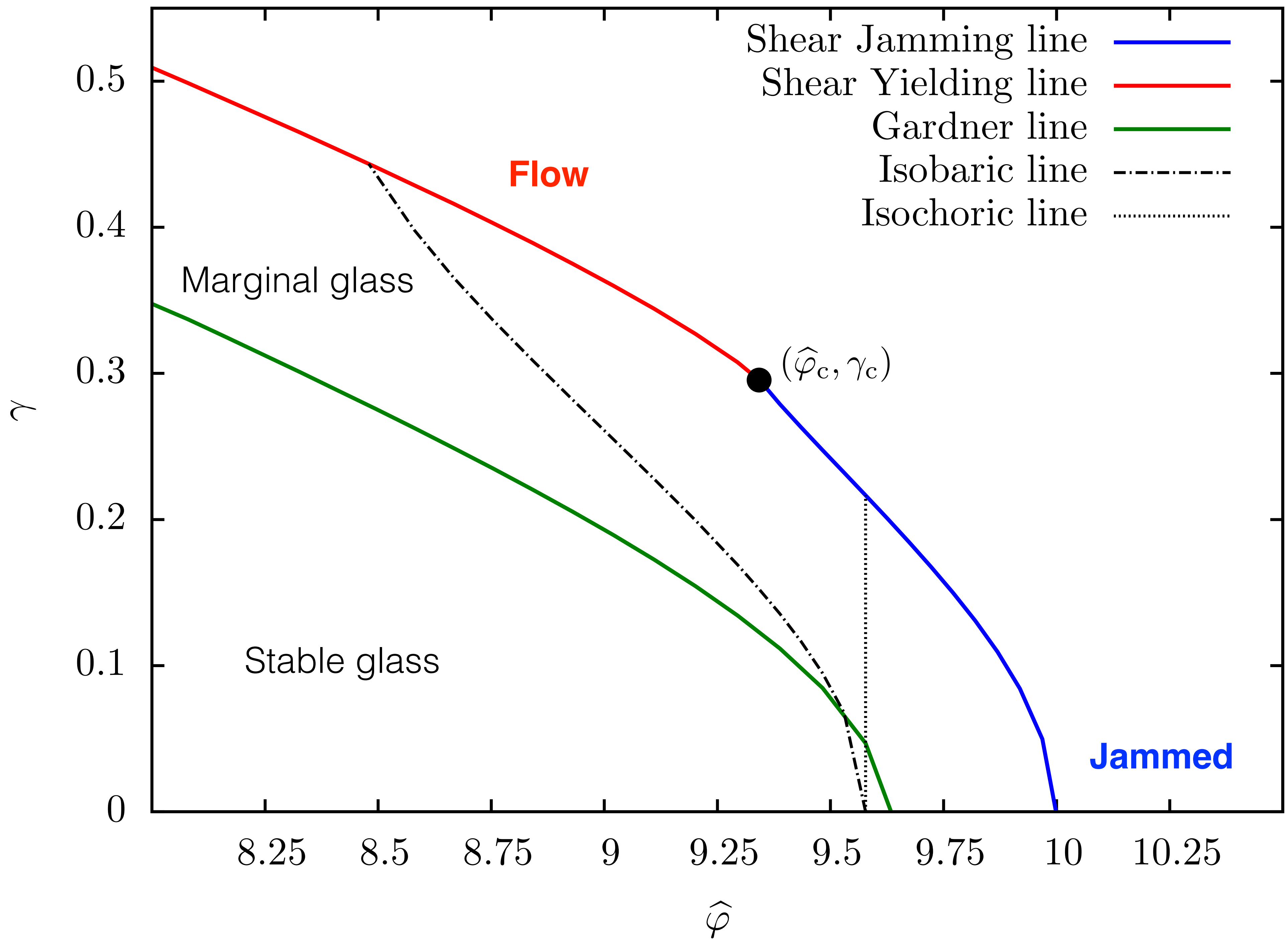
Gardner phase (plastic?)

Flow phase

The same picture holds for hard and soft spheres

Rainone, Urbani, Yoshino, Zamponi
PRL **114** (1) 015701 (2015)

Shear-Jamming *vs* Shear-Yielding



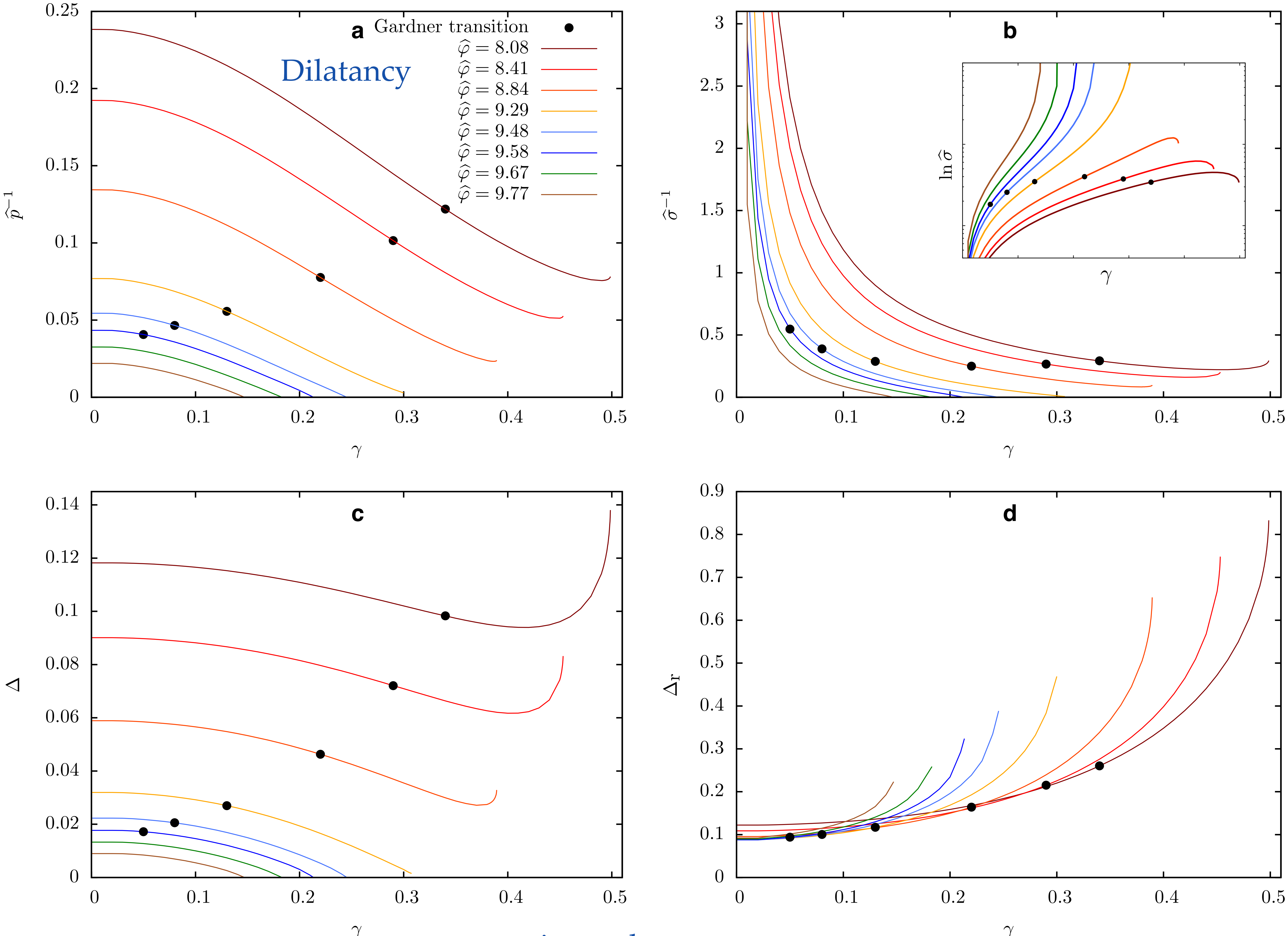
Urbani, Zamponi, Phys. Rev. Lett. 118 (3), 038001 (2017)
+ **Y. Jin, H. Yoshino**

+ **See talk by H. Yoshino!**

Shear-Jamming *vs* Shear-Yielding

Urbani, Zamponi, PRL **118**, 038001 (2017)

stress overshoot



caging order parameters

Order parameters

$$\{x_i(t=0,\gamma=0)\}$$

The collective coordinate of the system at zero strain and initial time

$$\{x_i(t,\gamma)\}$$

The collective coordinate of the system at strain γ and time t

$$\Delta_{\mathrm{r}}(t;\gamma) = \frac{1}{N}\sum_{i=1}^{N} \left|x_i(t,\gamma) - x_i(0,0)\right|^2$$

Distance from the initial config.

$$\Delta(t,t';\gamma) = \frac{1}{N}\sum_{i=1}^{N} \left|x_i(t;\gamma) - x_i(t',\gamma)\right|^2$$

Distance between two dynamical configs.

A dynamic→static dictionary

The elastic regime	$\lim_{t\to\infty} \Delta_{\mathrm{r}}(t,\gamma) = \Delta_r < \infty$ $\lim_{t'\to\infty} \Delta(t,t'\gamma) = \Delta(t-t')$ $\lim_{t-t'\to\infty} \Delta(t-t';\gamma) = \Delta < \infty$
The Gardner phase (plastic regime)	$\lim_{t\to\infty} \Delta_{\mathrm{r}}(t,\gamma) = \Delta_r < \infty$ $\lim_{t'\to\infty} \lim_{t\to\infty} \Delta(t,t') = \begin{cases} \Delta(t-t') \to \Delta_{EA} \\ \Delta(t,t') \quad \text{aging regime} \end{cases}$
The flow phase	$\lim_{t\to\infty} \Delta_{\mathrm{r}}(t;\gamma) = \infty$ $\lim_{t,t'\to\infty} \Delta(t,t';\gamma) = ?$

Susceptibilities

From the elastic to the Gardner phase

$$\chi_4 = N\left[\lim_{t-t'\to\infty}\langle\Delta^2(t-t';\gamma)\rangle - \Delta^2\right] \to \infty$$

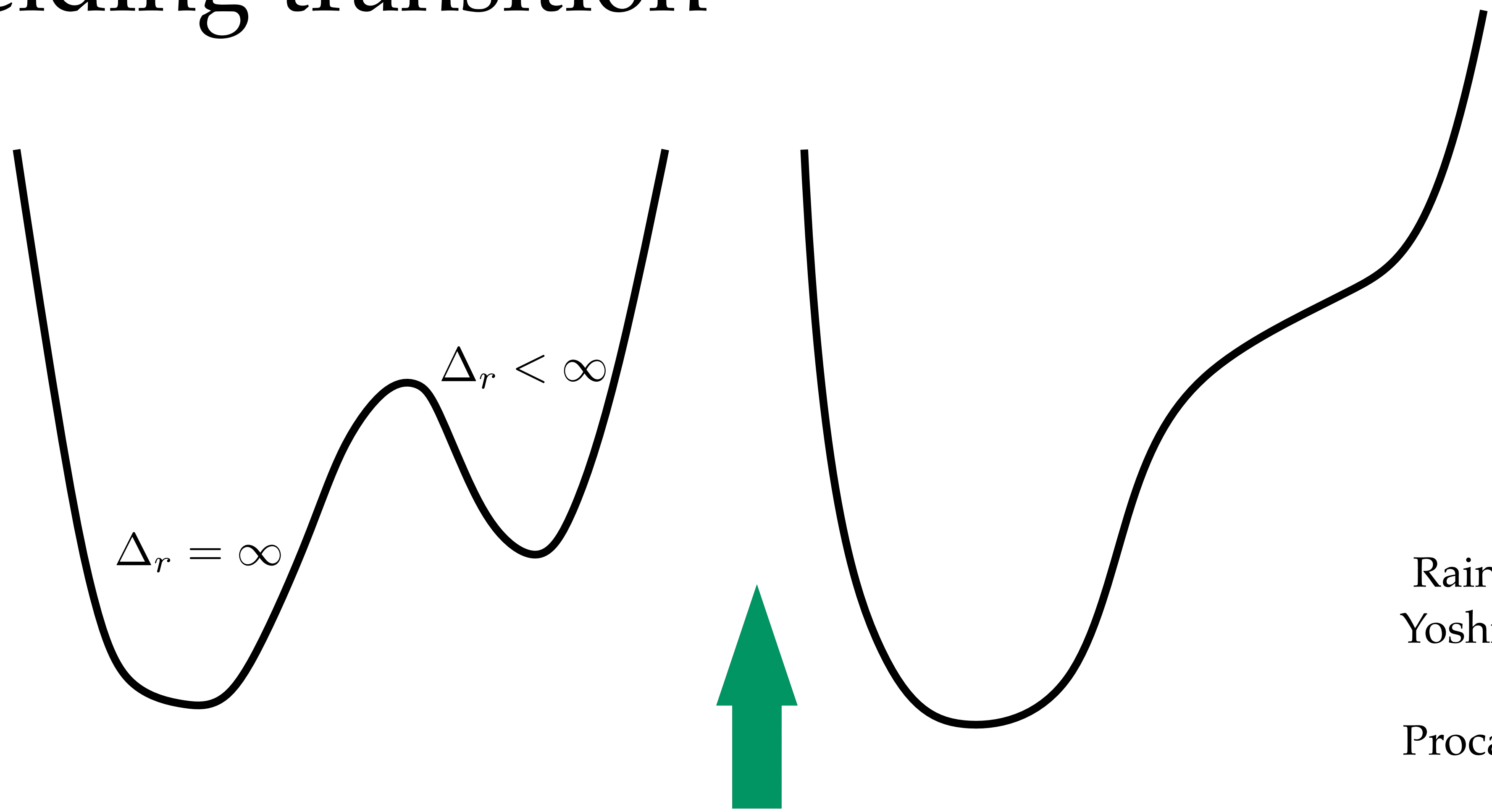
$$\Delta(t,t';\gamma) = \frac{1}{N}\sum_{i=1}^{N}\left|x_i(t;\gamma) - x_i(t',\gamma)\right|^2$$

when $t \to \infty$ two different replicas, say A and B

Therefore we can compute the MSD of two independent replicas evolving in the same glass metabasin

$$\Delta_{AB} = \lim_{t\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left|x_i^{(A)}(t) - x_i^{(B)}(t)\right|^2 \qquad \chi_4 \sim N\left[\langle\Delta_{AB}^2\rangle - \langle\Delta_{AB}\rangle^2\right]$$

The yielding transition



Rainone, Urbani, Yoshino, Zamponi
+
Procaccia's Group

spinodal transition

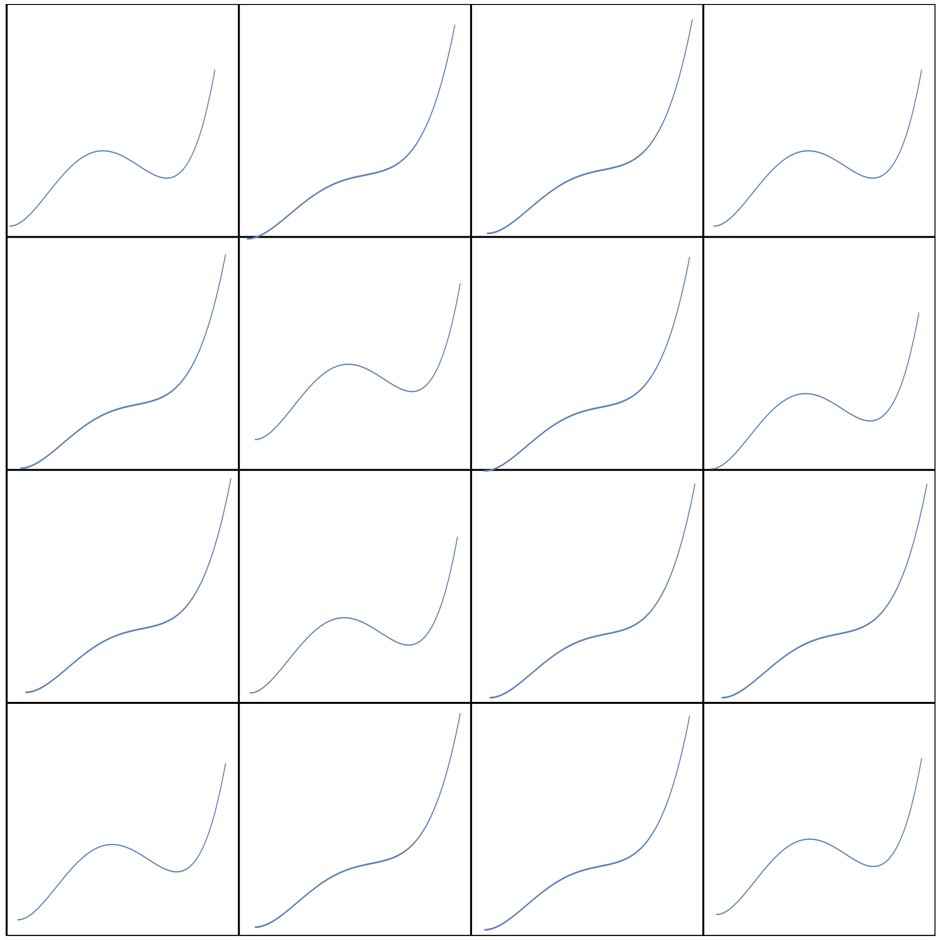
jump of the order parameter $\Delta_r \to \infty$

susceptibility $\chi_r \sim N \left[\langle \Delta_r^2 \rangle - \langle \Delta_r \rangle^2 \right]$

Universality classes

Perturbative regime. The simplest scenario

If yielding is reached coming from a stable glass phase (no plastic or Gardner phase before), one expects the criticality of a simple spinodal with disorder like the spinodal of the random field Ising model. This is analogous to MCT.



The local distance from the critical point is

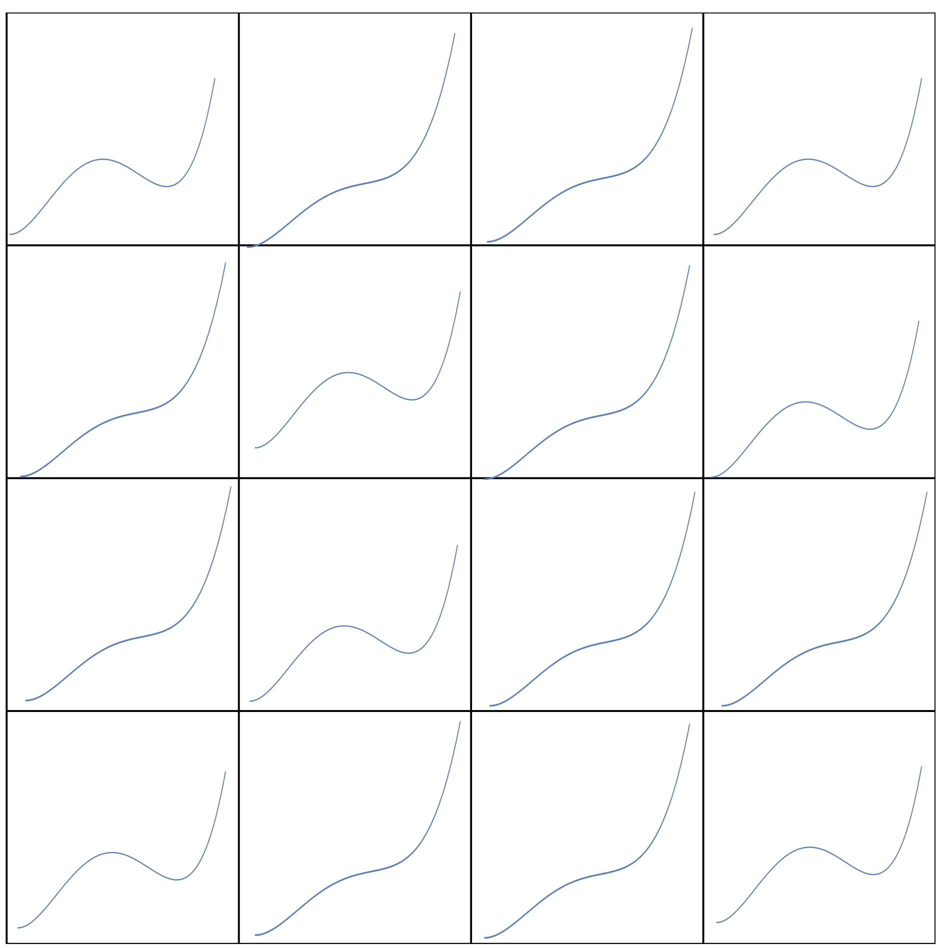
$$P(x) \sim x^{\theta} \qquad \theta = 0$$

Analogous to depinning

Universality classes

Perturbative regime. An alternative scenario:
<u>marginal stability</u>

If yielding is reached coming from a marginal glass phase (Gardner phase before), it could happen that marginal stability affects the critical behavior



The local distance from the critical point is

$$P(x) \sim x^{\theta} \qquad \theta > 0$$

The two scenarii are different and the avalanches are expected to be different

What we still miss at the MFT level

The study of avalanches within this theory is still in its infancy.
It has been done close to jamming or in the jammed phase but for infinitesimal strain when in the Gardner phase.

The flow regime is purely dynamical for the moment.
Some-time-ago ideas (Barrat, Berthier, Kurchan) suggest that the system surfs on threshold (marginal) states (like saddles).
Can we describe more carefully the landscape of glassy states in the flow region?

Non perturbative fluctuations: finite dimensions

?

First order transition at finite T? (what happens a T=0? non perturbative effects?). How study nucleation and include shear bands is still a big problem.

Conclusions

The infinite dimensional solution of structural glass models gives access to the theoretical understanding of the rheological properties of amorphous solids.

The Gardner transition signals the change from an elastic regime to a plastic one.

Yielding is a spinodal transition with an associated diverging susceptibility.

The universality class of the perturbative critical theory is under investigation and some news may be expected.

A characterization of the flow regime is still lacking both from the dynamical point of view and from the static perspective. What is the landscape of metastable states in the flow regime?

The theory completely misses the non-perturbative effects.